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Fitted Spectral Tau Jacobi Technique for Solving Certain Classes of Fractional Differential Equations

Abeer Adel Al-nana[1], *Omar Abu Arqub*[1], *Mohammed Al-Smadi*[2,*] *and Nabil Shawagfeh*[1]

[1] Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
[2] Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan

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Abstract: In this paper, an efficient numerical technique, so-called the fitted spectral tau Jacobi (FSTJ), is presented to obtain the solutions of a class of fractional differential equations (FDEs) based on the Jacobi polynomials utilized as natural basis functions under the Caputo sense of fractional derivative. The solution methodology is based on the matrix-vector–product technique in Tau formulation of the model. A comparative study was conducted between the gained results in FSTJ method and other existing methods. Convergence and error analysis are presented to confirm the validity and feasibility of the proposed method for solving such problems. Numerical applications are given which refer to the efficiency and effectiveness of the FSTJ method.

Keywords: Spectral method, fractional differential equation, Jacobi polynomials, numerical solutions

1 Introduction

Fractional differential equations (FDEs) are an important branch of mathematics and vital as well. FDE has ample applications because of its great ability to handle many practical problems arising in industrial engineering, computer science, physics, artificial intelligence, and so on. More recently, these types of equations have gained considerable attention by researchers for modeling and describing a lot of mysterious phenomena. So, research and studies have been constructed in terms of a fractional arrangement to deal with their generality and complexity. On the other hand, the theory of fractional calculus is, in fact, a generalization of classical calculus theory that deals with the well-known operations of integration and differentiation of non-integer fractional order, which has a long history dating back to the seventeenth century. The theory was primarily developed to confirm pure and theoretical mathematical predictions and still used effectively in various fields such as rheology, viscosity, electrochemistry and propagation processes [1]- [6].

The processes of fractional mathematical modeling and simulation depend on the properties of their fractional derivatives, which leads to dealing with the fractional order of differential equations with the essential need to solve these equations. Although it is impossible to find, in general, effective classical methods to handle those problems even in most appropriate processes that embedded derivatives or integrals of fractional order, the urgent need to find solutions calls for efficient alternative methods. Anyhow, pertaining to the codification of the generalization of differentiation within the context of fractional-orders, there are many advanced approaches such as Caputo, Comfortable, Riemann-Liouville, and Grunwald-Letnikov concepts that are found and used [7]- [11]. In addition, over the past decades, various initial and boundary value problems with constant or variable coefficients that include different kinds of fractional derivative operators, such as Riemann-Liouville, conformable and Caputo concepts, have been resolved by adopting different analytical and numerical methods, whereas such scientific applications have also included both linear and nonlinear FDEs [12]- [16]. This reason motivates us to consider effective numerical methods for the solution of these types of equations. However, among the recent accurate methods concerned with both linear and nonlinear initial and boundary value problems of fractional order, we refer to [17]- [22].

Furthermore, a wide range of applications in the area of the numerical analysis for various fractional operators have been found by using the collocation methods, which

* Corresponding author e-mail: mhm.smadi@bau.edu.jo

are choosing a finite-dimensional space of candidate solutions, orthogonal polynomials, with a number of collocation points in the domain of interest and then obtaining the solution that satisfies the given equation at these collocation points. The tau method is well presented to obtain effective approximations in the numerical treatment of the eigenvalue defined by the differential equations with spectral parameters, which includes the residual function's projection onto some appropriate set's span of basic functions that, naturally, appearing as eigenfunctions of a singular Sturm-Liouville problem. Besides, the extra conditions are enjoined as restrictions on the coefficients' expansion. It should be noted that the spectral tau method which depends on the classical Jacobi polynomials, called the Jacobi Tau method, does permit the infinitely smooth solutions' approximation of operator equations in which zero is approached by the truncation error quicker than any given negative power of the number of basis functions utilized in the approximation, where that number goes to infinity. The current paper chiefly and effectively aims at using the spectral tau Jacobi (STJ) method to construct the numerical solutions of a class of FDEs. More specifically, we discuss and provide numerical approximate solutions for FDEs of the following general form:

$$D^{v}u(x)=f(x,u(x),u'(x),...,u^{(n-1)}(x)),\ t\geq 0, \quad (1)$$

subject to the the constraints initial conditions:

$$u(x_0)=\mu_0,u'(x_0)=\mu_1,...,u^{(n-1)}(x_0)=\mu_{n-1}, \quad (2)$$

where $\mu_i\in\mathbb{R},\ i=0,1,2,...,n-1,\ x\in[0,\infty),\ D^{v}$ is the Caputo fractional derivative of order v such taht $n-1<v\leq n, f(\cdot)$ is a continuous real-valued function, and $u(x)$ is analytical unknown function to be determined under the assumption that the solution exists and is unique.

This paper is concerned with the investigation of the approximate solution of both linear and nonlinear fractional differential equations by using an efficient numerical method, so-called the spectral tau Jacobi method. The rest of this paper is structured as follows. In Section 2, we give basic definitions, theorems and properties required for subsequent sections. In Section 3, reviewing some essential facts about the Jacobi polynomials and shifted Jacobi polynomials with its properties. A spectral approximation method for Caputo fractional operators is proposed in Section 4. In Section 5, several numerical examples are presented to verify the accuracy and efficiency of the method. Finally, Section 6 ends with a short conclusion.

2 Preliminaries

In this section, we present certain essential definitions and properties of the fractional calculus theory, which we will use in this work [23]- [27].

Definition 2.1 A real function $f(x),\ x\geq 0$ is said to be in space $\mathbb{C}_\mu,\mu\in\mathbb{R}$, if there exist a real number $p>\mu$ such that $f(x)=x^{p}f_1(x)$ where f_1 is continuous on $[0,\infty)$. Obviously if $\beta<\mu$ then $\mathbb{C}_\mu\subset\mathbb{C}_\beta$. As well, we say that $f(x),\ x\geq 0$, in space $\mathbb{C}^{m}_{\mu},\ m\in\mathbb{N}\bigcup 0$, if $f^{m}\in\mathbb{C}_\mu$.

Definition 2.2 Suppose $v>0$ and $m-1<v<m$, the Riemann-Liouville fractional derivative is defined by:

$$J^{v}f(x)=\frac{1}{\Gamma(1-v)}\int_0^x(x-t)^{v-1}f(t)dt, \quad (3)$$

where $J^0f(x)=f(x)$ and Γ is gamma function.

Some of the basic properties of J^{v} are given as follows:

1. $J^{v}J^{w}f(x)=J^{w}J^{v}f(x),\ v,w>0,$
2. $J^{v}J^{w}f(x)=J^{v+w}f(x),\ v,w>0,$
3. $J^{v}c=\frac{c}{\Gamma(v+1)}x^{v},\ c\in\mathbb{R},$
4. $J^{v}x^{p}=\frac{\Gamma(p+v)}{\Gamma(p+v+1)}x^{p+v},\ p>-1$

Definition 2.3 For $x>0,m\in\mathbb{N}$, and $f\in\mathbb{C}^{m}_{-1}$. The fractional derivative operator D^{v} of $f(x)$ in the Caputo sense of order v, $m-1<v\leq m$, is defined as:

$$D^{v}f(x)=\begin{cases}\dfrac{1}{\Gamma(m-v)}\int_0^x(x-t)^{m-v-1}f^{(m)}(t)dt, & m\neq v,\\[2ex] \dfrac{d^m}{dt^m}f(t), & m=v.\end{cases} \quad (4)$$

The Caputo fractional derivative satisfies the following basic properties:

$D^{v}c=0$ (c is constsnt)

$$D^{v}x^{p}=\begin{cases}0, & p<v,\gamma\in N_0\\[2ex] \dfrac{\Gamma(p+1)}{\Gamma(p+1-v)}x^{p-v}, & otherwise.\end{cases} \quad (5)$$

Moreover, for constants μ and λ, the Caputo fractional derivative is a linear operation:

$$D^{v}(\mu f(x)+\lambda g(x))=\mu D^{v}f(x)+\lambda D^{v}g(x) \quad (6)$$

Lemma 2.1. If $m-1<\alpha\leq m,\ m\in\mathbb{N}$, and $f\in\mathbb{C}^{m}_{\mu}$, and $\mu\geq -1$, then
1) $D^{v}J^{v}f(x)=f(x)$
2) $J^{v}D^{v}f(x)=f(x)-\sum_{i=0}^{m-1}f^{i}(0)\frac{x^i}{i},\ x>0.$

3 Fractional-order Jacobi functions

In the present section, we provide certain basic properties of Jacobi polynomials and then we present Shifted Jacobi polynomials.

3.1 Jacobi polynomials

The well-known Jacobi polynomials are defined on the interval $[-1,1]$ and can be generated for $n \geq 2$ with the aid of the following recurrence formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{(2n+\alpha+\beta-1)\alpha^2+\beta^2+x((2n+\alpha+\beta)(2n+\alpha+\beta-2)}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)} P_{n-1}^{(\alpha,\beta)}(x) - \frac{(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)}{n(n+\alpha+\beta)(2n+\alpha+\beta-2)} P_{n-2}^{(\alpha,\beta)}(x) \tag{7}$$

where

$$P_0^{(\alpha,\beta)}(x) = 1,$$

and

$$P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta).$$

3.2 Shifted Jacobi polynomials

In order to use the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ on the interval $t \in [0,L]$, we defined the shifted Jacobi polynomials by setting $x = \frac{2t}{L} - 1$. Suppose that the shifted Jacobi polynomials $P_n^{(\alpha,\beta)}(\frac{2t}{L}-1)$ be denoted by $P_{L,n}^{(\alpha,\beta)}(t)$. Then, the shifted Jacobi polynomials $P_{L,n}^{(\alpha,\beta)}(t)$ can be generated from:

$$P_{L,n}^{(\alpha,\beta)}(t) = \frac{(2n+\alpha+\beta-1)\alpha^2+\beta^2+(\frac{2t}{L}-1)((2n+\alpha+\beta)(2n+\alpha+\beta-2))}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)} P_{n-1}^{(\alpha,\beta)}(t) - \frac{(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)}{n(n+\alpha+\beta)(2n+\alpha+\beta-2)} P_{n-2}^{(\alpha,\beta)}(t), \tag{8}$$

where

$$P_{L,0}^{(\alpha,\beta)}(t) = 1,$$

and

$$P_{L,1}^{(\alpha,\beta)}(t) = \frac{1}{2}(\alpha+\beta+2)(\frac{2t}{L}-1) + \frac{1}{2}(\alpha-\beta).$$

The analytic form of the shifted Jacobi polynomials $P_{L,n}^{(\alpha,\beta)}(t)$ of degree n can be given by:

$$P_{L,n}^{(\alpha,\beta)}(t) = \sum_{k=0}^{\infty} R_k^{(\alpha,\beta,n)}(t) t^{\mu k} \tag{9}$$

where:

$$R_k^{(\alpha,\beta,n)}(t) = (-1)^{n-k} \binom{n}{k} \frac{\Gamma(n+\alpha+1)}{n\,!\Gamma(n+\alpha+\beta+1)} \times \frac{\Gamma(k+n+\alpha+1)}{\Gamma(k+\alpha+1)}, \tag{10}$$

$$P_{L,n}^{(\alpha,\beta)}(0) = (-1)^n \frac{\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)},$$

and:

$$P_{L,n}^{(\alpha,\beta)}(L) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}.$$

From these polynomials, the most commonly used are the shifted Gengenbauer (ultraspherical) polynomials (symmetric shifted Jacobi polynomials) $C_{L,n}^{(\alpha,\mu)}(t)$, the shifted Chebyshev polynomials of the first kind $T_{L,n}^{(\mu)}(t)$, the shifted Legender polynomials $P_{L,n}^{(\mu)}(t)$, the shifted Chebyshev polynomials of the second kind $U_{L,n}^{(\mu)}(t)$, and for the nonsymmetric shifted Jacobi polynomials, the two important special cases of shifted Chebyshev polynomials of the third and fourth kinds $V_{L,n}^{(\mu)}(t)$ and $W_{L,n}^{(\mu)}(t)$, are also considered. These orthogonal polynomials are interrelated to the shifted Jacobi polynomials by the following relations:

$$C_{L,n}^{(\alpha,\mu)}(t) = \frac{n!\Gamma(\alpha+\frac{1}{2})}{\Gamma(n+\alpha+\frac{1}{2})} P_{L,n}^{(\alpha-\frac{1}{2},\beta-\frac{1}{2}),\mu}(t),$$

$$T_{L,n}^{(\mu)}(t) = \frac{n!\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} P_{L,n}^{(-\frac{1}{2},-\frac{1}{2}),\mu}(t),$$

$$P_{L,n}^{(\mu)}(t) = P_{L,n}^{(0,0,\mu)}(t),$$

$$U_{L,n}^{(\mu)}(t) = \frac{(n+1)!\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} P_{L,n}^{(\frac{1}{2},\frac{1}{2}),\mu}(t),$$

$$V_{L,n}^{(\mu)}(t) = \frac{(2n)!!}{(2n-1)!!} P_{L,n}^{(\frac{1}{2},-\frac{1}{2}),\mu}(t),$$

$$W_{L,n}^{(\mu)}(t) = \frac{(2n)!!}{(2n-1)!!} P_{L,n}^{(-\frac{1}{2},\frac{1}{2}),\mu}(t).$$

The orthogonality condition is:

$$\int_0^L P_{L,n}^{(\alpha,\beta)}(t) P_{L,m}^{(\alpha,\beta)}(t) w_L^{(\alpha,\beta)}(t)dt = \gamma_n^{(\alpha,\beta)} \delta_{nm}, \tag{11}$$

where the weight function $w_L^{(\alpha,\beta)}(t) = t^\beta (L-t)^\alpha$ and $\gamma_{L,n}^{(\alpha,\beta)} = \|p_{L,n}^{(\alpha,\beta)}\|^2_{w^{(\alpha,\beta)}} = \frac{L^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\beta+\alpha+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$.

Assume that a function $y(t)$ is a polynomial of degree n, and may be expressed in terms of shifted Jacobi polynomials as:

$$y(t) = \sum_{i=0}^{\infty} c_i P_{L,n}^{(\alpha,\beta)}(t) \tag{12}$$

where:

$$c_i = \frac{1}{\gamma_i^{(\alpha,\beta)}} \int_0^1 y(t) P_{L,n}^{(\alpha,\beta)}(t) w^{(\alpha,\beta)}(t)dt.$$

In practice, only the first $(N+1)$ terms shifted Jacobi polynomials are considered. Hence $y(t)$ can be expanded in the form:

$$y(t) \cong \sum_{i=0}^{N} c_i P_{L,n}^{(\alpha,\beta)}(t) = C^T \varphi(t). \tag{13}$$

The shifted Jacobi coefficient vector C and the shifted Jacobi vector $\varphi(t)$ are written as:

$$C^T = [c_0 \; c_1 \; \dots \; c_N],$$

and

$$\varphi(t) = \left[P_0^{(\alpha,\beta)} \; P_1^{(\alpha,\beta)} \; \dots \; P_N^{(\alpha,\beta)}\right]^T.$$

Theorem 3.1 For $v > 0$, the Caputo fractional derivative for the SFJFs is given by:

$$D^v P_n^{(\alpha,\beta,\mu)}(x) = \sum_{j=0}^{N} S_v(n,j,\alpha,\beta,\mu) P_j^{(\alpha,\beta,\mu)}(x) \tag{14}$$

where
$S_v(n,j,\alpha,\beta,\mu)(x) = \sum_{k=1}^{n} R_k^{(\alpha,\beta,n)}(x) \frac{\Gamma(\mu k+1)(\Gamma(\alpha+1))}{\gamma_l \Gamma(\mu k+v-1)} \times \sum_{s=0}^{j} R_s^{(\alpha,\beta,j)}(x) \frac{\Gamma(k+s+\beta-\frac{v}{\mu}+1)}{(2+k+s+\alpha+\beta-\frac{v}{\mu})}$.

Proof: The analytic form of the shifted fractional Jacobi polynomials $P_i^{(\alpha,\beta,\mu)}(x)$ of degree μ is given by (9). Using Eqs. (5) and (6) yields:

$$D^v P_n^{(\alpha,\beta,\mu)}(x) = \sum_{k=0}^{i} R_k^{(\alpha,\beta,i)} D^v x^{\mu k} = \sum_{k=1}^{i} R_k^{(\alpha,\beta,i)} \frac{\Gamma(\mu k+1)}{\Gamma(\mu k-v+1)} x^{\mu k-v}. \tag{15}$$

Now, by approximating $x^{\mu k-v}$ in $(N+1)$-terms of a shifted fractional-order Jacobi series, we obtain:

$$x^{\mu k-v} \cong \sum_{j=0}^{N} P_j^{(\alpha,\beta,\mu)}(x), \tag{16}$$

$$b_{k,j} = \sum_{s=0}^{j} R_s^{(\alpha,\beta,j)}(x) \frac{\Gamma(k+s+\beta-\frac{v}{\mu}+1)(\Gamma(\alpha+1))}{(2+k+s+\alpha+\beta-\frac{v}{\mu})\zeta_j}, \tag{17}$$

Now, combining Eqs.(17) and (15), it yields:

$$D^v P_n^{(\alpha,\beta,\mu)}(x) = \sum_{j=0}^{N} S_v(n,j,\alpha,\beta,\mu)(x) P_j^{(\alpha,\beta,\mu)}(x).$$

Corollary 3.1 If $\alpha=\beta=0$, we have the shifted fractional Legendere functions. Then, $S_v(i,j,\mu)$ is given as follows:

$$S_v(i,j,\mu) = \sum_{k=1}^{i} \frac{(-1)^{i-k}(2j)\Gamma(\mu k+1)\Gamma(i+k+1)}{(i-k)!k!\Gamma(k+1)\Gamma(\mu k-v+1)} \times \sum_{s=0}^{j} \frac{(-1)^{j-s}\Gamma(i+s+1)\Gamma(k+s+\frac{v}{\mu}+1)}{(j-s)!s!\Gamma(s+1)\Gamma(k+s-v+2)}.$$

Corollary 3.2 If $\alpha=\beta=\frac{-1}{2}$, the shifted fractional Chebyshev functions of the first kind are obtained, and then $S_v(i,j,\mu)$ is given as follows:

$$S_v(i,j,\mu) = \sum_{k=1}^{i} \frac{(-1)^{i-k}(2j)j\pi\Gamma(\mu k+1)i\Gamma(i+k+1)}{(i-k)!k!\Gamma(k+\frac{1}{2})\Gamma(j+\frac{1}{2})\Gamma(\mu k-v+1)} \times \sum_{s=0}^{j} \frac{(-1)^{j-s}\Gamma(i+s)\Gamma(k+s+\frac{v}{\mu}+\frac{1}{2})}{(j-s)!s!\Gamma(s+\frac{1}{2})\Gamma(k+s-v+1)}.$$

Corollary 3.3 If $\alpha=\beta=\frac{1}{2}$, we have the shifted fractional Chebyshev functions of the second kind. Then, $S_v(i,j,\mu)$ is given as follows:

$$S_v(i,j,\mu) = \sum_{k=1}^{i} \frac{(-1)^{i-k}(j+1)j\pi\Gamma(\mu k+1)\Gamma(i+k+1)}{(i-k)!k!\Gamma(k+\frac{3}{2})\Gamma(j+\frac{1}{2})\Gamma(\mu k-v+1)} \times \sum_{s=0}^{j} \frac{(-1)^{j-s}\Gamma(i+s+2)\Gamma(k+s+\frac{v}{\mu}+\frac{3}{2})}{(j-s)!s!\Gamma(s+\frac{3}{2})\Gamma(k+s-v+3)}.$$

4 Spectral methods for FDEs

In order to show the fundamental importance of the formula we proved in the last section, we apply it for solving linear FDEs with Tau method. Also, we propose the SFJCM method for solving nonlinear FDEs [28]- [32].

4.1 Tau method for Linear FDE

We are interested in using the SFJTM to solve the linear FDE:

$$D^v u(x) + \varepsilon u(x) = f(x), \; x \in [0,1] \tag{18}$$

subject to the initial condition:

$$u(0) = u_0 \tag{19}$$

The standard shifted fractional Jacobi-tau approximation to (18) and (19) requires that we find $u_N \in \mathscr{J}_N$ such that:

$$(D^v u_N P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}} + \varepsilon (u_N P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}} = (f, P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}}, \; \ell = 0,1,2,\dots,N-1, \quad u(0) = u_0, \tag{20}$$

where:

$$u_N(x) = \sum_{j=0}^{N} a_j P_j^{(\alpha,\beta,\mu)}(x),$$

$$f_\ell = (f, P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}}, \ell = 0,1,2,\dots,N-1,$$

$$F = \left(f_0 \; f_1 \; \dots \; f_{N-1}, u_0 \right)^T,$$

and:

$$A = \left(a_0 \; a_1 \; \dots \; a_{N-1}, a_N \right)^T.$$

Now, the vibrational formulation of Eqs. (18) and (19) is equivalent to:

$$\begin{cases} \sum_{j=0}^{N} a_j [(D^v P_j^{(\alpha,\beta,\mu)}(x) P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}} + \\ \varepsilon (P_j^{(\alpha,\beta,\mu)}(x) P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}}] = \\ (f, P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}}, \ell = 0,1,2,\dots,N-1 \\ \sum_{j=0}^{N} a_j [(D^{\ell-N} P_j^{(\alpha,\beta,\mu)}(x))] = u_{\ell-N}, \ell = N. \end{cases} \tag{21}$$

Let $a_{\ell j} = (D^v P_j^{(\alpha,\beta,\mu)}(x) P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}}$, and $b_{\ell j} = (P_j^{(\alpha,\beta,\mu)}(x) P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}}$. Then, (20) is equivalent to the following matrix equation:

$$(A + \gamma B)a = F,$$

where $A = (a_{\ell j})_{0 \leqslant \ell, j \leqslant N}$ and $B = (b_{\ell j})_{0 \leqslant \ell, j \leqslant N}$.

Theorem 4.1 If we denote:

$$a_{\ell j} = (D^v P_j^{(\alpha,\beta,\mu)}(x) P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}},$$

for $0 \le \ell \le N-1, 1 \le j \le N$,

$$a_{\ell j} = P_j^{(\alpha,\beta,\mu)}(0),$$

for $0 \le j \le N, l = N$, and:

$$b_{\ell j} = (P_j^{(\alpha,\beta,\mu)}(x) P_\ell^{(\alpha,\beta,\mu)}(x))_{w^{(\alpha,\beta,\mu)}},$$

for $0 \le l = j \le N-1$. Then the nonzero elements of $a_{\ell j}$ and $b_{\ell j}$ are given as follows:

$$a_{\ell j} = \begin{cases} \gamma_\ell S_(\ell, j, \alpha, \beta, \mu), & (0 \le \ell \le N-1, 1 \le j \le N) \\ \dfrac{(-1^j)\Gamma(j+\beta+1)}{\Gamma(\beta+1) j\,!}, & (0 \le j \le N, l = N). \end{cases}$$

$$b_{\ell j} = \gamma_\ell, 0 \le \ell = j \le N-1.$$

4.2 SFJ Collocation Method for Nonlinear FDE

Shifted fractional-order Jacobi collocation method (SFJCM) is used to solve a nonlinear FDE:

$$D^v u(x) = F(x, u(x))), v > 0, x \in [0,1], \tag{22}$$

subject to the initial condition:

$$u(0) = u_0. \tag{23}$$

Let

$$u_N(x) = \sum_{j=0}^{N} a_j P_j^{(\alpha,\beta,\mu)}(x)$$

The criterion employed by the SFJCM for solving (22) and (23) is to find $u_N \in \mathscr{J}_N$ such that:

$$D^v u_N(x) = F(x, u_N(x))), \tag{24}$$

is enforced at the collocation nodes $x_{NK}^{(\alpha,\beta,\mu)}$, $K = 0,1,\dots,N-1$. In other words, we must collocate equation (22) at the N-SFJ roots $x_{NK}^{(\alpha,\beta,\mu)}$, which yields:

$$\begin{aligned} &\sum_{j=0}^{N} a_j (D^v P_j^{(\alpha,\beta,\mu)}(x_{NK}^{(\alpha,\beta,\mu)}) \\ &= F((x_{NK}^{(\alpha,\beta,\mu)}), \sum_{j=0}^{N} a_j P_j^{(\alpha,\beta,\mu)}(x_{NK}^{(\alpha,\beta,\mu)})) \end{aligned} \tag{25}$$

with initial condition:

$$\sum_{j=0}^{N} a_j P_j^{(\alpha,\beta,\mu)}(0) = u_0 \tag{26}$$

Similarly, we use the SFJCM to solve the general form of systems of nonlinear FDEs:

$$\begin{aligned} D^{v i} u_i(x) &= F_i(x, u_1(x), u_2(x), \dots, u_n(x)), \\ & i = 1,\dots,n, v_i > 0, x \in [0,1], \end{aligned} \tag{27}$$

with initial conditions:

$$u_i(0) = u_{0i}. \tag{28}$$

Let

$$u_{iN}(x) = \sum_{j=0}^{N} a_{ij} P_j^{(\alpha,\beta,\mu)}(x).$$

The fractional derivative of the approximate solutions $D^{vi} u_i(x)$ can be expressed by means of SFJFs using Theorem 3.1. The SFJCM for solving (27) and (28) aims to find $u_{iN} \in \mathscr{J}_N$ such that it is satisfied exactly at the collocation points $x_{iNK}^{(\alpha,\beta,\mu)}$, $K = 0,1,\dots,N-1, i = 1,\dots,n$. This immediately gives:

$$\begin{aligned} &\sum_{j=0}^{N} a_{ij} (D^v P_j^{(\alpha,\beta,\mu)}(x_{iNK}^{(\alpha,\beta,\mu)}) \\ &= F_i((x_{iNK}^{(\alpha,\beta,\mu)}), \sum_{j=0}^{N} a_{1j} P_j^{(\alpha,\beta,\mu)}(x_{iNK}^{(\alpha,\beta,\mu)})), \end{aligned} \tag{29}$$

with initial condition:

$$\sum_{j=0}^{N} a_{ij} P_j^{(\alpha,\beta,\mu)}(0) = u_{i0}, i = 1, ..., n. \quad (30)$$

5 Numerical results

To illustrate the effectiveness of the proposed methods in the present paper, several test examples are carried out in this section.

Example 5.1: Consider the following initial value problem of fractional differential equation:

$$D^v u(x) + 2u'(x) = f(x), x \in (0,1),$$

subject to initial conditions:

$$u(0) = 1, u'(0) = 0,$$

where $1 < v \leq 2$, $f(x)$ satisfy the existence of the solution in which $u(x) = cos(x)$ is the exact solution at $v = 2$. The approximate solution of Example 5.1 is obtained by applying the FSJ technique, whereas the maximum absolute error for the various choices of v is shown in Table 1 and in Figure 1. From Table 1 and Figure 1, we can achieve a good approximation to the exact solution by using a few terms of fractional shifted Jacobi polynomials.

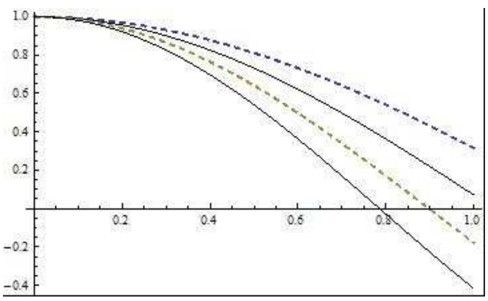

Fig. 1: The behavior of the approximate solutions of Example 5.1 at various choices of v

Example 5.2: Consider the following initial value problem of multi-term nonlinear FDE:

$$D^2 u(x) + [D^{\frac{4}{3}} u(x) D^{\frac{4}{3}} u(x)] + u(x) = v^2 e^{vx} + v^{\frac{8}{3}} + e^{2vx} + e^{\gamma x},$$

subject to initial conditions:

$$u(0) = 1, u'(0) = v,,$$

where $x \in [0,1]$, and the exact solution can be given by $u(x) = e^{vx}$. The approximate solution of Example 5.2 is obtained by applying the FSJ technique, whereas the maximum absolute error for the various choices of v is shown in Table 2 and in Figure 2. It is evident from Table 2 and Figure 2 that the SFJ method is accurate for solving such kind of problems and the approximate solution obtained is very close to the exact one.

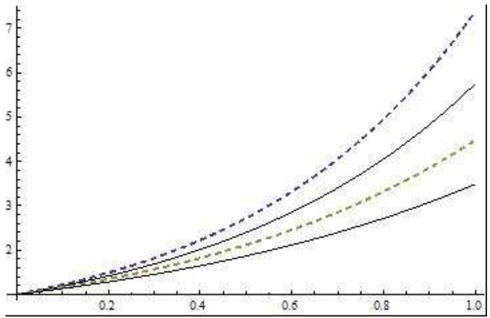

Fig. 2: The behavior of the approximate solutions of Example 5.2 at various choices of v

Example 5.3: Consider the following nonlinear initial value problem with fractional-order v:

$$D^v u(x) + u^2(x) = x + (\frac{x^{v+1}}{\Gamma(v+2)})^2, 0 \leq v \leq 2$$

subject to the initial conditions:

$$u(0) = 0, u'(0) = 0.$$

The exact solution of this problem is $u(x) = \frac{x^{v+1}}{\Gamma(v+2)}$. To see the effect of the fractional derivative to the proposed method, Example 5.3 is solved for different values and the maximum absolute error values are listed in Table 3. From these results, it can be observed that the approximate solution is in good agreement with the exact solution as soon as the fractional-order approaches to the integer-order.

6 Conclusion

The current paper chiefly aims to present a novel method to solve certain classes of FDEs with different order and types. Using the tau spectral method based on shifted Jacobi polynomials, the method was successfully applied to handle such problem. Moreover, the Caputo approach is used to describe the fractional derivatives. As a fact, the errors certainly will be smaller as soon as more terms of the shifted Jacobi polynomial are added. Some numerical examples were presented to show the accuracy and

Table 1: Absolute errors of Example 5.1 for different values of v

x	$v=2$	$v=1.75$	$v=1.5$	$v=1.25$
0.1	$2.51224517E-06$	$9.77512098E-06$	$6.02603148E-05$	$1.29984370E-04$
0.2	$2.99893446E-08$	$1.00238282E-05$	$5.99099939E-05$	$1.99617722E-04$
0.3	$5.98504152E-07$	$9.63937188E-06$	$9.56149388E-05$	$6.98119695E-05$
0.4	$8.23070830E-06$	$9.84747162E-06$	$8.05347437E-06$	$8.11280473E-05$
0.5	$6.58182091E-06$	$1.79582303E-05$	$1.93148884E-05$	$1.54100969E-04$
0.6	$7.85654782E-06$	$1.25741945E-05$	$1.34108687E-05$	$1.00151417E-04$
0.7	$4.68089820E-06$	$4.83300747E-07$	$8.89974305E-05$	$1.33090402E-04$
0.8	$6.87691171E-06$	$8.77701882E-06$	$1.44977019E-05$	$1.61611645E-04$
0.9	$1.82834942E-06$	$1.14577897E-05$	$5.86034191E-05$	$1.90852458E-04$
1	$8.39355827E-06$	$1.37699554E-05$	$2.94276589E-06$	$9.67933300E-05$

Table 2: Absolute errors of Example 5.2 for different values of v

x	$v=2$	$v=1.75$	$v=1.5$	$v=1.25$
0.1	$2.80168067E-06$	$9.75461535E-06$	$6.76583094E-05$	$1.33940748E-06$
0.2	$6.96957451E-07$	$3.19336755E-06$	$7.66886324E-06$	$1.75762151E-04$
0.3	$9.57252688E-06$	$1.65377508E-05$	$1.78856252E-05$	$5.94609810E-05$
0.4	$8.59707152E-06$	$1.12384408E-05$	$6.74724913E-05$	$1.68517230E-04$
0.5	$7.65725275E-06$	$1.11751289E-05$	$6.30514168E-05$	$5.07936653E-05$
0.6	$2.70599096E-06$	$1.38534678E-05$	$3.68223606E-05$	$3.18820213E-05$
0.7	$1.33799016E-06$	$1.76663781E-05$	$4.36870373E-05$	$2.26927188E-05$
0.8	$1.33799016E-06$	$1.11789609E-05$	$9.95447698E-05$	$1.15353706E-04$
0.9	$3.61355229E-07$	$3.45252846E-06$	$1.61195537E-06$	$1.96454923E-05$
1	$4.93235792E-06$	$1.81258867E-05$	$1.23827790E-05$	$2.84517207E-05$

Table 3: Absolute errors of Example 5.3 for different values of v

x	$v=2$	$v=1.75$	$v=1.5$	$v=1.25$
0.1	$4.27015162E-06$	$1.41922387E-05$	$1.83933644E-05$	$9.71585055E-05$
0.2	$5.57124663E-06$	$1.10875797E-05$	$3.23654451E-06$	$1.56400184E-04$
0.3	$1.53965749E-06$	$1.27955614E-05$	$8.29472872E-05$	$1.30180683E-04$
0.4	$9.13118516E-06$	$6.39742590E-06$	$6.52475168E-05$	$1.70845684E-04$
0.5	$2.82052438E-06$	$5.72272358E-06$	$2.93693974E-05$	$1.06165884E-04$
0.6	$4.28003296E-06$	$1.76768981E-05$	$9.90859443E-05$	$3.18820213E-05$
0.7	$4.22865326E-06$	$8.51658001E-06$	$5.20070121E-05$	$1.65872831E-04$
0.8	$4.45403553E-06$	$1.71527978E-05$	$2.79822939E-05$	$1.29687912E-04$
0.9	$5.43854393E-06$	$1.32587506E-05$	$3.64940763E-05$	$6.16532792E-05$
1	$6.75600047E-07$	$1.46679393E-05$	$5.08024066E-05$	$9.46561534E-05$

applicability of the suggested method. To explain the accurateness of the method, our problems' approximate solutions were compared with the exact solutions for each example. Most importantly, the recommended method may be effectively applied to solve numerous types of FDEs that are subject to nonhomogeneous conditions.

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Abeer Adel Al-nana received her Ph.D. degree in the Applied Mathematics from the University of Jordan in 2018. She received her MSc in mathematics from Al al-Bayt University, Jordan. Her main research interests are in differential equations, numerical Analysis and fractional calculus.

Omar Abu Arqub received his Ph.D. degree in the Applied Mathematics from the University of Jordan in 2008. He then began work at Al-Balqa Applied University as assistant professor of applied mathematics and promoted to full Professor in 2017. His research interests focus on the following fields: Numerical Analysis and Computational Optimization.

Mohammed Al-Smadi received his Ph.D. from the University of Jordan, Amman in 2011. He then began work as assistant professor of applied mathematics in Qassim University. Currently, Dr. Al-Smadi is working as assistant professor of applied mathematics (since Sep. 2013) at Al-Balqa Applied University and he promoted to associate professor in Sep. 2017. His research interests include applied mathematics, numerical analysis, and fractional calculus.

Nabil Shawagfeh earned his Ph. D. degree in mathematics from Clarkson university (USA) in 1983. He then began work at the department of mathematics, the university of Jordan in 1983 as an assistant professor of applied mathematics and promoted to full professor in 1995. His research interests are focused on approximate analytical methods, applied and computational mathematics, numerical analysis, and fractional differential and integral equations.