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Analytical Solutions of Incompressible Couple Stress Fluid Flows

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Abstract: In the present article, we intend to solve the Stokes' first and second problems for an incompressible couple stress fluid under isothermal conditions. The solutions of the considered problems are obtained by the Laplace transform (LT) as for the time variable t and the sine Fourier transform (FT) as for the y-variable. It ought be noticed that by suitable manipulations of the inverse integral transforms, fluid velocity expressions are written as the sum of steady-state (post-transient solution) and transient solutions. Further, we wish to give a comparison of the obtained results and the results obtained by Devakar and Lyengar [1] by using the four inverse Laplace transform algorithms (Stehfest's, Tzou's, Talbot, Fourier series) in the space time domain utilizing a numerical methodology. Moreover, velocity profiles are plotted and considered for different times and different values of couple stress Reynolds number. At the end, the outcomes are introduced through graphs and in tabular forms.

Keywords: Stoke's first and second problems, Couple stress fluid, Laplace and Fourier Transforms, velocity field, numerical inversion.

1 Introduction

Recently, researchers have demonstrated their enthusiasm for non-Newtonian liquids in view of their applications in numerous common, industrial and mechanical problems. One such type of fluid that has pulled in the consideration of research specialists in fluid mechanics during few decades is the couple stress fluid planned by Stokes [2,3]. The hypothesis of couple stress fluids is a simplification of the traditional hypotheses of viscous liquids, which takes into consideration the nearness of couple stresses and body couples in the fluid medium. Additionally, the couple stress hypothesis includes couple stresses and non-symmetric forces-tress in fluid.

Moreover, the couple stress fluid theory presented by Stokes' suggests models for those fluids whose microstructure is mechanically momentous [4]. To introduce a size dependent effect is one of the main features of couple stresses. The subject of continuum mechanics ignores the impact of magnitude of material particles inside the continua. This is unswerving with neglecting the rotational association between the particles of the fluid, which results in a symmetry of forces-tress tensor. However this cannot be true and a size dependent couple-stress hypotheses is needed in some important cases for fluid flow with suspended particles. The spin because of microrotation of these unreservedly suspended particles set up an antisymetric stress, which is known as couple stress, and in this way framing couple stress liquid. The couple stress fluids are proficient of describing different types of lubricants, suspension fluids, blood and so on.

These fluids have applications in various processes in industry, for example, solidification of liquid crystals, expulsion of polymer liquids, colloidal solutions and cooling of metallic plate in shower etc. Stokes has also written, review of couple stress fluid dynamics by the name "Theories of Fluids with Microstructure" [2] which contains an extensive study about these fluids. Essential thoughts and methods for both steady and unsteady flow problems of Newtonian and non-Newtonian fluids are given by Ellahi [5]. The essential equations overseeing the flow of couple stress liquids are non-linear in nature and even of higher order than the Navier Stokes equations. In this manner, an analytical solution of these equations is not easy to find. Diverse perturbation procedures are usually utilized for obtaining approximate solutions of these equations see [6,7,8,9] and references there in.

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In this paper, we establish the exact solutions of Stokes' first and second problems for an incompressible couple stress fluid by Laplace and Fourier transforms. The basic governing equations for couple stress fluids are given in section 2, and the formulation of the problem is given in section 3. Analytical solutions of Stocks first and second problems are obtained in section 4. In section 5, the conclusion and discussion is given. Also the comparison of our obtained results are shown with the results of Devakar and Lyengar [1] and four other results which, are obtained by employing the different numerical inversion techniques (Stehfests, Tzous, Tzous, Talbot, Fourier series) [10] for the inverse Laplace transform by graphical and tabular form.

2 Basic Equations

The simple equations governing the flow of an incompressible couple stress fluid are in [6,11,12,13] as follows

$$\nabla \cdot \mathbf{V}, \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \mathbf{T} - \eta \nabla^4 \mathbf{V} + \rho \mathbf{f}, \tag{2}$$

where $\mathbf{V}$ is the velocity vector, ρ is the constant density, $\mathbf{f}$ is the body force per unit mass, $\mathbf{T}$ is the Cauchy stress tensor, η is the couple stress parameter and the operator d/dt denotes the material derivative which is defined as:

$$\frac{d}{dt}(*) = \left(\frac{\partial}{\partial t} + V \cdot \nabla \right)(*).$$

The Cauchy stress tensor $\mathbf{T}$ can be defined as:

$$\mathbf{T} = -p\boldsymbol{I} + \tau, \quad \tau = \mu \mathbf{A}_1,$$

where p is the dynamic pressure, $\boldsymbol{I}$ is the unit tensor, μ is the coefficient of viscosity and $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor defined as: $\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T$, $\mathbf{L}$ is the gradient of $\mathbf{V}$ and $\mathbf{L}^T$ is the transpose of $\mathbf{L}$.
To solve the problem dealing with couple stress fluid flows, in addition to the assumption of no-slip condition, it is presumed that the couple stresses vanish at the boundary.

3 Formulation of the problem

Consider the unsteady flow of couple stress fluid which fills the half space $y > 0$ over an infinite extended flat (solid) plate possessing xy-plane. At first, we expect that both fluid and plate are at rest. At time $t = 0^+$, regardless of whether we enable the plate to begin with a constant velocity U along x-axis or oscillate with velocity $U\cos(\omega t)$ or $U\sin(\omega t)$. Along these lines, the velocity is expected to be in the form $\mathbf{V} = (u(y,t), 0, 0)$ and it consequently satisfies the continuity equation (1).
The equation governing $u(y,t)$, is now seen as

$$\rho \frac{\partial u(y,t)}{\partial t} = \mu \frac{\partial^2 u(y,t)}{\partial y^2} - \eta \frac{\partial^4 u(y,t)}{\partial y^4}. \tag{3}$$

Introducing the following non-dimensional quantities:

$$u^* = \frac{u}{U},\ y^* = \frac{y}{l},\ t^* = \frac{U}{l}t,\ \text{where } l^2 = \frac{\eta}{\mu},\ Re = \frac{\rho U l}{\mu},$$

into Eq. (3), yields the following dimensionless expression (drop out the star notation for simplicity)

$$Re \frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2} - \frac{\partial^4 u(y,t)}{\partial y^4}, \tag{4}$$

where, Re is the Reynold number.
It is simple that we need to solve the above equation utilizing the suitable boundary conditions relying upon whether we are managing Stokes' first or second problem.

4 Solution of the problem

4.1 Stokes' first problem

Initially, both fluid and plate are at rest. At time $t = 0^+$, the plate is all of a sudden set to move with constant velocity U. The non-dimensional initial and boundary conditions to be satisfied for this problem are

$$\begin{aligned} &u(y,0) = 0, \quad \text{for all } \ y, \quad \text{(initial condition)} \\ &u(0,t) = 1, \ \text{for all } \ t > 0, \ \text{(no-slip condition)}. \\ &u(y,t) \longrightarrow 0, \ \text{as } \ y \longrightarrow \infty, \text{for all } \ t > 0, \ \text{(natural condition)} \\ &\frac{\partial^2 (y,t)}{\partial y^2} = 0, \ \text{at } \ y = 0, \text{for any } \ t > 0. \end{aligned} \tag{5}$$

(vanishing of couple stress on the boundary)
Taking the LT to Eqs. (4), (5) and using the initial condition $(5)_1$, we have

$$\frac{\partial^4 \overline{u}(y,q)}{\partial y^4} - \frac{\partial^2 \overline{u}(y,q)}{\partial y^2} + Req\overline{u}(y,q) = 0, \tag{6}$$

with boundary conditions

$$\begin{aligned} &\overline{u}(0,q) = \frac{1}{q}, \\ &\overline{u}(y,q) \longrightarrow 0, \ \text{as } \ y \longrightarrow \infty, \\ &\frac{\partial^2 \overline{u}(y,q)}{\partial y^2} = 0, \ \text{at } \ y = 0. \end{aligned} \tag{7}$$

Applying the sine FT to Eq. (6) and taking into account the conditions (7), we find

$$\overline{u}_{sn}(\xi,q) = \sqrt{\frac{2}{\pi}} \frac{1}{\xi} \left\{ \frac{1}{q} - \frac{1}{q + \frac{\xi^2(\xi^2+1)}{Re}} \right\}.$$

Taking inverse LT to the last relation, we get

$$u_{sn}(\xi,t) = \sqrt{\frac{2}{\pi}}\frac{1}{\xi}\left\{1-\exp\left(-\frac{\xi^2(\xi^2+1)}{Re}t\right)\right\}. \quad (8)$$

Now, employing the inverse sine FT, we obtain

$$u(y,t) = 1-\frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\exp\left(-\frac{\xi^2(\xi^2+1)}{Re}t\right)d\xi. \quad (9)$$

Our final solution is the sum of post-transient (steady) solution and transient solution, where the transient solution is

$$u_t(y,t) = -\frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\exp\left(-\frac{\xi^2(\xi^2+1)}{Re}t\right)d\xi.$$

We hightailed the property $\lim_{t\longrightarrow\infty.} u_t(y,t) = 0$.

4.2 Stokes' second problem

Initially, both fluid and plate are at rest. At time $t = 0^+$, it is assumed that the plate begins to oscillate in its own plane with velocity $U\cos(\omega t)$ or $U\sin(\omega t)$, (where U is the amplitude of the motion and ω is the frequency of the vibration). Therefore, the non-dimensional conditions to be satisfied are

$$u(y,0) = 0, \text{for all } \ y, \quad \text{(initial condition)}$$

$$u(0,t) = \cos(\omega t) \text{ or } u(0,t) = \sin(\omega t), \text{for all } \ t>0, \quad (10)$$

(no-slip condition)

$$u(y,t) \longrightarrow 0, \ \text{ as } \ y\longrightarrow\infty, \text{for all } \ t>0, \ \text{(natural condition)}$$

$$\frac{\partial^2(y,t)}{\partial y^2} = 0, \ \text{ at } \ y=0, \text{for any } \ t>0.$$

(vanishing of couple stress on the boundary) As in the case of Stokes' first problem, taking LT of Eqs. (4), (10) and using initial condition $(10)_1$, we get Eq. (6) with the following boundary conditions

$$\overline{u}(0,q) = \frac{q}{q^2+\omega^2}, \text{ or } \overline{u}(0,q) = \frac{\omega}{q^2+\omega^2}$$
$$\overline{u}(y,q) \longrightarrow 0, \ \text{ as } \ y\longrightarrow\infty, \quad (11)$$
$$\frac{\partial^2\overline{u}(y,q)}{\partial y^2} = 0, \ \text{ at } \ y=0.$$

Employing the sine FT to Eq. (6) and taking into account the conditions (11), we have

$$\overline{u}_{sn}(\xi,q) = \sqrt{\frac{2}{\pi}}\xi(\xi^2+1)\frac{q}{(Req+\xi^2(\xi^2+1))(q^2+\omega^2)}.$$

An equivalent form is

$$\overline{u}_{sn}(\xi,q) = \sqrt{\frac{2}{\pi}}\frac{1}{\xi}\left\{\frac{q}{q^2+\omega^2}-\frac{1}{q+\frac{\xi^2(\xi^2+1)}{Re}} + \frac{\omega}{q^2+\omega^2}\frac{\omega}{q+\frac{\xi^2(\xi^2+1)}{Re}}\right\}.$$

Taking inverse LT to the above relation, we get

$$u_{sn}(\xi,t) = \sqrt{\frac{2}{\pi}}\frac{1}{\xi}\left\{\cos(\omega t)-\exp\left(-\frac{\xi^2(\xi^2+1)}{Re}t\right) + \omega\int_0^t \sin(\omega(t-\tau)) \times \exp\left(-\frac{\xi^2(\xi^2+1)}{Re}\tau\right)d\tau\right\}. \quad (12)$$

Now, employing the inverse sine Fourier transform, we obtain the solution corresponding to the cosine oscillation of the boundary

$$u_c(y,t) = \cos(\omega t)-\frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\exp\left(-\frac{\xi^2(\xi^2+1)}{Re}t\right)d\xi$$
$$+\frac{2\omega}{\pi}\int_0^\infty\int_0^t \frac{\sin(y\xi)}{\xi}\sin(\omega(t-\tau)) \quad (13)$$
$$\times\exp\left(-\frac{\xi^2(\xi^2+1)}{Re}\tau\right)d\tau. \quad (14)$$

With similar procedure as in this section, we find the following solution corresponding to sine oscillation of the boundary

$$u_s(y,t) = \sin(\omega t)-\frac{2\omega}{\pi}\int_0^\infty\int_0^t \frac{\sin(y\xi)}{\xi}\cos(\omega(t-\tau)) \times\exp\left(-\frac{\xi^2(\xi^2+1)}{Re}\tau\right)d\tau.$$

Table 1: Validation of the obtained numerical results with analytical solution (9)

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y	R=0.5, t=0.5					
	Exact solution $u(y,t)$	Honig and Hirdes	Stehfest's	Tzaou's	Talbot	Fouriesr Series
0.0	1	1	1	1	1	1
0.3	0.87	0.864	0.87	0.851	0.87	0.881
0.6	0.742	0.733	0.742	0.739	0.742	0.761
0.9	0.62	0.61	0.62	0.627	0.62	0.643
1.2	0.505	0.496	0.505	0.509	0.505	0.528
1.5	0.4	0.394	0.4	0.399	0.4	0.419
1.8	0.306	0.303	0.306	0.305	0.306	0.32
2.1	0.224	0.224	0.224	0.225	0.224	0.232
2.4	0.155	0.157	0.155	0.155	0.155	0.157
2.7	0.098	0.101	0.098	0.098	0.098	0.096
3.0	0.053	0.056	0.053	0.053	0.053	0.048

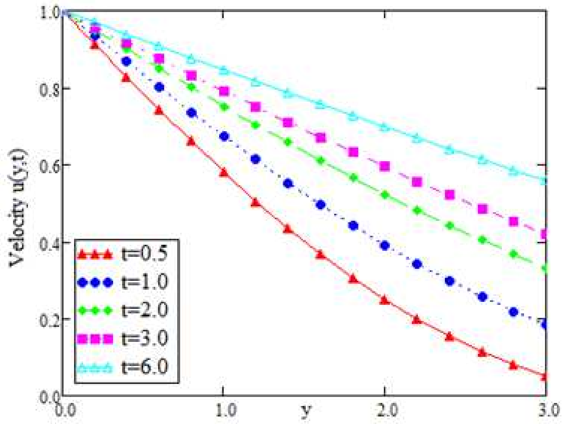

Fig. 1: Variation of the velocity field $u(y,t)$ with distance at different times for $Re = 0.5$.

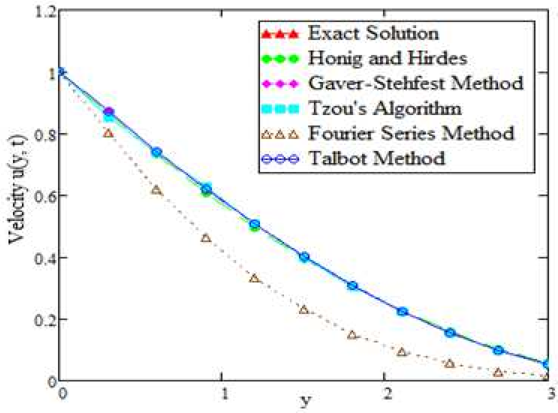

Fig. 3: Comparison of our result (9) with the results of [1, 10] for $Re = 0.5$ and $t = 0.5$

Table 2: Validation of the obtained numerical results with analytical solution (13)
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y	R=0.5, t=1, ω=π/7					
	Exact Solution $u_c(y,t)$	Honig and Hirdes	Stehfest's	Tzaou's	Fouries Series	Talbot
0.0	0.901	0.9	0.901	0.902	0.901	0.901
0.3	0.874	0.858	0.874	0.865	0.874	0.864
0.6	0.846	0.816	0.846	0.83	0.846	0.828
0.9	0.819	0.776	0.819	0.8	0.819	0.792
1.2	0.791	0.739	0.791	0.773	0.791	0.757
1.5	0.764	0.703	0.764	0.75	0.764	0.723
1.8	0.737	0.67	0.737	0.727	0.737	0.69
2.1	0.709	0.639	0.709	0.705	0.709	0.659
2.4	0.682	0.611	0.682	0.683	0.682	0.628
2.7	0.655	0.585	0.655	0.659	0.655	0.599
3.0	0.628	0.561	0.628	0.634	0.628	0.571

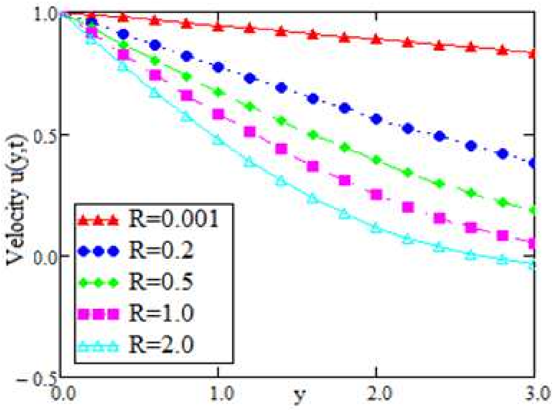

Fig. 2: Variation of the velocity field $u(y,t)$ for different values of Re at $t = 1$.

5 Conclusions and Numerical results

In the present study, the velocity field corresponding to the Stokes' first and second problems for an incompressible couple stress fluid under isothermal conditions are determine by using the LT and sine FT. Straightforward computations show that $u(y,t)$ given by Eq. (9), as well as $u_c(y,t)$ and $u_s(y,t)$ given by Eqs. (13) and (14), satisfy both the governing equation and all imposed initial and boundary conditions. Further, the obtained results are compared with the numerically evaluated results of Devakar and Lyengar [1] and four other inverse Laplace transform algorithms (Stehfest's, Tzou's, Fourier series, Talbot) in the space time domain using a numerical approach. In each case, physical aspects of the flow parameters on velocity field can be seen graphically and in tabular form. All the graphs and tables are presented in dimensional velocity profile and the special variable y.

Stokes' first problem. Fig. 1 is plotted against the velocity field and spatial variable y to see the effect of time t for fixed parameter (Reynolds number) Re. It is observed that, at a fixed distance y, as time increases, the fluid velocity increases. it means that velocity profile is directly proportional to the time t. Fig. 2, shows that for any fixed time t, as Re is increasing, the velocity profile is seen to be decreasing for a fixed distance y. Fig. 3, shows the comparison of our obtained result (9) and the numerically evaluated results of [1] and four other Inverse Laplace transform algorithms (Stehfest's, Tzou's, Fourier series, Talbot) in the space time domain using a numerical approach [10]. From Table 1, it can be observed that the Honig-Hirdes method [1], Stehfest's, Tzou's, and Talbot methods have good agreement with our obtained

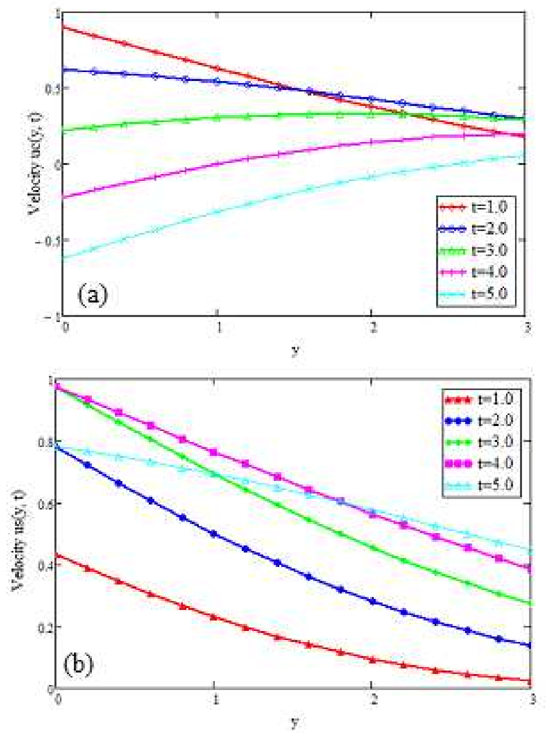

Fig. 4: Variation of the velocity fields $u_c(y,t)$ and $u_s(y,t)$ with distance at different times for $Re = 0.5$.

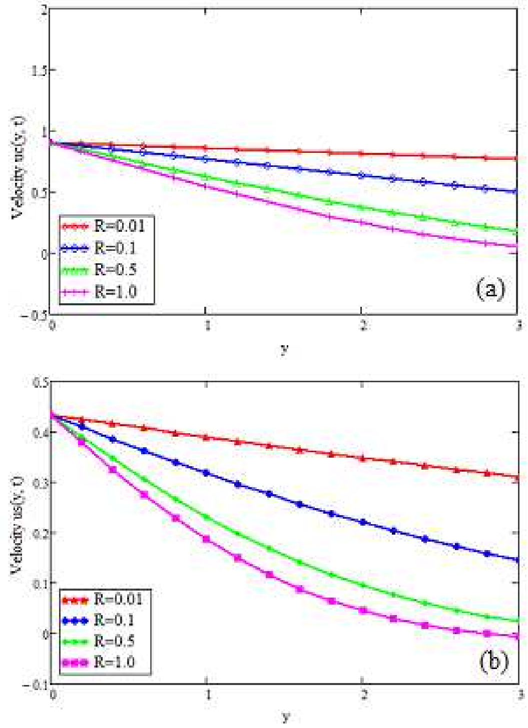

Fig. 5: Variation of the velocity fields $u_c(y,t)$ and $u_s(y,t)$ for different values of Re at $t = 1$.

Table 3: Validation of the obtained numerical results with analytical solution (14)
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y	R=0.5, t=1, ω=π/7					
	Exact Solution $u_s(y, t)$	Honig and Hirdes	Stehfest's	Tzaou's	Fouries Series	Talbot
0.0	0.434	0.434	0.434	0.434	0.434	0.434
0.3	0.412	0.412	0.412	0.412	0.412	0.413
0.6	0.39	0.39	0.39	0.39	0.39	0.391
0.9	0.369	0.368	0.369	0.369	0.369	0.37
1.2	0.347	0.347	0.347	0.348	0.347	0.349
1.5	0.327	0.326	0.327	0.327	0.327	0.329
1.8	0.306	0.305	0.306	0.307	0.306	0.309
2.1	0.287	0.286	0.287	0.287	0.287	0.289
2.4	0.268	0.266	0.268	0.268	0.268	0.27
2.7	0.249	0.248	0.249	0.249	0.249	0.252
3.0	0.232	0.23	0.232	0.232	0.232	0.235

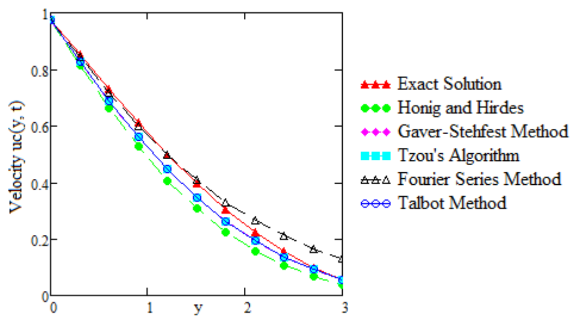

Fig. 6: Comparison of our result (13) with the results of [1, 10] for $Re = 0.5$, $t = 0.5$ and $\omega = \pi/7$.

analytical results. But Fourier Series method give errors.

Stokes' second problem. For fixed value of Reynolds number Re the oscillatory character of the velocity is seen in the Figs. 4(a) and 4(b), as can be expected. In Figs 5(a) and 5(b), the variation of fluid velocity is plotted for different values of couple stress Reynolds number Re at a fixed time t. As Reynolds number Re increasing, it can be seen that the velocity decreases for both cosine and sine oscillations. Figs. (6) and (7), give the comparison of our obtained analytical results (13) and (14), with the numerically evaluated results in [1, 10]. From Table 2, it can seen that all numerically evaluated results have good agreement with our obtained cosine results (13), but in Table.3, except of Honig-Hirdes method [1], all other algorithms show the error with the sine results (14).

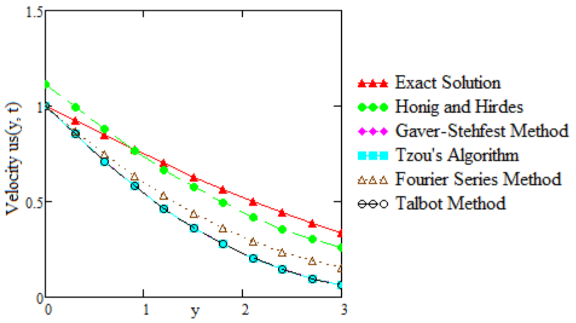

Fig. 7: Comparison of our result (14) with the results of [1, 10] for $Re = 0.5$, $t = 0.5$ and $\omega = \pi/7$.

By this discussion, we can see that all these numerically evaluated inverse Laplace transform algorithms are not suitable to find the solutions for all type of problems.

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