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# Exact Traveling Wave Solutions of the Whitham-Broer-Kaup-Like Equation with Time-Dependent Coefficients

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**Abstract:** The first integral method (FIM) is employed to solve the different type solutions of Whitham-Broer-Kaup-Like (WBKL) equation with time-dependent coefficient. We have acquired different types of solutions of this equation. We have also acquired the constraint conditions for the existence of the obtained solitons according to the parameters. It is shown that the method is effective and a direct method, based on the ring theory of commutative algebra.

**Keywords:** FIM, WBKL, Analytical and Soliton solutions.

## 1 Introduction

Nonlinear evolution equation is generally used to express the mathematical modeling of scientific system. Accordingly, it is decisive to generate general solutions of these analogous nonlinear equations. Consequently, the general solutions of these kinds of equations give a lot of an insight into character and the structure of the equations for authors. Many effective techniques have been modified to generate more ideas and knowledge for physicians and engineers. One important notice is that most of these techniques make use of the wave variable transformation to transform the NPDE to ODE in order to acquire the solitons. For these methods, we refer to look [1]-[17].

The FIM has firstly been proposed to the literature by acquiring solutions to the Burgers-KdV equation [7]. The technique has successfully been invented to NPDE and also some fractional differential equations. More recently, different types of research via this method have been proposed.

This article is organized as follows: In Section 2, the description of the FIM is given. In Section 3, the exact solutions of Eq. (3.1) using the FIM are obtained. In Section 4, some conclusions are presented.

## 2 Description of the FIM

The general description of the FIM is given below:
**Step 1.** Consider a NPDE in the form

$$W(q,q_t,q_x,q_{xt},q_{tt},q_{xx},\ldots)=0, \qquad (2.1)$$

where $W$ is a polynomial in its arguments and subscripts denote partial derivatives. Using the following wave transformation

$$\xi = x \mp \lambda t, \qquad (2.2)$$

to reduce Eq. (2.1) to the following nonlinear ODE:

$$H(Q,Q',Q'',Q''',\ldots)=0,\ Q'=\partial Q(\xi)/\partial\xi. \qquad (2.3)$$

**Step 2.** Assume that the solution of ODE (2.3) can be written as

$$q(x,t)=q(\xi). \qquad (2.4)$$

**Step 3.** We introduce a new independent variable

$$Q(\xi)=q(\xi), G(\xi)=\partial q(\xi)/\partial\xi \qquad (2.5)$$

which leads to a system of nonlinear ODEs

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$$\begin{aligned} \partial Q(\xi)/\partial \xi &= G(\xi) \\ \partial F(\xi)/\partial \xi &= P(Q(\xi),G(\xi)). \end{aligned} \quad (2.6)$$

**Step 4.** According to the qualitative principle of ODE [18], when we have the possibility of reaching the integrals for system (2.6), the solutions of (2.6) could immediently be obtained. On some certain independent plane system, no approximation exist giving the guidance on how to reach its first integrals. However, we receive an insight into how to reach the first integrals by the division theorem (DT) [19].

## 3 Application

In recent years, the WBKL equation has been studied by many researchers. Some of these studies are Guo and Zhou: $G'/G$ -expansion method [20], Aminikhah et al.: functional variable method [21], Song et al.: bifurcation method [22]. We have acquired WBKL equation with time-dependent coefficient as

$$p_t + pp_x + a(t)\,q_x + b(t)\,p_{xx} = 0, \quad (3.1)$$

$$q_t + (pq)_x + c(t)\,p_{xxx} - d(t)\,q_{xx} = 0. \quad (3.2)$$

For simplicity, we seek its usefulness to retype the WBKL equation, by using Miura type transformation, in single form. So if we take $p = w_x$ and then we have

$$q = -\frac{1}{a}\left(w_t + \frac{1}{2}w_x^2 + bw_{xx}\right)$$

Therefore we have the single form of the WBKL equation with the constraint $a = c$ and $b = d$ (for simplicity) as

$$w_{tt} + 2w_x w_{xt} + w_t w_{xx} + \frac{3}{2}w_x^2 w_{xx} - \left(a^2+b^2\right) w_{xxxx} = 0. \quad (3.3)$$

In Eq. (3.3),we use the wave transformation (2.2), we have

$$\lambda^2 w_{\xi\xi} + 3\lambda w_\xi w_{\xi\xi} + \frac{3}{2}w_\xi^2 w_{\xi\xi} - \left(a^2+b^2\right) w_{\xi\xi\xi\xi} = 0. \quad (3.4)$$

By integrating the Eq. (3.4) and using the transformation $Q = w_\xi$ we have

$$n + \lambda^2 Q + \frac{3}{2}\lambda Q^2 + \frac{1}{2}Q^3 - \left(a^2+b^2\right) Q_{\xi\xi} = 0, \quad (3.5)$$

where $n$ is an arbitrary integration constant. From (2.3) and (2.4) we acquire

$$\begin{aligned} Q_\xi &= G(\xi), \\ G_\xi(\xi) &= \frac{n}{a^2+b^2} + \frac{\lambda^2}{a^2+b^2}Q(\xi) + \frac{3\lambda}{2\left(a^2+b^2\right)}Q^2(\xi) \\ &+ \frac{1}{2\left(a^2+b^2\right)}Q^3(\xi). \end{aligned} \quad (3.6)$$

According to the FIM, we can write

$$F(Q(\xi),G(\xi)) = \sum_{i=0}^{r} a_i(Q)G^i = 0, \quad (3.7)$$

where $a_i(Q)$,$(i = 0,1,2,\ldots,r)$ are polynomials of $Q$ and $a_m(Q) \neq 0$. Eq. (3.7) is the first integral for system (3.6), thus we get

$$\begin{aligned} dF/d\xi &= \frac{dF}{dQ}\frac{dQ}{d\xi} + \frac{dF}{dG}\frac{dG}{d\xi} \\ &= [g(Q)+h(Q)G]\sum_{i=0}^{r} a_i(Q)G^i. \end{aligned} \quad (3.8)$$

We have considered two different cases, assuming that $r = 1$ and $r = 2$ cases in Eq. (3.8) :

**Case 1.** Suppose that $r = 1$, by equating the coefficients of $G^i (i = 0,1,2,\ldots,r)$, we have

$$\dot{a}_1(X) = h(Q)a_1(Q), \quad (3.9)$$

$$\dot{a}_0(X) = a_1(Q)g(Q) + h(Q)a_0(Q), \quad (3.10)$$

$$a_0(Q)g(Q) = a_1(Q)\xi, \quad (3.11)$$

where $\xi = \left[\frac{\lambda^2}{\beta}Q(\xi) + \frac{3\lambda}{2\beta}Q^2(\xi) + \frac{1}{2\beta}Q^3(\xi)\right]$ and $\beta = \left(a^2+b^2\right)$.

Since $a_1(Q)(i = 0,1)$ are polynomials of $Q$, then from (3.9) we have deduced that $a_1(Q)$ is a constant and $h(Q) = 0$. For simplicity, we have $a_1(Q) = 1$. By balancing, the degrees of $g(Q)$ and $a_0(Q)$ we acquire $\deg[g(Q)] = 0$.So, we suppose that $g(Q) = A_0 Q + A_1$. We acquire from Eq. (3.10)

$$a_0(Q) = \frac{A_0}{2}Q^2 + A_1 Q + A_2. \quad (3.12)$$

Replacing $a_0(Q)$,$a_1(Q)$ and $g(Q)$ in Eq. (3.11) then by balancing the coefficients of $Q^i$ to zero, we have

$$A_0 = \pm\frac{1}{\sqrt{a^2+b^2}}, A_1 = \pm\frac{\lambda}{\sqrt{a^2+b^2}}, A_2 = n = 0. \quad (3.13)$$

Setting (3.13) in (3.7), we have

$$Q_\xi = -\frac{A_0}{2}Q^2(\xi) - A_1 Q(\xi) - A_2. \quad (3.14)$$

If we solve Eq. (3.14) by using (2.3) and (2.4) respectively, we acquire

$$Q_1(\xi) = -\lambda\left(1+\tan\left[\frac{\lambda}{2\sqrt{a^2+b^2}}\xi-\lambda c_0\right]\right), \quad (3.15)$$
$$Q_2(\xi) = \lambda\left(-1+\cot\left[\frac{\lambda}{2\sqrt{a^2+b^2}}\xi-\lambda c_0\right]\right), \quad (3.16)$$

where $c_0$ is an arbitrary constant. For the solution (3.15), from the transformations $Q = w_\xi$, $p = w_x$, $q = -\frac{1}{a}\left(w_t+\frac{1}{2}w_x^2+bw_{xx}\right)$ and $\xi = x+\lambda t$ we get

$$p(x,t) = -\lambda\left(1+\tan\left[\frac{\lambda(x+\lambda t)}{2\sqrt{a^2+b^2}}-\lambda c_0\right]\right),$$

$$q(x,t) = \frac{-1}{a}\left(-\lambda^2-\frac{\lambda^2 b}{2\sqrt{a^2+b^2}}\sec^2\chi+\frac{1}{2}-\lambda(1+\tan\chi)^2 + \frac{2aa_t+2bb_t}{\sqrt{a^2+b^2}}\log[\cos\chi] - \left(\lambda^2+\frac{\lambda(x+\lambda t)(2aa_t+2bb_t)}{a^2+b^2}\right)\tan\chi\right),$$

where $\chi = \left[\frac{\lambda(x+\lambda t)}{2\sqrt{a^2+b^2}}-\lambda c_0\right]$.

**Case 2.** If we equate the coefficients of $G^i$ of Eq. (3.6) for $r = 2$, we have

$$\dot{a}_2(Q) = h(Q)a_2(Q), \quad (3.17)$$

$$\dot{a}_1(Q) = a_2(Q)g(Q)+h(Q)a_1(Q), \quad (3.18)$$

$$a_1(Q)g(Q)+h(Q)a_0(Q) = \dot{a}_0(Q)+2a_2(Q) \left(m+\left[c^2+nB(t)\right]Q(\xi)-\frac{c}{2}[A(t)+B(t)+C(t)]Q^2(\xi) + \frac{C(t)}{6}[2A(t)+B(t)]Q^3(\xi)\right), \quad (3.19)$$

$$a_1(Q)\dot{G} = a_0(Q)g(Q). \quad (3.20)$$

Since $a_2(Q)(i = 0,1,2)$ are polynomials of $Q$, we have deduced that $a_2(Q)$ is a constant and $h(Q) = 0$ from (3.17). For simplicity, we have $a_2(Q) = 1$. Balancing the degrees of $g(Q)$,$a_1(Q)$ and $a_2(Q)$ we have concluded that $\deg[g(Q)] = 1$.Suppose that $g(Q) = A_0Q+A_1$, and we acquire from Eq. (3.18)

$$a_1(Q) = \frac{A_0}{2}Q^2+A_1Q+A_2. \quad (3.21)$$

Replacing $a_0(Q), a_1(Q), a_2(Q)$ and $g(Q)$ in Eq. (3.18) and balancing the parameters of $Q^i$ to zero.

We acquire $\deg[g(Q)] = 0$. We assume that $g(Q) = A$, we consider $a_2(Q) = 1$, and by balancing the degrees of $g(Q), a_1(Q), a_2(Q)$, we acquire from Eq. (3.18)

$$a_0(Q) = A_4Q^4+A_3Q^3+A_2Q^2+A_1Q+A_0. \quad (3.22)$$

So, we have

$$A_0 = A_0,\ A_1 = \frac{-2n}{a^2+b^2},\ A_2 = \frac{-2\lambda^2}{a^2+b^2}, \quad (3.23)$$
$$A_3 = \frac{-\lambda}{a^2+b^2},\ A_4 = \frac{-1}{4(a^2+b^2)}.$$

Setting (3.23) in (3.7), we have

$$Q_\xi = \sqrt{-A_0+\frac{1}{a^2+b^2}\Psi}, \quad (3.24)$$

where $\Psi = \left[2nQ(\xi)+2\lambda^2Q^2(\xi)+\lambda Q^3(\xi)+\frac{1}{4}Q^4(\xi)\right]$.

Sirendaoreji [23] introduced an algebraic method to reach different forms of soliton solutions of NPDEs. Then Qingling and Xueqin [24] modified this method with the aid of an auxiliary ODE:

$$Q_\xi = \sqrt{AQ^2+BQ^3+CQ^4}. \quad (3.25)$$

Eq. (3.25) has the following solutions

$$Q(\xi) = \frac{2A\operatorname{sech}\left[\sqrt{A}\xi\right]}{\sqrt{B^2-4AC}-B\operatorname{sech}\left[\sqrt{A}\xi\right]}, \quad (3.26)$$
$$B^2-4AC>0,\ A>0,$$

$$Q(\xi) = -\frac{2A\operatorname{csch}\left[\sqrt{A}\xi\right]}{\sqrt{4AC-B^2}+B\operatorname{sech}\left[\sqrt{A}\xi\right]}, \quad (3.27)$$
$$B^2-4AC<0,\ A>0,$$

$$Q(\xi) = -\frac{2A\operatorname{sech}\left[\sqrt{A}\xi\right]}{B\operatorname{sech}\left[\sqrt{A}\xi\right]+\kappa\tanh\left[\sqrt{A}\xi\right]-2A}, \quad (3.28)$$
$$B^2-4AC-4A^2<0,\ A>0,$$

where $\kappa = \sqrt{4A^2+4AC-B^2}$.

$$Q(\xi) = -\frac{2A\operatorname{csch}\left[\sqrt{A}\xi\right]}{B\operatorname{csch}\left[\sqrt{A}\xi\right]-\Upsilon\coth\left[\sqrt{A}\xi\right]+2A}, \quad (3.29)$$
$$B^2-4AC-4A^2<0,\ A>0,$$

where $\Upsilon = \sqrt{B^2 - 4AC - 4A^2}$. For the solutions of Eq (3.24), if we take $A_0 = n = 0$ we have

$$Q_3(\xi) = -\frac{\frac{4\lambda^2}{a^2+b^2}\operatorname{csch}\left[2|\lambda|\sqrt{\frac{1}{a^2+b^2}}\xi\right]}{\left|\frac{\lambda}{a^2+b^2}\right| + \frac{\lambda}{a^2+b^2}\operatorname{sech}\left[2|\lambda|\sqrt{\frac{1}{a^2+b^2}}\xi\right]},$$
$$-\frac{\lambda^2}{(a^2+b^2)^2} < 0, \ \frac{2\lambda^2}{a^2+b^2} > 0. \qquad (3.30)$$

$$Q_4(\xi) = -\frac{4\lambda^2 F \operatorname{sech}\Psi}{\lambda F \operatorname{sech}\Psi + \sqrt{(16\lambda^4 - \lambda^2) F^2}\tanh\Psi - 4\lambda^2 F}, \qquad (3.31)$$
$$\left(\lambda^2 - 16\lambda^4\right) F^2 < 0, \ 2\lambda^2 F > 0.$$

where $\Psi = \left[2|\lambda|\sqrt{F}\xi\right]$, $F = \frac{1}{a^2+b^2}$ and we have

$$Q_5(\xi) = -\frac{4\lambda^2 F \operatorname{csch}\Psi}{\lambda F \operatorname{csch}\Psi + \sqrt{(16\lambda^4 - \lambda^2) F^2}\coth\Psi - 4\lambda^2 F}, \qquad (3.32)$$
$$\left(\lambda^2 - 16\lambda^4\right) F^2 < 0, \ 2\lambda^2 F > 0.$$

The $p$ and $q$ solutions for $(3.30)$, $(3.31)$ and $(3.32)$ could have been obtained from the transformations:

$$Q = w_\xi, p = w_x, q = -\frac{1}{a}\left(w_t + \frac{1}{2}w_x^2 + bw_{xx}\right) and \xi = x + \lambda t.$$

## 4 Conclusion

We have used the FIM with the extended tanh method for acquiring new exact solutions for the WBKL equation with time dependent coefficient. We are able to acquire different types of exact solutions. Our obtained solutions have included some new solitons that did not exist in the literature. Thus, the FIM is a very effecive and efficient method to establish different exact solutions for the NPDEs systems.

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