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A New Analytical-Approximate Solution for the Viscoelastic Squeezing Flow Between Two Parallel Plates

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Abstract: In this paper, a new approach is used to study analytically the axisymmetric fluid squeezed between two parallel plates. This new approach depends mainly on the coefficients of powers series resulting from integrating n^{th} order differential equation with known data. We obtianed an analytical-approximate solution for the squeezing flow between two parallel plates. The steady non-linear governing partial differential equations are converted by using the suitable similarity transformation into ordinary differential equation. In addition, some theorems are introduced to prove the convergence of a new approach theoretically and explain the verifications of these theorems computationally. The results domenstrate this new approach is efficient and reasonable which compare with the results of the other methods.

Keywords: squeezing flow, parallel plates, power series, analytical-approximate solution, convergence analysis.

1 Introduction

Squeezing flow occurs by the restriction of fluid particularly a viscoelastic fluid in the gap either between two parallel plates or coaxial disks which result in both shear and longitudinal deformation. Some practical examples of squeezing flow include polymer processing, modeling of synthetics transportation inside living bodies, hydro-mechanical machinery, injection molding and compression processes. Squeezing flows motivate when normal stresses or vertical velocities are externally applied by means of a mobile boundary. The squeezing flows have been studied and taken a considerable attention since 19^{th} century due to their wide range of practical applications in physical and biophysical fields. Stefan [1] published article on squeezing flow depended on lubrication approximation, and Reynolds [2] obtained a solution for elliptic plates.

Many researchers [3]-[6] have provided the theoretical and experimental studies of squeezing flows, which used to solve the governing nonlinear equation to find analytical-approximate solutions of the equations for the squeezing flow between two infinite plates. The importance of research includes the study of nonlinear differential equations which naturally describe the nonlinearity of many physical phenomena. In the recent years, several methods have been used to find analytical-approximate solutions to nonlinear differential equations such as; Adomians decomposition method (Sheikholeslami et al.[7], Birajdar [8]), differential transform method (Muhammad et al. [9]), homotopy perturbation method(Domairry and Aziz [10], Umar et al.[11]), homotopy analysis method (Bouremel[12], Ran et al. [13], Mustafa et al.[14] and Dayyan et al.[15]). The objective of this job is the search for an analytical-approximate solution for the nonlinear problem, which describes a viscous, incompressible fluid, squeezed between two infinite parallel plates, so that the plates are moving towards each other with a certain velocity V, see Figure 1.

In this paper, we introduce a new approach that depends on the coefficients of the power series as essential manner to find an analytical-approximate solution for the problem of squeezing flow between two parallel plates. The analytical-approximate solution is compared with Runge -Kutta of fourth order, Homotopy Analysis

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Method(HAM) [13] and Successive Linearization Method(SLM)[16]. The comparison shows that the solutions are compatible and have a good convergence. The organization of this paper is as follows: the governing equations are derived in section 2. Detailed derivation of the new approach has been written as steps in section 3. The performance of the new approach for the squeezing flow has been applied in section 4. In section 5 the analysis of converges are explained. Results and discussions are given in section 6. The paper ends with conclusions.

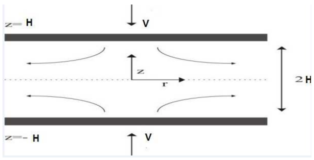

Fig. 1: squeezed between two infinite parallel plates

2 Governing Equations

The problem under consideration is that of a two-dimensional quasi -steady axisymmetric incompressible viscous flow between two infinite parallel plates as in [13]. The governing equations in the radial and axial coordinates (r,z) can be expressed as:

$$\frac{\partial p}{\partial r}+\frac{\rho}{r}\frac{\partial^2\psi}{\partial t\partial z}-\rho\frac{\partial\psi}{\partial r}\frac{E^2\psi}{r^2}-\frac{\mu}{r}\frac{\partial E^2\psi}{\partial z}=0, \quad (1)$$

$$\frac{\partial p}{\partial r}+\frac{\rho}{r}\frac{\partial^2\psi}{\partial t\partial z}-\rho\frac{\partial\psi}{\partial z}\frac{E^2\psi}{z^2}+\frac{\mu}{r}\frac{\partial E^2\psi}{\partial z}=0, \quad (2)$$

where ρ is the fluid density, μ is the coefficient of kinematic viscosity, p is the pressure

$$E^2=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \quad (3)$$

and $\psi(r,z)$ is the stokes stream function given by

$$u_r(r,z,t)=\frac{1}{r}\frac{\partial\psi}{\partial z},u_z(r,z,t)=-\frac{1}{r}\frac{\partial\psi}{\partial r} \quad (4)$$

Upon eliminating the generalized pressure p in Equations (1)and (2) we get

$$\rho[\frac{1}{r}\frac{\partial E^2\psi}{\partial t}-\frac{\partial(\psi,\frac{E^2\psi}{r^2})}{\partial(r,z)}]=\frac{\mu}{r}E^4\psi, \quad (5)$$

for small values of the approach velocity u of the two plates, the gap $2H$ changes slowly with time and can be assumed to be constant, hence from Equation (5), we have

$$-\rho[\frac{\partial(\psi,\frac{E^2\psi}{r^2})}{\partial(r,z)}]=\frac{\mu}{r}E^4\psi, \quad (6)$$

with the boundary conditions

$$u_r=0,\quad u_z=-V \quad at \quad z=H,$$

$$u_z=0,\quad \frac{\partial u_r}{\partial z}=0 \quad at \quad z=0, \quad (7)$$

the stream function can be expressed by similarity transformation as

$$\psi(r,z)=r^2f(z), \quad (8)$$

and introducing the non-dimensional parameters

$$f^*=\frac{f}{V/2},\quad z^*=\frac{z}{H},\quad M=\frac{\rho H}{\mu/V}, \quad (9)$$

After we use the definitions in Equation (9) and drop (*), Equation (6) with boundary conditions Equation (7) in non-dimensional forms becomes:

$$f^{iv}(z)+Mf(z)f'''(z)=0, \quad (10)$$

$$f(0)=0,\quad f''(0)=0,\quad f(1)=1,\quad f'(1)=0, \quad (11)$$

Recently, semi-analytical approximate methods HAM [13] and SLM [16] are used to solve this regime and compared with numerical method. These methods have limitations (linearizion, discretization, uses polynomials, choice auxiliary parameters and especially for SLM the determination of the number of collocation points is not straightforward) in the study of nonlinear problems. These drawbacks may produce unnecessary computations not only imply a divergence in the solution but more from that taking away long computation time. So, in this work we propose a new approach to overcome some of these limitations.

3 The basic steps of the new approach

This section describes how to obtain a new approach and to calculate the coefficients of the power series solution resulting from solving nonlinear ordinary differential equations to find analytical-approximate solution. These coefficients are important bases to construct the solution formula, therefore they can be computed recursively by differentiation ways. To illustrate the computation and operations for these coefficients and derivation the new approach, we summarize the detail of a new outlook in the following steps:
Step 1: Consider the non-linear differential equation as follows:

$$H(f(z),f'(z),f''(z),...,f^{(n-1)}(z),f^{(n)}(z))=0, \quad (12)$$

integrating Equation (12) with respect to z on $[0,z]$ acquire

$$f(z)=f(0)+f'(0)z+f''(0)\frac{z^2}{2!}+...+$$

$$f^{(n-1)}(0)\frac{z^{n-1}}{(n-1)!}+L^{-1}G[f(z)], \tag{13}$$

where,

$$G[f(z)]=H(f(z),f'(z),f''(z),...,f^{(n-1)}(z)),$$

$$L^{-1}=\int_0^z\int_0^z...\int_0^z(dz)^n, \tag{14}$$

Step 2 : Assume that

$$G[f(z)]=\sum_{n=1}^{\infty}\frac{d^{n-1}G(f_0(z))}{dz^{n-1}}, \tag{15}$$

rewriting the Equation (15)

$$G[f(z)]=G[f_0(z)]+G'[f_0(z)]+G''[f_0(z)]+..., \tag{16}$$

substituting Equation (16) in Equation (13), we obtain

$$f(z)=f_0+f_1+f_2+f_3+f_4+..., \tag{17}$$

where,

$$f_0=f(0)+f'(0)z+f''(0)\frac{z^2}{2!}...+f^{(n-1)}(0)\frac{z^{(n-1)}}{(n-1)!},$$

$$f_1=L^{-1}G[f_0(z)],\quad f_2=L^{-1}G'[f_0(z)],$$

$$f_3=L^{-1}G''[f_0(z)],\quad f_4=L^{-1}G'''[f_0(z)],... \tag{18}$$

Step 3 : We focus on computing the derivatives of G with respect to z which is the crucial part of the proposed method. Let start calculating $G[f(z)],G'[f(z)],G''[f(z)],G'''[f(z)],....$

$$G[f(z)]=H(f(z),f'(z),f''(z),...,f^{(n-1)}(z)), \tag{19}$$

$$G'[f(z)]=\frac{dG[f(z)]}{dz}=G_f.f_z+G_{f'}.(f_z)'$$
$$+...+G_{f^{(n-1)}}.(f_z)^{(n-1)}, \tag{20}$$

$$G''[f(z)]=\frac{d^2G[f(z)]}{dz^2}=G_{ff}.(f_z)^2+G_{ff'}.(f_z)'f_z$$
$$+G_{ff''}.f_z(f_z)''+...+G_{ff^{(n-1)}}.(f_z)(f_z)^{(n-1)}+G_f$$
$$.f_{zz}+G_{f'f}.(f_z)'.f_z+G_{f'f'}.(f_z)'^2+...+G_{f'f^{(n-1)}}$$
$$.(f_z)'(f_z)^{(n-1)}+G_{f'}.(f_{zz})'+G_{f''f}.(f_z)''.f_z+G_{f''f'}.$$
$$(f_z)'(f_z)''+G_{f''f''}.(f_z)''^2+G_{f''f'}.(f_z)'(f_z)''+$$
$$...+G_{f''f^{(n-1)}}.(f_z)''(f_z)^{(n-1)}+G_{f''}.(f_{zz})''+...+$$
$$G_{f^{(n-1)}f}.(f_z)^{(n-1)}.f_z+G_{f^{(n-1)}f'}(f_z)^{(n-1)}.(f_z)'+...+$$
$$G_{f^{(n-1)}f^{(n-1)}}.(f_z)^{(n-1)2}+G_{f^{(n-1)}}.(f_{zz})^{(n-1)}, \tag{21}$$

$$G'''[f(z)]=\frac{d^3G[f(z)]}{dz^3}=G_{fff}.(f_z)^3+G_{fff'}.(f_z)^2$$
$$(f_z)'+...+G_{fff^{(n-1)}}.(f_z)^2.(f_z)^{(n-1)}+G_{ff}.2(f_z).f_{zz}$$
$$+G_{ff'f}.(f_z)'(f_z)^2+G_{ff'f'}.(f_z)'^2(f_z)+...+G_{ff'f^{(n-1)}}$$
$$.(f_z)'(f_z).(f_z)^{(n-1)}+G_{ff'}.[(f_{zz})'.f_z+(f_z)'.f_{zz}]+G_{ff''f}$$
$$.(f_z)''(f_z)^2+G_{ff''f'}.(f_z)''(f_z).(f_z)'+...+G_{ff''f^{(n-1)}}$$
$$.(f_z)''(f_z).(f_z)^{(n-1)}+G_{ff''}.[f_{zz}.(f_z)''+f_z.(f_{zz})'']+...+$$
$$G_{ff^{(n-1)}f}.(f_z)^2.(f_z)^{(n-1)}+G_{ff^{(n-1)}f'}.(f_z).(f_z)'.(f_z)^{(n-1)}$$
$$+...+G_{ff^{(n-1)}f^{(n-1)}}.(f_z).(f_z)^{(n-1)2}+G_{ff^{(n-1)}}.[(f_{zz})$$
$$.(f_z)^{(n-1)}+(f_z)(f_{zz})^{(n-1)}]+G_{ff}.f_{zz}.(f_z)+G_{ff'}.f_{zz}.(f_z)'$$
$$+..+G_{ff^{(n-1)}}.f_{zz}.(f_z)^{(n-1)}+G_z.f_{zzz}+G_{f'ff}.(f_z)'(f_z)^2+$$
$$G_{f'f'f}.(f_z)'^2(f_z)+...+G_{f'ff^{(n-1)}}.(f_z)'(f_z).(f_z)^{(n-1)}$$
$$+G_{f'f}.[(f_{zz})'.f_z+(f_z)'.f_{zz}]+G_{f'f'f}.(f_z)'^2.f_z+G_{f'f'f'}$$
$$.(f_z)'^3+...+G_{f'f'f^{(n-1)}}.(f_z)'^2.(f_z)^{(n-1)}+G_{f'f'}.2(f_z)'$$
$$.(f_{zz})'+...+G_{f^{(n-1)}f^{(n-1)}f}.(f_z)^{(n-1)2}.f_z+G_{f^{(n-1)}f^{(n-1)}f'}.$$
$$(f_z)^{(n-1)2}.(f_z)'+...+G_{f^{(n-1)}f^{(n-1)}f^{(n-1)}}.(f_z)^{(n-1)3}$$
$$.(f_{zz})^{(n-1)}.f_z+G_{f^{(n-1)}f'}.(f_{zz})^{(n-1)}.(f_z)'+...+G_{f^{(n-1)}}$$
$$f^{(n-1)}.(f_{zz})^{(n-1)}.(f_z)^{(n-1)}+G_{f^{(n-1)}}.(f_{zzz})^{(n-1)}. \tag{22}$$

$$\vdots$$

The calculations are more complicated in the second and third derivatives because of the product rules. Consequently, the systematic structure on calculation is extremely important. Fortunately, due to the assumption that the operator G and the solution$f$ are analytic functions, then the mixed derivatives are equivalence.
We note that the derivatives function to f unknown, so we suggest the following hypothesis

$$f_z=f_1=L^{-1}G[f_0(z)],\quad f_{zz}=f_2=L^{-1}G'[f_0(z)],$$

$$f_{zzz}=f_3=L^{-1}G''[f_0(z)],f_{zzzz}=f_4=L^-1G'''[f_0(z)],... \tag{23}$$

Therefore Equations (19)- (22) are evaluated by

$$G[f_0(z)]=H(f_0(z),f_0'(z),...,f_0^{(n-1)}(z)), \tag{24}$$

$$G'[f_0(z)] = G_{f_0}.f_1 + G_{f_0'}.(f_1)' + ... + G_{f_0^{(n-1)}}.(f_1)^{(n-1)}, \quad (25)$$

$$G''[f_0(z)] = G_{f_0f_0}.(f_1)^2 + G_{f_0f_0'}.(f_1)'f_1 + G_{f_0f_0''}.f_1$$
$$(f_1)'' + ... + G_{f_0f_0^{(n-1)}}.(f_1)(f_1)^{(n-1)} + G_{f_0}.f_2 + G_{f_0'f_0}$$
$$.(f_1)'.f(_1) + G_{f_0'f_0'}.(f_1)'^2 + ... + G_{f_0'f_0^{(n-1)}}.(f_1)'$$
$$(f_1)^{(n-1)} + G_{f_0'}.(f_2)' + G_{f_0''f_0}.(f_1)''.f_1 + G_{f_0''f_0'}.$$
$$(f_1)'(f_1)'' + G_{f_0''f_0''}.(f_1)''^2 + ... + G_{f_0''f_0^{(n-1)}}.$$
$$(f_1)''(f_1)^{(n-1)} + G_{f_0''}.(f_2)'' + G_{f^{(n-1)}f}.(f_1)^{(n-1)}.f_1$$
$$+G_{f_0^{(n-1)}f_0'}.(f_1)^{(n-1)}.(f_1)' + ... + G_{f_0^{(n-1)}f_0^{(n-1)}}$$
$$.(f_1)^{(n-1)2} + ... + G_{f_0^{(n-1)}}.(f_2)^{(n-1)}, \quad (26)$$

$$G'''[f_0(z)] = G_{f_0f_0f_0}.(f_1)^3 + G_{f_0f_0f_0'}.(f_1)^2(f_1)' +$$
$$... + G_{f_0f_0f_0^{(n-1)}}.(f_1)^2.(f_1)^{(n-1)} + G_{f_0f_0}.2(f_1).f_2$$
$$+G_{f_0f_0'f_0}.(f_1)'(f_1)^2 + G_{f_0f_0'f_0'}.(f_1)'^2(f_1) + ...+$$
$$G_{f_0f_0'f_0^{(n-1)}}(f_1)^2 + G_{f_0f_0''f_0'}.(f_1)''.(f_1)'(f_1)$$
$$.(f_1)^{(n-1)} + G_{f_0f_0'}.[(f_2)'.f_z + (f_1)'.f_2] + G_{f_0f_0''f_0}.$$
$$(f_1)''(f_1).(f_1)' + ... + G_{f_0f_0''f_0^{(n-1)}}.(f_1)''f_1).(f((_1)^{(n-1)}$$
$$+G_{f_0f_0''}.[f_2.(f_1)'' + f_1.(f_2)''] + ... + G_{f_0f_0^{(n-1)}f_0}.(f_1)^2$$
$$.(f_1)^{(n-1)} + G_{f_0f_0^{(n-1)}f_0'}.(f_1).(f_1)'.(f_1)^{(n-1)} + ...+$$
$$G_{f_0f_0^{(n-1)}f_0^{(n-1)}}.(f_1).(f_1)^{2(n-1)} + G_{f_0f_0^{(n-1)}}$$
$$.[(f_2).(f_2)^{(n-1)} + (f_1)(f_2)^{(n-1)}] + G_{f_0f_0}.f_2.(f_1) + G_{f_0f_0'}$$
$$.f_2.(f_1)' + .. + G_{f_0f_0^{(n-1)}}.f_2.(f_1)^{(n-1)} + G_{f_0}.f_3 + G_{f_0'f_0f_0}$$
$$.(f_1)'(f_1)^2 + G_{f_0'f_0'f_0}.(f_1)'^2(f_1) + ... + G_{f_0'f_0f_0^{(n-1)}}.(f_1)'$$
$$(f_1).(f_1)^{(n-1)} + G_{f_0'f_0}.[(f_2)'.f_1 + (f_1)'.f_1] + G_{f_0'f_0'f_0}.$$
$$(f_1)'^2.f_1 + G_{f_0'f_0'f_0'}.(f_1)'^3 + ... + G_{f_0'f_0'f_0^{(n-1)}}.(f_1)'^2.$$
$$(f_1)^{(n-1)} + G_{f_0'f_0'}.2(f_1)'.(f_2)' + ... + G_{f_0^{(n-1)}f_0^{(n-1)}f_0}.(f_1)^{(n-1)2}$$
$$.f_1 + G_{f_0^{(n-1)}f_0^{(n-1)}f_0'}.(f_1)^{(n-1)2}.(f_1)' + ... + G_{f_0^{(n-1)}f_0^{(n-1)}f_0^{(n-1)}}$$
$$.(f_1)^{(n-1)3} + G_{f_0^{(n-1)}f_0^{(n-1)}}.2.(f_1)^{(n-1)}.(f_2)^{(n-1)} + G_{f_0^{(n-1)}f_0}$$
$$.(f_2^{(n-1)}.f_1 + G_{f_0^{(n-1)}f_0'}.(f_2)^{(n-1)}.(f_1)' + ... + G_{f_0^{(n-1)}f_0^{(n-1)}}$$
$$.(f_2)^{(n-1)}.(f_1)^{(n-1)} + G_{f_0^{(n-1)}}.(f_3)^{(n-1)}, \quad (27)$$

$$\vdots$$

Step4: Substitute of Equations(24)-(27) in Equation (17) gives the required analytical-approximate solution for the Equation (12).

4 Application

The new approach described in the previous section can be used as a powerful solver to the nonlinear differential equations of squeezing flow between two parallel plates (10) - (11) in order to find new an analytical-approximate solution.

From step 1 we have

$$f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + f'''(0)\frac{z^3}{3!}$$
$$+L^{-1}[-Mf(z)f'''(z)], \quad (28)$$

rewrite the Equation(28) as follows

$$f(z) = A_1 + A_2z + A_3\frac{z^2}{2!} + A_4\frac{z^3}{3!} + L^{-1}[G[f(z)]], \quad (29)$$

where,

$$A_1 = f(0), \quad A_2 = f'(0), \quad A_3 = f''(0), \quad A_4 = f'''(0),$$

$$G[f] = -Mf(z)f'''(z), L^{-1}(.) = \int_0^z\int_0^z\int_0^z\int_0^z(.)(dz)^4. \quad (30)$$

From the boundary conditions the Equation (29) can be written as follow

$$f(z) = A_2z + A_4\frac{z^3}{3!} + L^{-1}[G[f(z)]]. \quad (31)$$

From step 2, we have

$$f_0 = A_2z + A_4\frac{z^3}{3!}, f_1 = L^{-1}G[f_0(z)], f_2 = L^{-1}G'[f_0(z)],$$

$$f_3 = L^{-1}G''[f_0(z)], f_4 = L^{-1}G'''[f_0(z)], \ldots. \quad (32)$$

From step 3, yields

$$G[f(z)] = -Mf(z).f'''(z), \quad (33)$$

$$G'[f(z)] = \frac{dG[f(z)]}{dz} = G_f.f_z + G_{f'''}.(f_z)''', \quad (34)$$

$$G''[f(z)] = \frac{d^2G[f(z)]}{dz^2} = G_{ff}.(f_z)^2 + G_{f'''f'''}.(f_z)^{'''2}$$
$$+2.G_{ff'''}.f_z(f_z)''' + G_f.f_{zz} + G_{f'''}.(f_{zz})''', \quad (35)$$

$$G'''[f(z)] = \frac{d^3G[f(z)]}{dz^3} = G_{fff}.(f_z)^3 + G_{fff'''}.(f_z)^2$$
$$.(f_z)''' + 3.G_{ff}.f_z.f_{zz} + G_{f'''f'''f}.(f_z)^{'''2}.f_z + G_{f'''f'''f'''}.$$
$$(f_z)^{'''3} + \quad 3.G_{f'''f'''}.(f_z)'''.(f_{zz})''' + 2.G_{ff'''f}.$$
$$(f_z)^2.(f_z)''' + 2.G_{ff'''f'''}.f_z.(f_z)^{'''2} + 3.G_{ff'''}.f_{zz}.(f_z)'''$$
$$+3.G_{ff'''}.(f_{zz})'''.f_z + G_f.f_{zzz} + G_{f'''}.(f_{zzz})''', \quad (36)$$

$$\vdots$$

We note that the derivatives of f with respect z that are given in (23), can be computed by Equations(33)-(36) as

$$G[f_0(z)] = -M.f_0.f_0''', \quad (37)$$

$$G'[f_0(z)] = G_{f_0}.f_1 + G_{f_0'''}.(f_1)''', \quad (38)$$

$$G''[f_0(z)] = G_{f_0f_0}.(f_1)^2 + G_{f_0'''f_0'''}.(f_1)^{'''2} + 2.G_{f_0f_0'''}.$$
$$f_1.(f_1)''', + G_{f_0}.f_2 + G_{f_0'''}.(f_2)''', \quad (39)$$

$$G'''[f_0(z)] = G_{f_0f_0f_0}.(f_1)^3 + G_{f_0f_0f_0'''}.(f_1)^2.(f_1)'''$$
$$+3.G_{f_0f_0}.f_1f_2 + +G_{f_0'''f_0'''f_0}.(f_1)^{'''2}.f_1 + G_{f_0'''f_0'''f_0'''}.$$
$$(f_1)^{'''3} + 3.G_{f_0'''f_0'''}.(f_1)'''(f_2)''' + 2.G_{f_0f_0'''f_0}$$
$$.(f_1)^2.(f_1)''' + 2.G_{f_0f_0'''f_0'''}.f_1.(f_1)^{'''2} + 3.G_{f_0f_0'''}.f_2.$$
$$(f_1)''' + 3.G_{f_0f_0'''}.f_1.(f_2)''' + G_{f_0}.f_3 + G_{f_0'''}.(f_3)''', \quad (40)$$

$$\vdots$$

Now, we need to extract the first derivatives of G as follows

$$G_{f_0} = -Mf_0'''(z), \quad G_{f_0f_0} = 0, \quad G_{f_0f_0'''} = -M,$$

$$G_{f_0f_0f_0} = 0, \quad G_{f_0'''} = -Mf_0(z), \quad G_{f_0'''f_0'''} = 0,$$

$$G_{f_0'''f_0} = -M, \quad G_{f_0'''f_0'''f_0'''} = 0, \quad (41)$$

from Equation(32) by using Equations (37)-(40), we obtain

$$f_0 = \frac{1}{6}A_4z^3 + A_2z, \quad (42)$$

$$f_1 = -\frac{1}{5040}MA_4^2z^7 - \frac{1}{120}MA_2A_4z^5, \quad (43)$$

$$f_2 = \frac{1}{1108800}M^2A_4^3z^{11} + \frac{1}{22680}M^2A_2$$
$$A_4^2z^9 + \frac{1}{1680}M^2A_2^2A_4z^7, \quad (44)$$

$$f_3 = -\frac{79}{1556755000}M^3A_4^4z^{15} - \frac{251}{778377600}M^3A_2A_4^3z^{13}$$
$$-\frac{131}{19958400}M^3A_2^2A_4^2z^{11} - \frac{1}{24192}M^3A_2^3A_4z^9. \quad (45)$$

$$\vdots$$

Substituting Equations(42) - (45) in Equation (17), we get the analytical approximate solution:

$$f(z) = A_2z + \frac{1}{6}A_4z^3 - \frac{1}{120}MA_2A_4z^5 - (\frac{1}{5040}MA_4^2$$
$$-\frac{1}{1680}M^2A_2^2A_4)z^7 - (\frac{1}{24192}M^3A_2^3A_4 - \frac{1}{22680}M^2A_2$$
$$A_4^2)z^9 - (\frac{131}{19958400}M^3A_2^2A_4^2 - \frac{1}{1108800}M^2A_4^3)z^{11} +$$
$$-\frac{251}{778377600}M^3A_2A_4^3z^{13} - \frac{79}{1556755000}M^3A_4^4z^{15} + \ldots., \quad (46)$$

5 The analysis of convergence

Here, we study the analysis of convergence for the analytical-approximate solution that are resulted from the application of new power series approach for solving the problem of the squeezing flow between two parallel plates.

Definition 5.1. Suppose that H is Banach space, R is the real numbers and G[F] is a nonlinear operator defined by $G[F]: H \longrightarrow R$. Then the sequence of the solutions generated by a new approach can be written as

$$F_{n+1} = G[F_n], \quad F_n = \sum_{k=0}^{n} f_k, \quad n = 0, 1, 2, 3, \ldots \quad (47)$$

where G[F] satisfies Lipschitz condition such that for $\gamma > 0$, $\gamma \in R$, we have

$$\| G[F_n] - G[F_{n-1}] \leqslant \gamma \| F_n - F_{n-1} \|, \quad (48)$$

Theorem 5.1. The series of the analytical-approximate solution $f(z)=\sum_{k=0}^{\infty} f_k(z)$ generated by new approach converge if the following condition is satisfied:

$$\| F_n - F_m \| \longrightarrow 0, \quad m \longrightarrow \infty \quad for \quad 0 \leqslant \gamma < 1, \tag{49}$$

Proof. From the above definition, we have

$$\| F_n - F_m \| = \| \sum_{k=0}^{n} f_k - \sum_{k=0}^{m} f_k \|,$$

$$= \| [f_0 + L^{-1}\textstyle\sum_{k=1}^{n} \frac{d^{(k-1)}G[f_0(z)]}{dz^{(k-1)}}] - [f_0 + L^{-1}\sum_{k=1}^{m} \frac{d^{(k-1)}G[f_0(z)]}{dz^{(k-1)}}] \|,$$

$$= \| [L^{-1}G[\textstyle\sum_{k=0}^{n-1} f_k]] - [L^{-1}G[\sum_{k=0}^{m-1} f_k]] \|, (since\ F_n = G[F_{n-1}])$$

$$= \| [L^{-1}G[\sum_{k=0}^{n-1} f_k]] - [L^{-1}G[\sum_{k=0}^{m-1} f_k]] \|,$$

$$\leqslant |Ł^{-1}| \; \| G[\sum_{k=0}^{n-1} f_k] - G[\sum_{k=0}^{m-1} f_k] \|,$$

$$\leqslant |L^{-1}| \; \| G[F_{n-1}] - G[F_{m-1}] \|$$

$$\leqslant \gamma \| F_{n-1} - F_{m-1} \|, \tag{50}$$

since G[F] satisfies Lipchitz condition. Let $n = m+1$, then

$$\| F_{m+1} - F_m \| \leqslant \gamma \| F_m - F_{m-1} \|, \tag{51}$$

hence,

$$\| F_m - F_{m-1} \| \leqslant \gamma \| F_{m-1} - F_{m-2} \| \leqslant \dots \leqslant \gamma^{m-1} \| F_1 - F_0 \|, \tag{52}$$

from Equation (52) we get

$$\| F_2 - F_1 \| \leqslant \gamma \| F_1 - F_0 \|,$$

Using triangle inequality

$$\| F_n - F_m \| = \| F_n - F_{n-1} - F_{n-2} - \dots - F_{m+1} - F_m \|,$$

$$\leqslant \| F_n + F_{n-1} \| + \dots + \| F_{m+1} - F_m \|,$$

$$\leqslant [\gamma^{n-1} + \gamma^{n-2} + \dots + \gamma^m] \| F_1 - F_0 \|,$$

$$= \gamma^m [\gamma^{n-m-1} + \gamma^{n-m-2} + \dots + 1] \| F_1 - F_0 \|,$$

$$\leqslant \frac{\gamma^m}{1-\gamma} \| F_1 - F_0 \|,$$

as $m \longrightarrow \infty$, we have $\| F_n - F_m \| \longrightarrow 0$, then F_n is a Cauchy sequence in Banach space H. □

Theorem 5.2. The convergence of the analytical-approximate solution $\sum_{k=0}^{\infty}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4(k+3)!})$ generated by the new procedure will be verified when

$$\exists 0 \leqslant \gamma < 1, \| F_{n+1} - F_n \| \longrightarrow 0, \quad as n \longrightarrow \infty. \tag{53}$$

Proof. The series solution can be indicated by F_k where k is the nth term of solution (46), we get that

$$F_0 = f_0 = a_{00}z + a_{01}\frac{z^3}{3!},$$

$$F_1 = f_0 + f_1 = a_{00}z + a_{01}\frac{z^3}{3!} + a_{10}\frac{z^5}{5!} + a_{11}\frac{z^7}{7!},$$

$$F_2 = f_0 + f_1 + f_2 = a_{00}z + a_{01}\frac{z^3}{3!} + \dots + a_{20}\frac{z^9}{9!} + a_{21}\frac{z^{11}}{11!},$$

$$= a_{00}z + a_{01}\frac{z^3}{3!} + \dots + a_{21}\frac{z^{11}}{11!} + a_{30}\frac{z^{13}}{13!} + a_{31}\frac{z^{15}}{15!},$$

$$\vdots$$

$$F_n = f_0 + f_1 + f_2 + f_3 + f_4 + f_5 \dots + f_{n-1} + f_n,$$

$$= a_{00}z + a_{01}\frac{z^3}{3!} + \dots + a_{n0}\frac{z^{4n+1}}{(4n+1)!} + a_{n1}\frac{z^{4n+3}}{(4n+3)!},$$

$$\| F_{n+1} - F_n \| = \| \sum_{k=0}^{n+1}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4k+3)!})$$

$$- \sum_{k=0}^{n}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4k+3)!}) \|,$$

$$\leqslant \gamma \| \sum_{k=0}^{n}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4(k+3)!})$$

$$- \sum_{k=0}^{n-1}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4k+3)!}) \|$$

$$\leqslant \gamma^2 \| \sum_{k=0}^{n-1}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4(k+3)!})$$

$$- \sum_{k=0}^{n-2}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4k+3)!}) \|,$$

$$\vdots$$

$$\leqslant \gamma^n \| \sum_{k=0}^{1}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4(k+3)!})$$

$$- \sum_{k=0}^{0}(a_{k0}\frac{z^{4k+1}}{(4k+1)!} + a_{k1}\frac{z^{4k+3}}{(4k+3)!}) \|,$$

$$= \gamma^n \| a_{00}z + a_{01}\frac{z^3}{3!} + a_{10}\frac{z^5}{5!} + a_{11}\frac{z^7}{7!} - z - \frac{z^3}{3!} \|$$

$$= \gamma^n \| F_1 - F_0 \|, \tag{54}$$

as $n \longrightarrow \infty$, then $\| F_{n+1} - F_n \| \longrightarrow 0$ for $0 \leqslant \gamma < 1$. □
In practice, the theorems (5.1)-(5.2) suggest to compute the value of γ as described in the following definition
Definition 5.2. For $k = 1, 2, 3, \dots$

$$\gamma^k = \begin{cases} \frac{\|F_{k+1}-F_k\|}{\|F_1-F_0\|} = \frac{\|f_{k+1}\|}{\|f_1\|}, & \| f_1 \| \neq 0, \\ 0, & \| f_1 \| = 0, \end{cases} \tag{55}$$

Now, we apply definition (5.2) on the squeezing flow between two parallel plates to find convergence, then we obtain for examples:

if we put $M = 0.2, A_2 = 1.506733871, A_4 = -3.083763288$, the value of γ will be

$$\| F_2 - F_1 \|_2 \leqslant \gamma \| F_1 - F_0 \|_2 \Longrightarrow \gamma = 0.02174535 < 1,$$

$$\| F_3 - F_2 \|_2 \leqslant \gamma^2 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^2 = 0.00047330 < 1,$$

$$\| F_4 - F_3 \|_2 \leqslant \gamma^3 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^3 = 0.00001028 < 1,$$

$$\vdots$$

$$\| F_2 - F_1 \|_{+\infty} \leqslant \gamma \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma = 0.02152477 < 1,$$

$$\| F_3 - F_2 \|_{+\infty} \leqslant \gamma^2 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^2 = 0.00045044 < 1,$$

$$\| F_4 - F_3 \|_{+\infty} \leqslant \gamma^3 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^3 = 0.00000864 < 1,$$

$$\vdots$$

Also, if we choose $M = 1, A_2 = 1.532547, A_4 = -3.42130$, then obtain

$$\| F_2 - F_1 \|_2 \leqslant \gamma \| F_1 - F_0 \|_2 \Longrightarrow \gamma = 0.11080095 < 1,$$

$$\| F_3 - F_2 \|_2 \leqslant \gamma^2 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^2 = 0.01235149 < 1,$$

$$\| F_4 - F_3 \|_2 \leqslant \gamma^3 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^3 = 0.00139008 < 1,$$

$$\vdots$$

$$\| F_2 - F_1 \|_{+\infty} \leqslant \gamma \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma = 0.10946764 < 1,$$

$$\| F_3 - F_2 \|_{+\infty} \leqslant \gamma^2 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^2 = 0.01165021 < 1,$$

$$\| F_4 - F_3 \|_{+\infty} \leqslant \gamma^3 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^3 = 0.00113620 < 1,$$

$$\vdots$$

And when compensation for $M = 2, A_2 = -3.8433, A_4 = 1.5620$, so that the result is

$$\| F_2 - F_1 \|_2 \leqslant \gamma \| F_1 - F_0 \|_2 \Longrightarrow \gamma = 0.22643980 < 1,$$

$$\| F_3 - F_2 \|_2 \leqslant \gamma^2 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^2 = 0.05193447 < 1,$$

$$\| F_4 - F_3 \|_2 \leqslant \gamma^3 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^3 = 0.01220471 < 1,$$

$$\vdots$$

$$\| F_2 - F_1 \|_{+\infty} \leqslant \gamma \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma = 0.22314285 < 1,$$

$$\| F_3 - F_2 \|_{+\infty} \leqslant \gamma^2 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^2 = 0.04840960 < 1,$$

$$\| F_4 - F_3 \|_{+\infty} \leqslant \gamma^3 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^3 = 0.00962383 < 1,$$

$$\vdots$$

Then $\sum_{k=0}^{\infty} f_k(z)$ converges to the solution $f(z)$ when $0 \leqslant \gamma^k < 1, k = 1, 2, ...$

6 Results and discussions

In this section the influences of the magnetic number M on the axial $f(z)$ and radial velocities $f'(z)$ are characterized. A comparison between the analytical-approximate solutions obtained by the new power series approach and the solutions that are obtained by fourth-order Runge-Kutta method(RK-4), the Homotopy Analysis Method (HAM) [13] and Successive Linearization Method(SLM)[16] are shown in Table 1 for the axial f(z). It confirms that, there is acceptable agreement between analytical- approximate solution obtained by the new power series approach and these methods. The numerical values of the unknown constants A_2 and A_4 are important to find the analytical-approximate solutions and their convergence analysis, where we note that in Table 2 the values of A_2 and A_4 are constant at the fifth approximations for $M = 0.2, 1$ while for $M = 2$ is constant at the sixth approximations. It explains that $f(z)$ reaches its steady state value with small iterations for various values of M. Exactly, according to fixity of A_2 and A_4 which increases with increase of the values M. A comparison between the analytical-approximate solutions and the numerical solutions of RK-4 for the regime studied represents the squeezing flow between two parallel plates are given in Table 3 for the axial $f(z)$and radial velocities $f'(z)$ are in agreement with small errors. To demonstrate the efficiency and accuracy of the new approach for the regime studied, we have presented in Table 3 the values of $f(z)$ and$f'(z)$ obtained by sixth order of the new approach in comparison with numerical solutions of RK-4 method for $M = 0.2, 1$. We observe that the agreement is verified and the values of $f(z)$ and $f'(z)$ are identical with 5 or 6 decimal. The effects for various values of M on the functions $f(z)$ and $f'(z)$ are illustrated in Figures 2 and 3 . In Figure 2 the analytical approximate solution $f(z)$ increases with increasing M. Also the effects of increasing M on the radial velocity $f'(z)$ is demonstrated in Figure 3 where the curves of $f'(z)$ are pushing towards the upper plate for $z \leqslant 0.5$ with increasing M, while this case inversely change for $0.5 < z \leqslant 1$. Also we conclude that the distribution of radial velocity $f'(z)$ is parabolic due to the axisymmetric of region for $-1 \leqslant z \leqslant 1$, that is the nature of the curves is the same for positive and negative values of z, moreover $f'(z)$ increases monotonically. The distributions of the non-dimensional stream function $\psi(r,z)$ and velocities $u_r(r,z)$ and $u_z(r,z)$ are shown in Figures 4-6. In these figures, we see that when $M = 2, 10, 15$ the $8th$ approximation magnitude of $\psi(r,z)$ increases with increasing in the values r and z, and the magnitude of $u_r(r,z)$ increases with r and decreases with z, while the magnitude of $u_z(r,z)$ decreases with increasing of z. From these figures, we note that the change of the squeezing flow between two infinite parallel plates happen about $M = 10$ and it is clear more about $M = 15$. This change may imply that the influences of forces domination that enter in constructing the magnetic number M on each other. Eventually, we note that the convergence of analytical solutions resulting from using the new approach depends on the values of M, if the value of M is small in the analytical-approximate solution with the components A_2 , A_2 give very good convergence.

Table 1: Camparion of f(z) for $M = 2$

z	present results	HAM [13]	SLM [16]	$Rk-4$
0.10	0.15556045	-	0.15558330	0.155559450
0.15	0.23214561	0.2321790	-	0.232138143
0.20	0.30730726	-	0.30735107	0.307305382
0.30	0.45154229	0.4516030	0.45160336	0.451539813
0.40	0.58478538	-	0.5847854	0.584782721
0.45	0.64627714	0.6463530	-	0.646262691
0.50	0.70385384	-	0.703993178	0.703851389
0.60	0.80591233	0.8059870	0.805987500	0.805910921
0.70	0.88849440	-	0.888558530	0.888495366
0.75	0.92181061	0.9218660	-	0.921811113
0.80	0.94950550	-	0.94954223	0.949502875
0.90	0.98717143	0.9871760	0.98717592	0.987171428

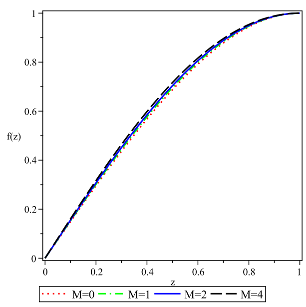

Fig. 2: Effect of the values M on $f(z)$

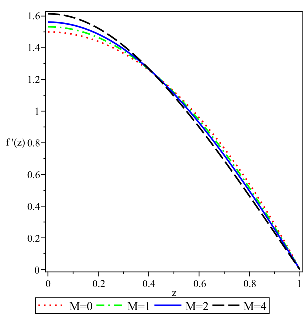

Fig. 3: Effect of the values M on$f'(z)$

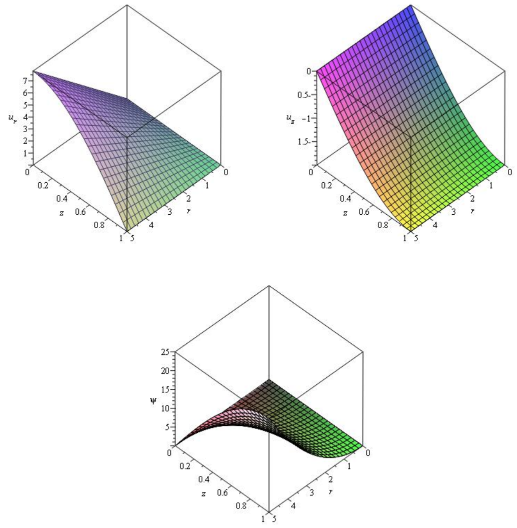

Fig. 4: Effect of the value $M = 2$ on analytical- approximate solution

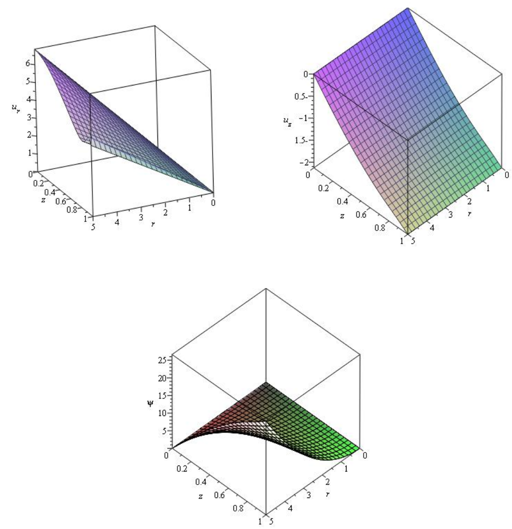

Fig. 5: Effect of the value $M = 10$ on analytical- approximate solution

7 Conclusions

In this paper, a new approach based on the coefficients of power series resulting from integrating differential equations with appropriate conditions is proposed, and it is applied to obtain a new analytical-approximate solution for non-linear squeezing flow between two infinite parallel plates successfully. It has been found that the construction of this approach possess good convergent series. According to invariability of A_2 and A_4 we

Table 2: convergence of analytical- approximate solution for $M = 0.2, 1, 2$

order of approximations	$M = 0.2$ A_2	$M = 0.2$ A_4	$M = 1$ A_2	$M = 1$ A_4	$M = 2$ A_2	$M = 2$ A_4
2terms	1.506995590	-3.086214870	1.540001	-3.49455	1.5983	-4.2224
3terms	1.506727353	-3.086214870	1.531626	-3.41318	1.5539	-3.7671
4terms	1.506733971	-3.083763688	1.532618	-3.42190	1.5633	-3.8436
5terms	1.506733871	-3.083763288	1.532547	-3.42130	1.5621	-3.8425
6terms	1.506733871	-3.083763288	1.532547	-3.42130	1.5620	-3.8433
7terms	1.506733871	-3.083763288	1.532547	-3.42130	1.5620	-3.8433
8terms	1.506733871	-3.083763288	1.532547	-3.42130	1.5620	-3.8433

Table 3: Comparison between the analytical- approximate and numerical solutions for M=0.2,1

	$M = 0.2$				$M = 1$			
z	$f(z)$	$RK-4$	$f'(z)$	$RK-4$	$f(z)$	$RK-4$	$f'(z)$	$RK-4$
0.1	0.150159504	0.150159427	1.49131892	1.49131893	0.152684911	0.152684483	1.51546221	1.51546235
0.2	0.297237561	0.297237409	1.44512031	1.44512036	0.301961559	0.301960717	1.46446740	1.46446775
0.3	0.438161926	0.438161708	1.36827531	1.36827551	0.444472898	0.444471734	1.38032261	1.38032349
0.4	0.569878488	0.569878217	1.26100861	1.26100883	0.576961050	0.576959689	1.26423941	1.26424109
0.5	0.689359672	0.689359368	1.12362495	1.12362521	0.696308764	0.696307366	1.11780478	1.11780730
0.6	0.793612070	0.793611756	0.95650112	0.95650164	0.799572356	0.799571121	0.94288653	0.94289054
0.7	0.879683091	0.879682795	0.76007475	0.76007547	0.884004796	0.884004031	0.74153261	0.74154006
0.8	0.944666434	0.944666190	0.53483192	0.53483291	0.947068648	0.947068978	0.51587102	0.51588691
0.9	0.985706262	0.985706117	0.28129336	0.28129548	0.947068648	0.947068978	0.26801778	0.26805529

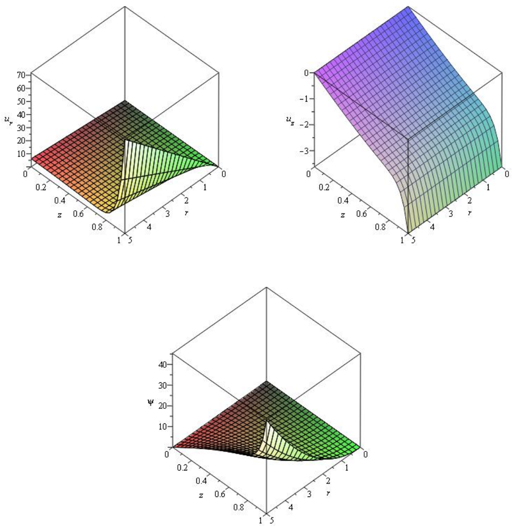

Fig. 6: Effect of the value $M = 15$ on analytical- approximate solution

conclude that the iterations that attain to steady state solution of $f(z)$ increase with an increase in the value of M. The results obtained by the application of proposed approach are well-founded with good accuracy and convergence. Its application is simple and it has superiority over the other methods in case of computation nonlinearity. Furthermore, the resulting solution is near to the numerical solution rapidly. Analysis of convergence confirm that the new approach is an efficient technique as compared to other methods introduced in this paper. As can be seen from the comparison, the present solution and other solutions are identical with 5th or 6th decimal places. Even, they reach to steady state with 5terms for $M = 0.2, 1$ and 6terms with $M = 2$, that means the reach ability to steady state is verified and related with values of M and the fixity of the series components A_2 and A_4. Finally, from analysis of results, we can conclude that the new approach is easy and straightforward to solve nonlinear problems, and give chance to employ it for handle unsteady state non-linear squeezing flow between two infinite parallel plates which present as complicate problems in future works.

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