Appl. Math. Inf. Sci. **13**, No. 2, 163-171 (2019)

Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/130203

On Some Stationary Inar Models with Discrete Laplace Marginals

Ahmed M. Agwa[1] , *Emad-Eldin A A Aly*[1] *and M. M. Gabr*[2,*]

[1] Department of Statistics and Operations Research, Kuwait University, Kuwait
[2] Department of Mathematics, Alexandria University, Alexandria, Egypt

Received: 28 Nov. 2018, Revised: 22 Dec. 2018, Accepted: 28 Dec. 2018
Published online: 1 Mar. 2019

Abstract: We propose and study integer-valued time series models with the discrete Laplace marginal distribution. These models allow for positive and negative values. The model with symmetric discrete Laplace marginal allows for positive and negative autocorrelation. As an illustration, we have applied the proposed models to real-life data sets.

Keywords: Discrete Laplace distribution, Integer-valued autoregressive models

1 Introduction

In the last three decades, integer-valued time series models have received considerable attention in the literature. Integer-valued time series can be used to model count data, for example, the number of patients in a hospital at the end of the day and the number of claims an insurance company receives during each day. In many applications in real life we may encounter time series data with negative and positive integer values. Some of these data are obtained when the difference operator is applied to a non-stationary count data. In addition, most of the proposed integer-valued time series models have positive autocorrelation functions. Kozubowski and Podgŏrstki (2000) introduced and studied the asymmetric Laplace distributions. Jayakumar and Kuttykrishnan (2006) introduced and studied time-series models with asymmetric Laplace distribution marginals. Krishna and Jose (2011) introduced and studied the generalized Marshall-Olkin asymmetric Laplace distribution. In this paper we introduce and study stationary integer-valued autoregressive models with discrete Laplace (DL) and skew DL (SDL) marginals. Based on the results of Freeland (2010), Barreto-Souza and Bourguignon (2015) introduced and studied a stationary integer-valued autoregressive model with SDL marginal $(SDL-INAR(1,\theta_1,\theta_2))$ and Nastić et al. (2016) introduced and studied a stationary integer-valued autoregressive model with DL marginal $(DL-INAR(1,\theta))$. These models are essentially developed by taking the difference of two independent versions of Ristić et al. (2009) geometric $INAR$ model. Using a totally different approach, in this paper we introduce and study new stationary $(SDL-INAR(1,\theta_1,\theta_2))$ and $DL-INAR(1,\theta)$ models. These models allow for positive- and negative-integer values. The stationary integer-valued autoregressive model with DL marginal allows for positive and negative autocorrelation function.

In Section 2 we review important results of the DL distribution. In Section 3 we introduce and study stationary integer-valued autoregressive models with SDL marginals. In Section 4 we introduce and study stationary integer-valued autoregressive models with DL marginals. In Sections 5 we consider the problem of estimating the parameters of the models of Sections 3 and 4. In Sections 6 and 7 we report the results of Monte Carlo studies and give some applications of the proposed models.

2 The DL distribution

First we present briefly some results of Inusah and Kozubowski (2006) and Kozubowski and Inusah (2006) regarding the Discrete Laplace (DL) and the Skew DL distributions (SDL). Assume that Z has the SDL

* Corresponding author e-mail: mahgabr@yahoo.com

distribution ($SDL(\theta_1,\theta_2)$) with parameters $0<\theta_1,\theta_2<1$. Then, the following results hold.

$$P(Z=z)=\begin{cases}\frac{\theta_1\theta_2\overline{\theta}_2^{z}}{1-\overline{\theta}_1\overline{\theta}_2}, & \text{if } z=0,-1,-2,\ldots\\ \frac{\theta_1\theta_2\overline{\theta}_1^{|z|}}{1-\overline{\theta}_1\overline{\theta}_2}, & \text{if } z=0,1,2,\ldots\end{cases}, \quad (1)$$

where $\overline{\theta}_1=1-\theta_1, \overline{\theta}_2=1-\theta_2$.

$$M_Z(t)=E\left(e^{tZ}\right)=\frac{\theta_1\theta_2}{\theta_1\theta_2-\overline{\theta}_1\xi(t)-\overline{\theta}_2\xi(-t)}, \quad (2)$$

where $\xi(t)=e^t-1$

$$\mu=E(Z)=\frac{\overline{\theta}_1}{\theta_1}-\frac{\overline{\theta}_2}{\theta_2}=\frac{\theta_2-\theta_1}{\theta_1\theta_2}, \quad (3)$$

and

$$\sigma^2=V(Z)=\left(\frac{\theta_2-\theta_1}{\theta_1\theta_2}\right)^2+\frac{\overline{\theta}_1+\overline{\theta}_2}{\theta_1\theta_2}. \quad (4)$$

The special case of $SDL(\theta_1,\theta_2)$ when $\theta_1=\theta_2=\theta$ is referred to as the DL distribution and is denoted by $DL(\theta)$. By taking $\theta_1=\theta_2=\theta$ in (1)-(4) we obtain the corresponding results for the $DL(\theta)$ distribution.

Note that if $Z=X_1-X_2$, where X_1 and X_2 are independent random variables such that $X_1\sim Geo(\theta_1)$ and $X_2\sim Geo(\theta_2)$, then, Z has $SDL(\theta_1,\theta_2)$.

Assume that $M(t)$ is the MGF of a random variable. Following Marshall-Olkin (1997), the corresponding Marshall-Olkin family of distributions, with moment generating function $\Psi(t)$, is defined by

$$\Psi(t)=\frac{\beta M(t)}{1-\overline{\beta}M(t)}, \beta>0. \quad (5)$$

Assume that $X_1,X_2,\cdots$ are *iidrv*, $N(\beta)$ is a geometric random variable with $P(N(\beta)=k)=\beta\overline{\beta}^{k-1}, k=1,2,3,\ldots$ and $N(\beta)$ and the $X_i's$ are independent. For $0<\beta\le 1, \Psi(t)$ is the MGF of $\sum_{i=1}^{N(\beta)}X_i$ when the MGF of X_1 is $M(t)$. For $\beta>1, M(t)$ is the MGF of $\sum_{i=1}^{N(\frac{1}{\beta})}X_i$ when the MGF of X_1 is $\Psi(t)$.

Applying (5) to the $SDL(\theta_1,\theta_2)$ with MGF (2) we obtain the Marshall-Olkin SDL distribution, $MOSDL(\beta,\theta_1,\theta_2)$, with MGF,

$$\Psi(t)=\frac{\theta_1\theta_2}{\theta_1\theta_2-\frac{1}{\beta}\left[\overline{\theta}_1\xi(t)+\overline{\theta}_2\xi(-t)\right]}. \quad (6)$$

We can show that $MOSDL(\beta,\theta_1,\theta_2)\overset{D}{=}SDL(\delta_1,\delta_2)$, where for $i=1,2; 0\le\delta_i\le 1$ and

$$\delta_i=\frac{2\theta_1\theta_2}{\theta_1\theta_2+\frac{(-1)^i}{\beta}(\theta_1-\theta_2)+\sqrt{\Delta}}.$$

where $\Delta=\theta_1^2\theta_2^2+\frac{2\theta_1\theta_2(\theta_1-\theta_2)}{\beta}+\frac{(\theta_1-\theta_2)^2}{\beta^2}+\frac{4\overline{\theta}_1\theta_1\theta_2}{\beta}$

Note that if $X\sim MOSDL(\beta,\theta_1,\theta_2)$, then

$$X\overset{D}{=}\sum_{i=1}^{N(p)}Y_{p,i},$$

where $Y_{p,i}$,$i=1,2,\ldots$ are i.i.d $MOSDL\left(\frac{\beta}{p},\theta_1,\theta_2\right)$ and independent of $N(p)$.

The Marshall-Olkin discrete Laplace distribution ($MODL(\beta,\theta)$) with parameters $\beta>0$ and $0<\theta<1$ is a special case of (2) when $\theta_1=\theta_2=\theta$. We can show that $MODL(\beta,\theta)\overset{D}{=}DL\left(2\theta/(\theta+\sqrt{\theta^2+\frac{4\overline{\theta}}{\beta}})\right)$.

3 The $SDL-INAR(1,\theta_1,\theta_2)$ model

We introduce and study a stationary $INAR(1)$ time series with $SDL(\theta_1,\theta_2)$ which is denoted by $SDL-INAR(1,\theta_1,\theta_2)$. Consider the time-series model

$$Z_t=I_t(\alpha)Z_{t-1}+\varepsilon_t, t=1,2,\ldots, \quad (7)$$

where $I_t(\alpha), Z_{t-1}$ and ε_t are independent random variables, $I_t(\alpha)$ is Bernoulli with parameter $\alpha, 0<\alpha<1$

Theorem 1. The process Z_t of (7) is a stationary $SDL-INAR(1,\theta_1,\theta_2)$ if and only if $\varepsilon_t\overset{D}{=}MODL(\frac{1}{\overline{\alpha}},\theta_1,\theta_2)$ and Z_0 is $SDL(\theta_1,\theta_2)$.

Proof. Let $M_{Z_t}(t)$ be the MGF of Z_t of (7). Then,

$$M_{Z_t}(t)=\left\{\overline{\alpha}+\alpha M_{Z_{t-1}}(t)\right\}M_\varepsilon(t). \quad (8)$$

By the stationarity of Z_t, we obtain

$$M_\varepsilon(t)=\frac{M_{Z_t}(t)}{\overline{\alpha}+\alpha M_{Z_t}(t)}. \quad (9)$$

If $Z_t\sim SDL(\theta_1,\theta_2)$, then

$$M_\varepsilon(t)=\frac{\theta_1\theta_2}{\theta_1\theta_2-\overline{\alpha}\left[\overline{\theta}_1\xi(t)+\overline{\theta}_2\xi(-t)\right]},$$

i.e., $\varepsilon_t\overset{D}{=}MOSDL(\frac{1}{\overline{\alpha}},\theta_1,\theta_2)$.

For the only if part, assume $\varepsilon_t\overset{D}{=}MOSDL(\frac{1}{\overline{\alpha}},\theta_1,\theta_2)$ and Z_0 is $SDL(\theta_1,\theta_2)$. Then, by (8)

$$M_{Z_1}(t)=\frac{\theta_1\theta_2}{\theta_1\theta_2-\left[\overline{\theta}_1\xi(t)+\overline{\theta}_2\xi(-t)\right]}.$$

Hence Z_1 is $SDL(\theta_1,\theta_2)$. Similarly, we can show that if Z_t is $SDL(\theta_1,\theta_2)$, then Z_{t+1} is $SDL(\theta_1,\theta_2)$. Hence the required result follows by induction.220e

Note that the mean and the variance of ε_t are given by

$$\mu_\varepsilon=\overline{\alpha}\mu, \sigma_\varepsilon^2=\overline{\alpha}\sigma^2-\alpha\overline{\alpha}\mu^2,$$

where μ and σ^2 are as in (3) and (4)

Theorem 2. The process Z_t can be written as

$$Z_t \overset{D}{=} \varepsilon_t + \sum_{j=1}^{\infty} \left(\prod_{i=0}^{j-1} I_{t-i}(\alpha) \right) \varepsilon_{t-j}.$$

Proof.

$$\begin{aligned} Z_t &= I_t(\alpha) Z_{t-1} + \varepsilon_t = I_t(\alpha)(I_{t-1}(\alpha) Z_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= I_t(\alpha) I_{t-1}(\alpha) Z_{t-2} + I_t(\alpha)\varepsilon_{t-1} + \varepsilon_t = \cdots \\ &= \varepsilon_t + \left(\prod_{i=0}^{k-1} I_{t-i}(\alpha) \right) Z_{t-k} + \sum_{j=1}^{k-1} \left(\prod_{i=0}^{j-1} I_{t-i}(\alpha) \right) \varepsilon_{t-j} \\ &\vdots \\ &\overset{D}{=} \varepsilon_t + \sum_{j=1}^{\infty} \left(\prod_{i=0}^{j-1} I_{t-i}(\alpha) \right) \varepsilon_{t-j}. \end{aligned}$$

Theorem 3. The autocorrelation function ρ_k of Z_t is given by

$$\rho_k = \alpha^k, k \geq 0.$$

Proof. From the definition of Z_t we obtain

$$\begin{aligned} \gamma_k &= Cov(Z_t, Z_{t-k}) = \alpha Cov(Z_{t-1} + \varepsilon_t, Z_{t-k}) \\ &= \alpha Cov(Z_{t-1}, Z_{t-k}) = \alpha^2 Cov(Z_{t-2}, Z_{t-k}) \\ &\vdots \\ &= \alpha^k V(Z_{t-k}) = \alpha^k \gamma_0. \end{aligned}$$

Theorem 4. The conditional mean and variance of $\{Z_t\}$ are given by

$$E(Z_t|Z_{t-1}) = \alpha Z_{t-1} + \mu_\varepsilon$$

and

$$V(Z_t|Z_{t-1}) = \alpha(1-\alpha) Z_{t-1}^2 + \sigma_\varepsilon^2. \quad (10)$$

Proof.

$$\begin{aligned} E(Z_t|Z_{t-1}) &= E(I_t(\alpha) Z_{t-1} + \varepsilon_t | Z_{t-1}) \\ &= E(I_t(\alpha) Z_{t-1} | Z_{t-1}) + \mu_\varepsilon \\ &= Z_{t-1} E(I_t(\alpha)) + \mu_\varepsilon \\ &= \alpha Z_{t-1} + \mu_\varepsilon. \end{aligned}$$

For the conditional variance of $\{Z_t\}$ we have

$$\begin{aligned} V(Z_t|Z_{t-1}) &= E(Z_t^2|Z_{t-1}) - [E(Z_t|Z_{t-1})]^2 \\ &= E\left((I_t(\alpha) Z_{t-1} + \varepsilon_t)^2 | Z_{t-1} \right) - [\alpha Z_{t-1} + \mu_\varepsilon]^2 \\ &= E\left(\left((I_t(\alpha))^2 Z_{t-1}^2 + 2 I_t(\alpha) Z_{t-1} \varepsilon_t + \varepsilon_t^2 \right) | Z_{t-1} \right) \\ &- \alpha^2 Z_{t-1}^2 - 2\alpha\mu_\varepsilon Z_{t-1} - \mu_\varepsilon^2 \\ &= \alpha Z_{t-1}^2 + E(\varepsilon_t^2) - \alpha^2 Z_{t-1}^2 - \mu_\varepsilon^2 \\ &= \alpha\overline{\alpha} Z_{t-1}^2 + E(\varepsilon_t^2) - \mu_\varepsilon^2 = \alpha\overline{\alpha} Z_{t-1}^2 + \sigma_\varepsilon^2. \end{aligned}$$

Note that the conditional mean is linear in Z_{t-1}. The conditional variance is quadratic in Z_{t-1} where as in Barreto-Souza and Bourguignon (2013) the conditional variance is linear in $|Z_{t-1}|$.

Theorem 5. The joint MGF of $\{Z_t, Z_{t-1}\}$ is given by

$$M_{Z_t, Z_{t-1}}(t_1, t_2) = M_{\varepsilon_t}(t_1)[M_1(t_1, t_2) + M_2(t_2)]. \quad (11)$$

where

$$M_1(t_1, t_2) = \frac{\alpha\theta_1\theta_2}{\theta_1\theta_2 - \overline{\theta}_1 \xi(t_1 + t_2) - \overline{\theta}_2 \xi(-(t_1 + t_2))},$$

$$M_2(t_2) \frac{\overline{\alpha}\theta_1\theta_2}{\theta_1\theta_2 - \overline{\theta}_1 \xi(t_2) - \overline{\theta}_2 \xi(-t_2)}.$$

Proof: (11) follows from

$$\begin{aligned} M_{Z_t, Z_{t-1}}(t_1, t_2) &= E\left(e^{t_1 Z_t} e^{t_2 Z_{t-1}} \right) \\ &= E\left(e^{t_1 (I_t(\alpha) Z_{t-1} + \varepsilon_t)} e^{t_2 Z_t} \right) \\ &= M_{\varepsilon_t}(t_1) E\left(e^{t_1 I_t(\alpha) Z_{t-1}} e^{t_2 Z_{t-1}} \right) \\ &= M_{\varepsilon_t}(t_1) E\left(e^{(t_1 I_t + t_2) Z_{t-1}} \right) \\ &= M_{\varepsilon_t}(t_1) E\left(e^{(t_1 I_t + t_2) Z_{t-1}} \right). \end{aligned}$$

The integer-valued autoregressive model of order p is defined as

$$Z_t = \begin{cases} \varepsilon_t & \text{w.p. } \alpha_0 \\ Z_{t-1} + \varepsilon_t & \text{w.p. } \alpha_1 \\ Z_{t-2} + \varepsilon_t & \text{w.p. } \alpha_2 \\ \ldots & \ldots \\ Z_{t-p} + \varepsilon_t & \text{w.p. } \alpha_p \end{cases}, \quad (12)$$

where $\{\varepsilon_t\}$ is a sequence of $iidrv, \sum_{i=0}^{p} \alpha_i = 1, 0 < \alpha_i < 1, i = 1, 2, , ..., p$. Note that Z_t of (12) is a stationary $INAR(p)$ process with $MOSDL(\beta, \theta_1, \theta_2)$ marginal if and only if $\varepsilon_t \overset{D}{=} MODL(\frac{\beta}{\alpha_\circ}, \theta_1, \theta_2)$ and Z_0 is $MOSDL(\beta, \theta_1, \theta_2)$.

4 The $DL - INAR(1, \theta)$ model

Consider the time series model

$$Z_t = \kappa I_t(\alpha) Z_{t-1} + \varepsilon_t, \quad (13)$$

where $I_t(\alpha), Z_{t-1}$ and ε_t are independent random variables, $I_t(\alpha)$ is Bernoulli with parameter $\alpha, 0 < \alpha < 1$ and the constant $\kappa = 1$ if the time series has positive lag 1 correlation and $\kappa = -1$ if the time series has negative lag 1 correlation.

Theorem 6. The process Z_t of (13) is a stationary $DL - INAR(1, \theta)$ time series if and only if ε_t is $MODL\left(\frac{1}{\overline{\alpha}}, \theta\right)$ and Z_0 is $DL(\theta)$.

Proof. The proof is similar to that of Theorem 1.

The process Z_t can be written as

$$Z_t \stackrel{D}{=} \varepsilon_t + \sum_{j=1}^{\infty} (\kappa)^j \left(\prod_{i=0}^{j-1} I_{t-i}(\alpha) \right) \varepsilon_{t-j}.$$

Theorem 7. Let $\gamma_k = Cov(Z_t, Z_{t-k})$. The autocorrelation function ρ_k of Z_t is given by

$$\rho_k = \frac{\gamma_k}{\gamma_0} = (\kappa\alpha)^k, k = 1, 2, ..., \tag{14}$$

Proof. The proof is similar to that of Theorem 3.

Theorem 8. The conditional mean and variance of $\{Z_t\}$ are given by

$$E(Z_t|Z_{t-1}) = \kappa\alpha Z_{t-1} \tag{15}$$

and

$$V(Z_t|Z_{t-1}) = \alpha\overline{\alpha}Z_{t-1}^2 + \frac{2\overline{\alpha}\overline{\theta}}{\theta^2}. \tag{16}$$

Note that the conditional mean is linear in Z_{t-1}. The conditional variance is quadratic in Z_{t-1} where as in Nastić et al. (2016) the conditional variance is linear in $|Z_{t-1}|$.

Proof: The proof is similar to that of Theorem 4.

Theorem 9. The joint MGF of $\{Z_t, Z_{t-1}\}$ is given by

$$M_{Z_t,Z_{t-1}}(t_1,t_2) = \left(\frac{\theta^2}{\theta^2 - \overline{\alpha}\overline{\theta}\left[\xi(t_1) + \xi(-t_1)\right]} \right) \times [\Delta_1 + \Delta_2] \tag{17}$$

where

$$\Delta_1 = \frac{\alpha\theta^2}{\theta^2 - \overline{\theta}\left[\xi(\kappa t_1 + t_2) + \xi(-(\kappa t_1 + t_2))\right]},$$

$$\Delta_2 = \frac{\overline{\alpha}\theta^2}{\theta^2 - \overline{\theta}\left[\xi(t_2) + \xi(-t_2)\right]}.$$

By the lack of symmetry of (17) the process is not time reversible.

Remark. Note that

$$Z_t = \begin{cases} \varepsilon_t & \text{w.p. } \alpha_0 \\ \kappa Z_{t-1} + \varepsilon_t & \text{w.p. } \alpha_1 \\ \kappa Z_{t-2} + \varepsilon_t & \text{w.p. } \alpha_2 \\ \cdots & \cdots \\ \kappa Z_{t-p} + \varepsilon_t & \text{w.p. } \alpha_p \end{cases} \tag{18}$$

is a stationary $INAR(p)$ process with $MODL(\beta,\theta)$ marginal if and only if $\varepsilon_t \stackrel{D}{=} MODL(\frac{\beta}{\alpha_\circ},\theta)$ and Z_0 is $MODL(\beta,\theta)$. Note also that Z_t of (18) is a stationary $INAR(1)$ process with $MODL(\beta,\theta)$ marginal $(MODL - INAR(1,\beta,\theta))$ if and only if $\varepsilon_t \stackrel{D}{=} MODL(\frac{\beta}{\overline{\alpha}},\theta)$ and Z_0 is $MODL(\beta,\theta)$.

5 Parameters estimation

5.1 Conditional least squares estimators for the $SDL - INAR(1,\theta_1,\theta_2)$ model

Theorem 10. The two step conditional least squares estimators of α, θ_1 and θ_2 are given by

$$\widehat{\alpha}_{cls} = \frac{(N-1)\sum_{t=2}^{N} Z_t Z_{t-1} - \left(\sum_{t=2}^{N} Z_t\right)\left(\sum_{t=2}^{N} Z_{t-1}\right)}{(N-1)\sum_{t=2}^{N} Z_{t-1}^2 - \left(\sum_{t=2}^{N} Z_{t-1}\right)^2},$$

$$\widehat{\theta}_{1cls} = \frac{(1+\widehat{\mu}_{cls}) - \sqrt{(1+\widehat{\mu}_{cls})^2 - 2\left(\widehat{\mu}_{cls}(1+\widehat{\mu}_{cls}) - \widehat{\sigma}_{cls}^2\right)}}{\widehat{\mu}_{cls}(1+\widehat{\mu}_{cls}) - \widehat{\sigma}_{cls}^2}$$

and

$$\widehat{\theta}_{2cls} = \left(\frac{1}{\widehat{\theta}_1} - \widehat{\mu}_{cls} \right)^{-1},$$

where

$$\widehat{\mu}_{cls} = \frac{\sum_{t=2}^{N} Z_t - \widehat{\alpha}\sum_{t=2}^{N} Z_{t-1}}{(N-1)(1-\widehat{\alpha})}.$$

Proof. First we estimate α and μ. Note that

$$E(Z_t|Z_{t-1}) = \alpha Z_{t-1} + \mu_\varepsilon, \mu_\varepsilon = \overline{\alpha}\mu$$

We minimize the quadratic function

$$\begin{aligned} Q &= \sum_{t=2}^{N} \left[Z_t - E(Z_t|Z_{t-1})\right]^2 \\ &= \sum_{t=2}^{N} \left[Z_t - \alpha Z_{t-1} - (1-\alpha)\mu\right]^2. \end{aligned}$$

By solving $\frac{\partial Q}{\partial \alpha} = 0$ and $\frac{\partial Q}{\partial \mu} = 0$ we obtain

$$\widehat{\alpha}_{cls} = \frac{(N-1)\sum_{t=2}^{N} Z_t Z_{t-1} - \left(\sum_{t=2}^{N} Z_t\right)\left(\sum_{t=2}^{N} Z_{t-1}\right)}{(N-1)\sum_{t=2}^{N} Z_{t-1}^2 - \left(\sum_{t=2}^{N} Z_{t-1}\right)^2} \tag{19}$$

and

$$\widehat{\mu}_{cls} = \frac{\sum_{t=2}^{N} Z_t - \widehat{\alpha}_{cls}\sum_{t=2}^{N} Z_{t-1}}{(N-1)(1-\widehat{\alpha}_{cls})}. \tag{20}$$

In the second step, we estimate σ^2. Define the random variable V_t as

$$\begin{aligned} V_t &= \left(Z_t - E(Z_t|Z_{t-1})\right)^2 \\ &= \left(Z_t - \alpha Z_{t-1} - (1-\alpha)\mu\right)^2. \end{aligned}$$

Note that

$$\begin{aligned} E(V_t|Z_{t-1}) &= V(Z_t|Z_{t-1}) = \alpha(1-\alpha)Z_{t-1}^2 + \sigma_\varepsilon^2 \\ &= \alpha(1-\alpha)Z_{t-1}^2 + \overline{\alpha}\sigma^2 - \alpha\overline{\alpha}\mu^2. \end{aligned}$$

Now, the conditional least squares estimator of σ^2 is obtained by minimizing the quadratic function

$$\begin{aligned} Q_N(\sigma^2) &= \sum_{t=2}^{N} \left[V_t - E(V_t|Z_{t-1})\right]^2 \\ &= \sum_{t=2}^{N} \left[V_t - \alpha(1-\alpha)Z_{t-1}^2 - \overline{\alpha}\sigma^2 + \alpha\overline{\alpha}\mu^2\right]^2 \end{aligned}$$

By solving $\frac{\partial Q}{\partial \sigma^2} = 0$ we obtain

$$\widehat{\sigma}^2_{cls} = \frac{\sum_{t=2}^N \left[Z_t - \widehat{\alpha}_{cls} Z_{t-1} - (1-\widehat{\alpha}_{cls})\,\widehat{\mu}_{cls}\right]^2}{(N-1)(1-\widehat{\alpha}_{cls})} - \frac{\sum_{t=2}^N \widehat{\alpha}_{cls}(1-\widehat{\alpha}_{cls})\left(Z_{t-1}^2 - \widehat{\mu}_{cls}^2\right)}{(N-1)(1-\widehat{\alpha}_{cls})}.$$

Using the fact that the estimators $\widehat{\alpha}_{cls}$ and $\widehat{\mu}_{cls}$ are the solutions of the normal equations, we can simplify the estimator $\widehat{\sigma}^2_{cls}$ as follows

$$\widehat{\sigma}^2_{cls} = \frac{\sum_{t=2}^N Z_t^2 - \widehat{\alpha}_{cls}\sum_{t=2}^N Z_t Z_{t-1}}{(N-1)(1-\widehat{\alpha}_{cls})} - \frac{\widehat{\mu}_{cls}\sum_{t=2}^N Z_t + \widehat{\alpha}_{cls}\sum_{t=2}^N Z_{t-1}^2}{(N-1)} + \widehat{\alpha}_{cls}\widehat{\mu}^2_{cls}. \quad (21)$$

Finally, by

$$\frac{1}{\theta_2} = \frac{1}{\theta_1} - \mu$$

and

$$\sigma^2 = \frac{1}{\theta_1}\left(\frac{1}{\theta_1} - 1\right) + \frac{1}{\theta_2}\left(\frac{1}{\theta_2} - 1\right)$$

The conditional least squares estimators of θ_1 and θ_2 are

$$\widehat{\theta}_{1cls} = \frac{(1+\widehat{\mu}_{cls}) - \sqrt{(1+\widehat{\mu}_{cls})^2 - 2\left(\widehat{\mu}_{cls}(1+\widehat{\mu}_{cls}) - \widehat{\sigma}^2_{cls}\right)}}{\widehat{\mu}_{cls}(1+\widehat{\mu}_{cls}) - \widehat{\sigma}^2_{cls}}.$$

and

$$\widehat{\theta}_{2cls} = \left(\frac{1}{\widehat{\theta}_{1cls}} - \widehat{\mu}_{cls}\right)^{-1}.$$

5.2 Yule-Walker estimators for the $SDL-INAR(1,\theta_1,\theta_2)$ model

The Yule-Walker estimator of α is the sample autocorrelation at lag 1,

$$\widehat{\alpha}_{yw} = \frac{\sum_{t=2}^N (Z_t - \overline{Z})(Z_{t-1} - \overline{Z})}{\sum_{t=1}^N (Z_t - \overline{Z})^2}.$$

To obtain the Yule-Walker estimators of θ_1, θ_2 we solve

$$\overline{Z} = \frac{1}{\widehat{\theta}_{1yw}} - \frac{1}{\widehat{\theta}_{2yw}}$$

and

$$s_z^2 = \frac{\widehat{\overline{\theta}}_{1yw}}{\widehat{\theta}^2_{1yw}} + \frac{\widehat{\overline{\theta}}_{2yw}}{\widehat{\theta}^2_{2yw}},$$

where

$$\overline{Z} = \frac{\sum_{t=1}^N Z_t}{N},$$

and

$$s_z^2 = \frac{\sum_{t=1}^N (Z_t - \overline{Z})^2}{N-1}.$$

The solutions of the above two equations are

$$\widehat{\theta}_{1yw} = \frac{(1+\overline{Z}) - \sqrt{(1+\overline{Z})^2 - 2\left(\overline{Z}(1+\overline{Z}) - s_z^2\right)}}{\overline{Z}(1+\overline{Z}) - s_z^2}$$

and

$$\widehat{\theta}_{2yw} = \left(\frac{1}{\widehat{\theta}_{1yw}} - \overline{Z}\right)^{-1}.$$

5.3 Conditional least squares estimators for the $DL-INAR(1,\theta)$ model

Theorem 11. The conditional least squares estimators of α and θ are given by

$$\widehat{\alpha}_{cls} = \kappa \frac{\sum_{t=2}^N Z_t Z_{t-1}}{\sum_{t=2}^N Z_{t-1}^2},$$

and

$$\widehat{\theta}_{cls} = \frac{2}{1+\sqrt{1+\frac{2\sum_{t=1}^N Z_t^2}{N}}}.$$

Proof. First, we minimize the quadratic function

$$Q = \sum_{t=2}^N \left[Z_t - E(Z_t|Z_{t-1})\right]^2 = \sum_{t=2}^N \left[Z_t - \kappa\alpha Z_{t-1}\right]^2.$$

By solving $\frac{\partial Q}{\partial \alpha} = 0$ for α we obtain

$$\widehat{\alpha}_{cls} = \kappa \frac{\sum_{t=2}^N Z_t Z_{t-1}}{\sum_{t=2}^N Z_{t-1}^2}.$$

To estimate θ we note that $V(Z_t) = \frac{2\overline{\theta}}{\theta^2}$. By solving

$$V(Z_t) = \frac{2\overline{\theta}}{\theta^2} = \frac{\sum_{t=1}^N Z_t^2}{N},$$

we obtain

$$\widehat{\theta}_{cls} = \frac{2}{1+\sqrt{1+\frac{2\sum_{t=1}^N Z_t^2}{N}}}.$$

Theorem 12. $\widehat{\alpha}_{cls}$ has the following asymptotic distribution

$$\sqrt{N}\left(\widehat{\alpha}_{cls} - \alpha\right) \xrightarrow{D} N\left(0, \sigma_0^2\right), \quad (22)$$

where

$$\sigma_0^2 = \overline{\alpha} + \frac{\alpha\overline{\alpha}\left(\theta^3 + 14\overline{\theta}\theta^2 + 36\overline{\theta}^2\theta + 24\overline{\theta}^3\right)}{2\overline{\theta}\theta^4\left(1+\overline{\theta}\right)}. \quad (23)$$

Proof. We will only prove the result in the case $\kappa = 1$. Note that

$$\sqrt{N}\left(\widehat{\alpha}_{cls}-\alpha\right)=\frac{N^{-\frac{1}{2}}\sum_{t=2}^{N}Z_{t-1}\left(Z_t-\alpha Z_{t-1}\right)}{N^{-1}\sum_{t=2}^{N}Z_{t-1}^2}.$$

Note that $\{Z_t\}$ is a stationary ergodic Markov chain. Hence, by the ergodic Theorem,

$$N^{-1}\sum_{t=2}^{N}Z_{t-1}^2 \xrightarrow{a.s.} V\left(Z_t\right)=\sigma^2=2\overline{\theta}\theta^{-2}. \quad (24)$$

Hence, by Slutsky's Theorem, $\sqrt{N}\left(\widehat{\alpha}_{cls}-\alpha\right)$ has the same asymptotic distribution as

$$\frac{1}{\sigma^2}N^{-\frac{1}{2}}M_N \text{ with } M_N=\sum_{t=2}^{N}Z_{t-1}\left(Z_t-\alpha Z_{t-1}\right). \quad (25)$$

Next we prove that M_N is a discrete time martingale. Let $F_N=(Z_1,\ldots,Z_N)$ be the σ field generated by $Z_1,Z_2,\ldots,Z_N$. Note that

$$\begin{aligned}E\left(M_{N+1}|_N\right)&=E\left(\{M_N+Z_N\left(Z_{N+1}-\alpha Z_N\right)\}|_N\right)\\&=M_N+E\left(Z_N\left(Z_{N+1}-\alpha Z_N\right)|_N\right)\\&=M_N+Z_NE\left(Z_{N+1}|_N\right)-\alpha Z_N^2\\&=M_N+Z_N\left(\alpha Z_N\right)-\alpha Z_N^2\\&=M_N+\alpha Z_N^2-\alpha Z_N^2=M_N\end{aligned}$$

Following the proof of Theorem 1 of Freeland (2010) we obtain

$$N^{-\frac{1}{2}}M_N \xrightarrow{D} N\left(0,E\left\{Z_{t-1}^2(Z_t-\alpha Z_{t-1})^2\right\}\right). \quad (26)$$

By (25) and (26) we obtain

$$\sqrt{N}\left(\widehat{\alpha}_{cls}-\alpha\right)\xrightarrow{D}N\left(0,\left(\frac{1}{\sigma^2}\right)^2E\left\{Z_{t-1}^2(Z_t-\alpha Z_{t-1})^2\right\}\right). \quad (27)$$

Next we compute the expected value,

$$\begin{aligned}E\left\{Z_{t-1}^2(Z_t-\alpha Z_{t-1})^2\right\}&=E\left\{E\left(Z_{t-1}^2(Z_t-\alpha Z_{t-1})^2|Z_{t-1}\right)\right\}\\&=E\left\{Z_{t-1}^2E\left((Z_t-\alpha Z_{t-1})^2|Z_{t-1}\right)\right\}\\&=E\left\{Z_{t-1}^2V\left(Z_t|Z_{t-1}\right)\right\}\\&=\alpha\overline{\alpha}E\left(Z_{t-1}^4\right)+\sigma_\varepsilon^2V\left(Z_{t-1}\right)\\&=\sigma^2\sigma_\varepsilon^2+\alpha\overline{\alpha}E\left(Z_{t-1}^4\right). \quad (28)\end{aligned}$$

By the results of Inusah and Kozubowski (2006) we obtain

$$E\left(Z_{t-1}^4\right)=\frac{2\overline{\theta}}{\left(1+\overline{\theta}\right)\theta^4}\left\{\theta^3+14\overline{\theta}\theta^2+36\overline{\theta}^2\theta+24\overline{\theta}^3\right\}. \quad (29)$$

By (27), (28) and (29) we obtain (22).

Remark. Note that, by (24), $\widehat{\theta}_{cls}$ satisfies

$$\widehat{\theta}_{cls}=\frac{2}{1+\sqrt{1+2\frac{\sum_{t=1}^{N}Z_t^2}{N}}},$$

$$\widehat{\theta}_{cls}\xrightarrow{a.s.}\frac{2}{1+\sqrt{1+4\overline{\theta}\theta^{-2}}}=\theta.$$

5.4 Yule-Walker estimators for the $DL-INAR(1,\theta)$ model

Since $\sigma^2=\frac{2\overline{\theta}}{\theta^2}$ and $\alpha=\kappa\rho_1$,we can derive estimators of α and θ as

$$\widehat{\alpha}_{yw}=\kappa\frac{\sum_{t=2}^{N}\left(Z_t-\overline{Z}_N\right)\left(Z_{t-1}-\overline{Z}_N\right)}{\sum_{t=1}^{N}\left(Z_t-\overline{Z}_N\right)^2},$$

and

$$\widehat{\theta}_{yw}=\widehat{\theta}_{cls}.$$

5.5 Prediction for the $DL-INAR(1,\theta)$ model

For $m\geq 1$,

$$\begin{aligned}\widehat{Z}_N\left(m\right)&=E\left(Z_{N+m}|\mathscr{F}_N\right)=E\left(\kappa I_tZ_{N+m-1}|\mathscr{F}_N\right)\\&=\kappa\alpha E\left(Z_{N+m-1}|\mathscr{F}_N\right)=\kappa\alpha\widehat{Z}_N\left(m-1\right)\\&=\left(\kappa\right)^{m-1}\alpha^{m-1}\widehat{Z}_N\left(1\right)=\left(\kappa\right)^m\alpha^mZ_N,\end{aligned}$$

The prediction formula of $DL-INAR(1,\theta)$ is the same as that of $AR(1)$.

6 Monte Carlo Results

We have simulated 1000 samples of size $N=100,500$ and 1000 from the $DL-INAR(1,\theta)$ process for $\theta=0.4,0.6,0.8$ and $\alpha=0.3,0.5$ and 0.7. In each case we have computed the mean and Standard Error (SE) of $\widehat{\alpha}_{cls},\widehat{\alpha}_{yw}$ and $\widehat{\theta}_{cls}$. The results are presented in Table 1-3. The results of Tables 1-3 show that $\widehat{\alpha}_{cls},\widehat{\alpha}_{yw}$ and $\widehat{\theta}_{cls}$ become very close to their true values as the sample size N increases.

Table 1: Mean(SE) of estimators of α and θ for $\alpha=0.3$

θ	N	$\widehat{\theta}$	$\widehat{\alpha}_{YW}$	$\widehat{\alpha}_{CLS}$
0.4	100	0.41(0.045)	0.27(0.125)	0.28(0.122)
	500	0.40(0.020)	0.29(0.061)	0.30(0.062)
	1000	0.40(0.013)	0.30(0.044)	0.30(0.044)
0.6	100	0.61(0.049)	0.26(0.117)	0.28(0.118)
	500	0.60(0.022)	0.31(0.061)	0.31(0.061)
	1000	0.60(0.017)	0.30(0.050)	0.30(0.050)
0.8	100	0.81(0.041)	0.27(0.123)	0.28(0.120)
	500	0.80(0.020)	0.30(0.062)	0.30(0.062)
	1000	0.80(0.014)	0.29(0.047)	0.29(0.047)

Table 2: Mean (SE) of estimators of α and θ for $\alpha = 0.5$

θ	N	$\widehat{\theta}$	$\widehat{\alpha}_{YW}$	$\widehat{\alpha}_{CLS}$
0.4	100	0.42(0.051)	0.46(0.118)	0.47(0.118)
	500	0.40(0.018)	0.47(0.046)	0.48(0.046)
	1000	0.40(0.002)	0.49(0.006)	0.49(0.006)
0.6	100	0.61(0.058)	0.46(0.112)	0.47(0.113)
	500	0.60(0.020)	0.49(0.043)	0.50(0.043)
	1000	0.60(0.003)	0.49(0.006)	0.50(0.006)
0.8	100	0.80(0.058)	0.45(0.133)	0.46(0.133)
	500	0.81(0.020)	0.47(0.043)	0.48(0.043)
	1000	0.80(0.003)	0.49(0.006)	0.50(0.006)

Table 3: Mean (SE) of estimators of α and θ for $\alpha = 0.7$

θ	N	$\widehat{\theta}$	$\widehat{\alpha}_{YW}$	$\widehat{\alpha}_{CLS}$
0.4	100	0.42(0.070)	0.64(0.108)	0.66(0.108)
	500	0.40(0.024)	0.69(0.038)	0.70(0.038)
	1000	0.40(0.003)	0.69(0.005)	0.69(0.005)
0.6	100	0.63(0.077)	0.62(0.102)	0.66(0.103)
	500	0.61(0.030)	0.68(0.039)	0.68(0.039)
	1000	0.60(0.004)	0.69(0.005)	0.69(0.005)
0.8	100	0.82(0.062)	0.63(0.108)	0.65(0.111)
	500	0.81(0.023)	0.68(0.041)	0.69(0.041)
	1000	0.80(0.003)	0.70(0.005)	0.70(0.005)

Secondly we simulated 1000 samples of size $N = 100, 500$ and 1000 from the $SDL-INAR(1,\theta_1,\theta_2)$ process for $\theta_1 = 0.4, 0.6, 0.8, \theta_2 = 0.3, 0.5, 0.7$ and $\alpha = 0.3, 0.5$ and 0.7. In each case we have computed the mean and standard error (SE) of $\widehat{\alpha}_{cls}, \widehat{\alpha}_{yw}, \widehat{\theta}_{1cls}, \widehat{\theta}_{2cls}, \widehat{\theta}_{1yw}$ and $\widehat{\theta}_{2yw}$. The results are presented in Tables 4-6. The results of Tables 4-6 show that all the estimators become very close to their true values as the sample size N increases.

7 Applications

In this section we present two applications of the $DL-INAR(1,\theta)$ model using the data for the Saudi Telecommunication Company (STC) stock and the electricity stock of the Saudi Stock Market TASI in 2007. Note that the minimum amount of change (a tick) is SR 0.25 for all stocks. The daily close number of ticks of any Stock equals the close price times 4.

The graphs of the two series show that they are nonstationary indicating that differencing is needed. The two differenced series are stationary in the mean. For the STC data the lag-one correlation is positive and significant hence a $DL-INAR(1,\theta)$ with $\kappa = 1$ is proposed to model the differenced series of the STC data. For the electricity data the lag one correlation is negative and significant hence a $DL-INAR(1,\theta)$ with $\kappa = -1$ is proposed to model the differenced series of the electricity data. In Table 7, we give the Yule-Walker and the conditional least squares estimates of α and θ for both data sets.

To study the adequacy of the model, in Figures 1 and 2, we plotted each data set and the corresponding fitted DL distribution. Clearly the STC data and the electricity data can be fitted by discrete Laplace distribution. The residuals plots indicate that residuals are white noise and the proposed model is a good fit for each data set.

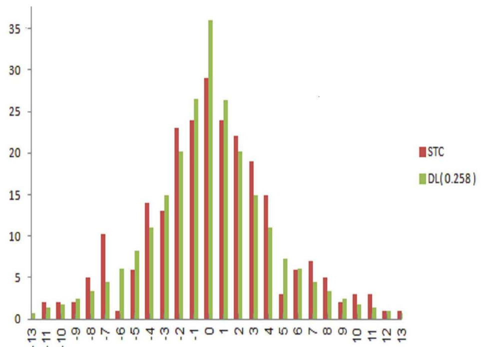

Fig. 1: Relative frequency of STC and fitted discrete Laplace.

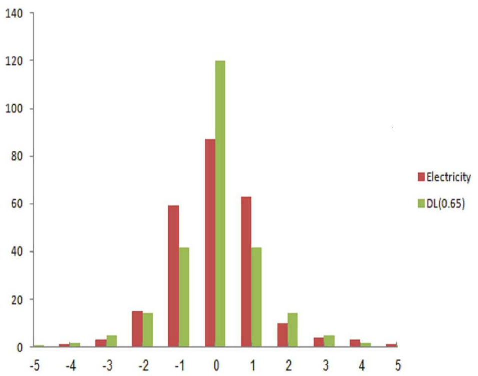

Fig. 2: Relative frequency of electricity and fitted discrete Laplace.

Next, we re-examine the STC data using the $SDL-INAR(1,\theta_1,\theta_2)$ model. The conditional least squares and the Yule-Walker estimators are reported in Table 8.

In Figure 3 the STC data is plotted together with the fitted SDL distribution. Clearly the proposed model is a good fit for the STC data. The residuals plot indicate that the residuals are white noise. We notice from Table 8 that the estimated values of θ_1 and θ_2 are almost equal

Table 4: Mean(SE) of estimators ofα ,θ_1 and θ_2 for $\alpha = \theta_2 = 0.3$

θ_1	N	$\widehat{\theta}_{1YW}$	$\widehat{\theta}_{2YW}$	$\widehat{\alpha}_{YW}$	$\widehat{\theta}_{1CLS}$	$\widehat{\theta}_{2CLS}$	$\widehat{\alpha}_{CLS}$
0.4	100	0.41(0.062)	0.31(0.038)	0.28(0.123)	0.42(0.063)	0.31(0.039)	0.28(0.108)
	500	0.40(0.030)	0.30(0.019)	0.29(0.059)	0.41(0.027)	0.30(0.018)	0.29(0.059)
	1000	0.41(0.019)	0.30(0.014)	0.30(0.047)	0.40(0.018)	0.30(0.012)	0.30(0.046)
0.6	100	0.62(0.100)	0.31(0.037)	0.29(0.126)	0.64(0.132)	0.31(0.045)	0.29(0.125)
	500	0.60(0.053)	0.30(0.024)	0.30(0.069)	0.61(0.049)	0.31(0.019)	0.30(0.069)
	1000	0.60(0.035)	0.30(0.014)	0.30(0.050)	0.60(0.035)	0.30(0.013)	0.30(0.052)
0.8	100	0.88(0.201)	0.31(0.038)	0.25(0.122)	0.89(0.176)	0.32(0.042)	0.24(0.121)
	500	0.81(0.075)	0.30(0.019)	0.29(0.072)	0.81(0.078)	0.30(0.021)	0.29(0.072)
	1000	0.81(0.060)	0.30(0.015)	0.29(0.051)	0.81(0.064)	0.30(0.015)	0.29(0.061)

Table 5: Mean(SE) of estimators ofα, θ_1 and θ_2 for $\alpha = \theta_2 = 0.5$

θ_1	N	$\widehat{\theta}_{1YW}$	$\widehat{\theta}_{2YW}$	$\widehat{\alpha}_{YW}$	$\widehat{\theta}_{1CLS}$	$\widehat{\theta}_{2CLS}$	$\widehat{\alpha}_{CLS}$
0.4	100	0.43(0.062)	0.54(0.083)	0.44(0.118)	0.42(0.055)	0.53(0.078)	0.44(0.118)
	500	0.41(0.027)	0.51(0.032)	0.49(0.054)	0.41(0.030)	0.50(0.039)	0.49(0.065)
	1000	0.40(0.021)	0.50(0.027)	0.50(0.042)	0.40(0.019)	0.50(0.025)	0.50(0.046)
0.6	100	0.63(0.080)	0.52(0.064)	0.46(0.102)	0.64(0.076)	0.51(0.069)	0.45(0.124)
	500	0.60(0.043)	0.51(0.035)	0.50(0.069)	0.61(0.038)	0.51(0.037)	0.50(0.067)
	1000	0.60(0.028)	0.50(0.025)	0.49(0.048)	0.60(0.025)	0.50(0.024)	0.50(0.043)
0.8	100	0.85(0.122)	0.52(0.075)	0.44(0.123)	0.84(0.101)	0.52(0.066)	0.46(0.125)
	500	0.81(0.054)	0.50(0.032)	0.49(0.071)	0.81(0.056)	0.50(0.032)	0.50(0.067)
	1000	0.80(0.035)	0.50(0.024)	0.49(0.044)	0.80(0.035)	0.50(0.025)	0.49(0.053)

Table 6: Mean(SE) of estimators of α, θ_1 and θ_2 for $alpha = \theta_2 = 0.7$

θ_1	N	$\widehat{\theta}_{1YW}$	$\widehat{\theta}_{2YW}$	$\widehat{\alpha}_{YW}$	$\widehat{\theta}_{1CLS}$	$\widehat{\theta}_{2CLS}$	$\widehat{\alpha}_{CLS}$
0.4	100	0.43(0.083)	0.79(0.241)	0.65(0.110)	0.43(0.081)	0.84(0.253)	0.63(0.120)
	500	0.41(0.036)	0.72(0.075)	0.68(0.055)	0.41(0.041)	0.72(0.080)	0.68(0.062)
	1000	0.40(0.031)	0.71(0.059)	0.69(0.048)	0.41(0.029)	0.72(0.052)	0.68(0.043)
0.6	100	0.63(0.091)	0.73(0.118)	0.65(0.112)	0.63(0.091)	0.75(0.121)	0.64(0.117)
	500	0.61(0.041)	0.71(0.046)	0.68(0.047)	0.61(0.051)	0.71(0.051)	0.69(0.052)
	1000	0.60(0.036)	0.70(0.037)	0.70(0.044)	0.61(0.037)	0.70(0.035)	0.69(0.040)
0.8	100	0.85(0.095)	0.74(0.080)	0.60(0.124)	0.83(0.102)	0.71(0.098)	0.65(0.112)
	500	0.81(0.042)	0.71(0.045)	0.68(0.052)	0.81(0.040)	0.70(0.050)	0.69(0.056)
	1000	0.80(0.033)	0.70(0.034)	0.69(0.037)	0.81(0.032)	0.71(0.035)	0.69(0.046)

Table 7: Estimation result for STC and electricity

Stock	Parameter	YW	CLS
STC	α	0.218139	0.218139
	θ	0.258756	0.258756
electricity	α	0.243627	0.243108
	θ	0.654261	0.654261

Table 8: Estimation result for STC data

Stock	Parameter	YW	CLS
STC	α	0.21813	0.21814
	θ_1	0.25817	0.25775
	θ_2	0.25844	0.25888

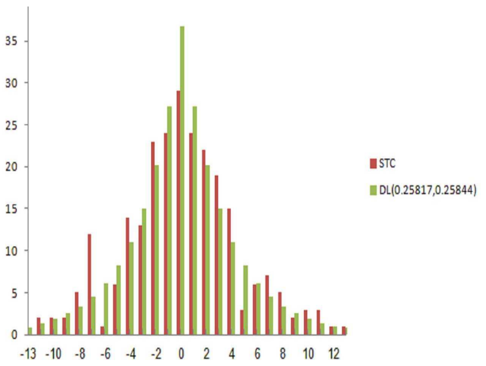

Fig. 3: Relative frequency of STC and fitted discrete Laplace.

suggesting that the $DL-INAR(1,\theta)$ model might be more appropriate for this data.

8 Perspective

In this paper we introduce and study stationary integer-valued autoregressive models with discrete Laplace (DL) and Skew DL (SDL) marginals. These models allow for positive- and negative-integer values. The stationary integer-valued autoregressive model with DL marginal allows positive and negative autocorrelation function.

we can make an extension of this model to higher order autoregressive model of order p with discrete Laplace marginal and we think in the moving average model of order 1 and higher order of order q.

Acknowledgement

The first author acknowledges the financial support by the FIRB project-RBID08PP3J-Metodi matematici e relativi strumenti per la modellizzazione e la simulazione della formazione di tumori, competizione con il sistema immunitario, e conseguenti suggerimenti terapeutici.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] A.S. Nastić, M.M. Ristićand M.S. Djordjević, An INAR model with discrete Laplace marginal distributions. Brazilian Journal of probability and statistics, Vol. 30, pp. 107-126(2016).
[2] A.W. Marshall and I. Olkin, A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. Biometrika, Vol. 84, pp. 641-652(1997).
[3] E. Krishna and K.K. Jose, Marshall-Olkin generalized asymmetric Laplace distribution and process. Statistica, Vol. 71, pp. 453-467(2011).
[4] F.W. Steutel, , K. van Harn, Discrete analogues of self-decomposability and stability, The Annals of probability, Vol 7, pp. 893-899(1979).
[5] H. Karlsen, and D. Tjφstheim, Consistent estimates for the NEAR(2) and NLAR(2) time series models, Journal of the Royal Statistical Society, Series B: Methodological, Vol. 50, pp. 313-320(1988).
[6] K. Jayakumar, and A.P. Kuttykrishnan, Time series model using asymmetric Laplace distribution, Statistics and Probability Letters, Vol. 76, pp. 813-820(2006).
[7] M.M. Ristić, H.S. Bakouch and A.S. Nastić, A new geometric first-order integer-valued autoregressive (NGINAR(1)) process, Journal of Statistical Planning and Inference, Vol. 139, pp. 2218-2226(2009).
[8] R.K. Freeland, True integer value time series, Advances in Statistical Analysis, Vol. 94, pp. 217-229(2010).
[9] S. Inusah and T.J.Kozubowski, A discrete analogue of the Laplace distribution, Journal of Statistical Planning and Inference, Vol. 136, pp. 1090-1102(2006).
[10] T.J. Kozubowski and K. Podgŏrstki, Asymmetric Laplace distributions, Journal of Mathematical Sciences, Vol. 25, pp. 37-46(2000).
[11] T.J. Kozubowski and S. Inusah, A skew Laplace distribution on integers, Annals of the Institute of Statistical Mathematics, Vol. 58, pp. 555-571(2006).
[12] W.Barreto-Souza and M.Bourguignon, A skew true INAR(1) process on Z, Advances in Statistical Analysis, Vol. 99, pp. 189-208(2015).

Ahmed M. Agwa I received the PhD degree in Mathematics department at Alexandria University, Alexandria, Egypt. I work in Statistics and Operations Research department, Kuwait University, Kuwait.

Emad-Eidin A A Aly is Professor of Statistics and Operations Research department, Kuwait University, Kuwait.

M. M. Gabr is Professor of Mathematics department, Alexandria University, Alexandria, Egypt. He is Chief Information Officer of Alexandria University.