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Conformable Decomposition for Analytical Solutions of a Time-Fractional One-Factor Markovian Model for Bond Pricing

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Abstract: In financial and option pricing setting, one-factor model denotes the notion that there exists one Wiener process in the definition of the short-rate process indicating one source of randomness. In this paper, approximate-analytical solution of a time-fractional one-factor Markovian model for bond pricing is considered using the approach of conformable decomposition. The method is a modified version of Adomian decomposition coupled with fractional derivative defined in conformable sense. Illustrative examples are presented in order to clarify the effectiveness of the proposed solution method, and the solutions are presented graphically based on some financial parameters at different values of the time-fractional order. This approach can be extended to multi-factor models formulated in terms of stochastic dynamics.

Keywords: Option pricing, Black Scholes model, Adomian decomposition, conformable fractional derivative, analytical solutions

1 Introduction

In any financial setting, interest rate implies a certain rate (amount) that is charged upon the use of financial item, mainly money [1]. In financial market, interest rates are non-tradable assets but are derived from the prices of tradable assets such as bonds and swaps. An Interest Rate Model (IRM) describes the evolution of the interest rates in relation to their dependency at maturity known as Term Structure of Interest Rates (TSIR) [2, 3]. Meanwhile, bond prices are popularly modeled using short-rate models [1, 4]. It is basically assumed that the short rate, ξ is governed by the general stochastic differential equation (SDE):

$$d\xi = \mu(\xi,t)\,dt + \sigma(\xi,t)\,dW \tag{1}$$

where $\mu(\xi,t)$, $\sigma(\xi,t)$, and $W = W(t)$ are trend process, random fluctuation determinant function, and Wiener process respectively. Equation (1) yields Markovian models if $\mu(\xi,t) = \mu(\xi)$ and $\sigma(\xi,t) = \sigma(\xi)$; meaning that μ and σ are strictly functions of variable ξ (not depending on t at all). We speak of one-factor models or multi-factor models if $\xi = X$ in (1) is a scalar or a vector respectively [5]. This work considers one-factor model (denoting the existence of one Wiener process-for one source of randomness).

The choice of the volatility function with respect to (w.r.t.) correctness has been a topic of concern. Chen, Karolyi, Longstaff, and Sanders proposed a general Short Rate Model (SRM) defined in terms of single stochastic dynamics of the form:

$$d\xi = (\beta_1 + \beta_2\xi)\,dt + \sigma\xi^{\phi}\,dW \tag{2}$$

known as CKLS model; where β_1, β_2, ϕ are constants, and $W = W(t)$ is a Wiener process [6]. In this regard, related references include those of [7-16]. By considering only corresponding Markov models, and taking $\tau = T - t$ (for convenient) to denote the remaining time to maturity, it is therefore claimed that the bond price, $P(\xi,\tau)$ solves the parabolic partial differential model evolving from (2) expressed in the form:

$$\begin{cases} \frac{\partial P}{\partial \tau} = \frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P}{\partial \xi} - \xi P \\ P(\xi,0) = h(\xi) \end{cases} \tag{3}$$

for $P(\xi,\tau) = P$, $\xi > 0$, $\tau \in (0,T]$.

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Solution of PDEs such as (3) can be considered by semi-analytical, numerical, and approximate methods [17-20]. Goard [21] considered bond-pricing model w.r.t. group invariant solutions using classical Lie method approach. In computing the zero-coupon bonds, Sinkala *et al.* [22] applied symmetry analysis for the Vasicek and Cox-Ingersoll-Roll (CIR) models. Pooe *et al.* [23] transformed the one-factor bond pricing model to one-dimensional heat equation in order to obtain fundamental solutions of zero-coupon bond models. Recently, Khalique and Motsepa [24] analysed the one-factor term structure model; thereafter, they constructed new group invariant solutions to the corresponding equation.
In this paper, the extension of (3) to time-fractional order is considered. This takes the form:

$$\begin{cases} \frac{\partial^\alpha P}{\partial \tau^\alpha} = \frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P}{\partial \xi^2} + (\beta_1+\beta_2\xi)\frac{\partial P}{\partial \xi} - \xi P, \\ P(\xi,0) = h(\xi),\ \alpha\in(0,1]. \end{cases} \tag{4}$$

The fractional derivative in (4) is defined in conformable sense. For related researches on fractional, integer, financial models and solution methods, we make reference to [25-35].
In terms of organization in the remaining parts of the paper, we have in section 2 a brief note on the basic notions of conformable differential operators, section 3 is on the proposed solution method (CADM). In section 4, the proposed method is applied. Thereafter, concluding remark is presented in section 5.

2 Basic Notions of Conformable Differential Operators

For a function $\psi : [0,\infty) \to \mathrm{R}$, the conformable derivative of ψ of order $\alpha \in [0,\infty)$ is defined as:

$$C_\alpha(\psi)(t) = \lim_{\varepsilon\to 0}\left(\frac{\psi\left(t+\varepsilon t^{1-\alpha}\right)-\psi(t)}{\varepsilon}\right),\ \forall t\geq 0. \tag{5}$$

Reference is made to [25-28, 31] for details regarding conformable differential operators.

2.1 Properties of Conformable Differential Operators CDOs

Suppose $\psi=\psi(t)$, $\psi_1=\psi_1(t)$ and $\psi_2=\psi_2(t)$ are $\alpha-$ differentiable functions at $t>0$, then the following holds:

(C1):$C_\alpha(\lambda_1\psi_1 \pm \lambda_2\psi_2) = \begin{Bmatrix} \lambda_1 C_\alpha(\psi_1) \\ \pm\lambda_2 C_\alpha(\psi_2),\ \lambda_1,\lambda_2\in\mathrm{R} \end{Bmatrix}$.

(C2):$C_\alpha(\psi=\lambda)=0,\ \lambda\in\mathrm{R}$.

(C3):$C_\alpha(\psi_1\psi_2)=\psi_1 C_\alpha(\psi_2)+\psi_2 C_\alpha(\psi_1)$.

(C4):$C_\alpha\left(\frac{\psi_1}{\psi_2}\right) = \frac{\psi_2 C_\alpha(\psi_1)-\psi_1 C_\alpha(\psi_2)}{\psi_2^2}$.

(C5):$C_\alpha(t^\gamma)=\gamma t^{\gamma-\alpha},\ \forall\gamma\in\mathrm{R}$.

(C6):$C_\alpha(\psi(t)) = t^{\gamma-1}\psi'(t),\ \psi'(t)=\frac{d\psi(\cdot)}{dt}$.

(C7):Suppose further that $\psi(t)$ is an n-times differentiable function at t, then:

$$C_\alpha(\psi(t)) = \begin{cases} t^{[\alpha]-\alpha}\psi^{([\alpha])}(t) \\ \alpha\in(n,n+1], \end{cases}$$

where $[\alpha]$ denotes the smallest integer such that $[\alpha]\geq\alpha$.

3 The Conformable Sense of the Decomposition Method

Let us consider a general nonlinear fractional partial differential equation (NLFDE) of the form [25-28]:

$$L_\alpha(\psi(x,t))+R(\psi(x,t))+N(\psi(x,t))=\eta(x,t) \tag{6}$$

where $L_\alpha(\cdot)$ represents a linear operator based on conformable derivative of order α, with respect to t, such that $\alpha\in(n,n+1]$, R is the remaining part of the linear conformable differential operator, N denotes the nonlinear operator, while $\eta(x,t)$ is the associated non-homogeneous part (source term).
Suppose $L_\alpha(\cdot)=C_\alpha(\cdot)$ is invertible such that $L_\alpha^{-1}(\cdot)$ exists, then (6) becomes:

$$C_\alpha(\psi(x,t))+R(\psi(x,t))+N(\psi(x,t))=\eta(x,t). \tag{7}$$

Hence, by the differential property of the conformable derivative (C7), we have:

$$t^{[\alpha]-\alpha}\psi^{([\alpha])}(t)+R(\psi(x,t))+N(\psi(x,t))=\eta(x,t). \tag{8}$$

$$\therefore\ t^{[\alpha]-\alpha}\frac{\partial^{[\alpha]}\psi(x,t)}{\partial t^\alpha} = \eta(x,t)-\begin{pmatrix} R(\psi(x,t)) \\ +N(\psi(x,t)) \end{pmatrix}. \tag{9}$$

The inverse operator is defined as follows:

$$L_\alpha^{-1}(\cdot) = \int_o^t\int_o^{\tau_1}\cdots\int_o^{\tau_{n-1}}\frac{1}{\tau^{[\alpha]-\alpha}}(\cdot)\,d\tau_n d\tau_{n-1}\cdots d\tau_1. \tag{10}$$

So, applying (10) to both sides of (9) gives:

$$L_\alpha^{-1}\left(t^{[\alpha]-\alpha}\frac{\partial^{[\alpha]}\psi(x,t)}{\partial t^\alpha}\right) = L_\alpha^{-1}\begin{pmatrix} \eta(x,t) \\ -\begin{pmatrix} R(\psi(x,t)) \\ +N(\psi(x,t)) \end{pmatrix} \end{pmatrix}.$$

$$\Rightarrow \psi(x,t) = \begin{Bmatrix} \psi(x,0)+L_\alpha^{-1}(\eta(x,t)) \\ -L_\alpha^{-1}(R(\psi(x,t))+N(\psi(x,t))) \end{Bmatrix}. \tag{11}$$

For the decomposition of the solution, we write:

$$\psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t) \tag{12}$$

while the nonlinear term with the Adomian polynomials A_n is defined as:

$$N(\psi(x,t)) = \sum_{n=0}^{\infty} A_n. \tag{13}$$

Note that A_n (the Adomian polynomials) is given as:

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \varsigma^n}\left[N\left(\sum_{i=0}^{n}\varsigma^i h_i\right)\right]_{\varsigma=0}. \tag{14}$$

Thus, using (12-14) in (11) gives:

$$\sum_{n=0}^{\infty}\psi_n(x,t) = \left\{\begin{array}{l} \psi(x,0) + L_t^{-1}\{\eta(x,t)\} \\ -L_t^{-1}\left\{\left(\begin{array}{l} R\left(\sum\limits_{n=0}^{\infty}\psi_n(x,t)\right) \\ +N\left(\sum\limits_{n=0}^{\infty}A_n\right)\end{array}\right)\right\}\end{array}\right.. \tag{15}$$

Hence, in recursive relation, we have:

$$\begin{cases} \psi_0 = \psi(x,0) + L_t^{-1}\{\eta(x,t)\} \\ \psi_{n+1} = -L_t^{-1}\{(R(\psi_n) + A_n)\}, n \geq 0. \end{cases} \tag{16}$$

The solution, $\psi(x,t)$ is therefore confirmed as:

$$\psi(x,t) = \lim_{n\to\infty}\sum_{n=0}^{\infty}\psi_n \tag{17}$$

4 Applications and Illustrative Examples

Here, the CADM as proposed above is applied to a Time-Fractional One-Factor Markovian Model (TF1FMM) for bond pricing as follows [3]:

$$\begin{cases} \frac{\partial^\alpha P}{\partial \tau^\alpha} = \frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P}{\partial \xi} - \xi P, \\ P(\xi,0) = 1,\ \alpha \in (0,1]. \end{cases} \tag{18}$$

Procedure: Let $D_t^\alpha P(\cdot) = C_\alpha P(\cdot)$ be applied to (18). Thus,

$$C_\alpha P(\xi,\tau) = \frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P}{\partial \xi} - \xi P. \tag{19}$$

By the property: (C6), we have:

$$t^{1-\alpha}\frac{\partial P(\xi,\tau)}{\partial t} = \frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P}{\partial \xi} - \xi P. \tag{20}$$

So, operating $L_\alpha^{-1}(\cdot) = \int\limits_0^t \frac{1}{\tau^{1-\alpha}}(\cdot)\,d\tau$ on both sides of (20) gives:

$$P(\xi,\tau) = P(\xi,0) + L_\alpha^{-1}\left(\begin{array}{l} \frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P}{\partial \xi^2} \\ +(\beta_1 + \beta_2\xi)\frac{\partial P}{\partial \xi} - \xi P\end{array}\right). \tag{21}$$

By decomposing $P(x,t)$, we have:

$$\sum_{n=0}^{\infty}P_n(\xi,\tau) = \left\{\begin{array}{l} P(\xi,0) \\ +L_\alpha^{-1}\left\{\begin{array}{l} \frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2}{\partial \xi^2}\left(\sum\limits_{n=0}^{\infty}P_n\right) - \xi\sum\limits_{n=0}^{\infty}P_n \\ +(\beta_1 + \beta_2\xi)\frac{\partial}{\partial \xi}\left(\sum\limits_{n=0}^{\infty}P_n\right)\end{array}\right\}\end{array}\right.. \tag{22}$$

Thus,

$$\begin{cases} P_0 = P(\xi,0) \\ P_{n+1} = L_\alpha^{-1}\left\{\begin{array}{l} \frac{1}{2}\ sigma^2\xi^{2\phi}\frac{\partial^2}{\partial \xi^2}P_n - \xi P_n \\ +(\beta_1 + \beta_2\xi)\frac{\partial}{\partial \xi}P_n\end{array}\right\}, n \geq 0. \end{cases} \tag{23}$$

As a result, the recursive relation in (23) yields:

$$\begin{cases} P_0 = P(\xi,0) \\ P_1 = L_\alpha^{-1}\left\{\frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P_0}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P_0}{\partial \xi} - \xi P_0\right\} \\ P_2 = L_\alpha^{-1}\left\{\frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P_1}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P_1}{\partial \xi} - \xi P_1\right\} \\ P_3 = L_\alpha^{-1}\left\{\frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P_2}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P_2}{\partial \xi} - \xi P_2\right\} \\ P_4 = L_\alpha^{-1}\left\{\frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P_3}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P_3}{\partial \xi} - \xi P_3\right\} \\ \vdots \\ P_k = L_\alpha^{-1}\left\{\begin{array}{l} \frac{1}{2}\sigma^2\xi^{2\phi}\frac{\partial^2 P_{k-1}}{\partial \xi^2} + (\beta_1 + \beta_2\xi)\frac{\partial P_{k-1}}{\partial \xi} \\ -\xi P_{k-1}\end{array}\right\}, k \in \mathrm{N}. \end{cases} \tag{24}$$

Whence, for: $P(\xi,0) = 1$, the following are obtained:

$$\begin{cases} P_0 = 1, \\ P_1 = -\frac{\tau^\alpha\xi}{\alpha}, \\ P_2 = -\left(\beta_2\xi - \xi^2 + \beta_1\right)\frac{\tau^{2\alpha}}{2!\alpha^2}, \end{cases}$$

$$P_3 = -\left(\beta_2^2\xi - 3\beta_2\xi^2 - \xi\sigma^2 + \xi^3 + \beta_1\beta_2 - 3\beta_1\xi\right)\frac{\tau^{3\alpha}}{3!\alpha^3},$$

$$P_4 = \left(\begin{array}{l} -\beta_2^3\xi + 7\beta_2^2\xi^2 + 4\xi\sigma^2\beta_2 \\ -6\beta_2\xi^3 - 4\xi^2\sigma^2 + \xi^4 - \beta_1\beta_2^2 \\ +10\beta_1\beta_2\xi + \beta_1\sigma^2 - 6\beta_1\xi^2 + 3\beta_1^2\end{array}\right)\frac{\tau^{4\alpha}}{4!\alpha^4},$$

$$P_5 = \left(\begin{array}{l} -\beta_2^4\xi + 15\beta_2^3\xi^2 + 11\beta_2^2\sigma^2\xi \\ -25\beta_2^2\xi^3 - 30\beta_2\sigma^2\xi^2 + 10\beta_2\xi^4 \\ -4\sigma^4\xi + 10\sigma^2\xi^3 - \xi^5 - \beta_1\beta_2^3 \\ +25\beta_1\beta_2^2\xi + 4\beta_1\beta_2\sigma^2 \\ -40\beta_1\beta_2\xi^2 - 15\beta_1\sigma^2\xi + 10\beta_1\xi^3 \\ +10\beta_1^2\beta_2 - 15\beta_1^2\xi\end{array}\right)\frac{\tau^{5\alpha}}{5!\alpha^5},$$

$$P_6 = -\left(\begin{array}{l} \beta_2^5\xi - 31\beta_2^4\xi^2 - 26\beta_2^3\sigma^2\xi \\ +90\beta_2^3\xi^3 + 146\beta_2^2\sigma^2\xi^2 \\ -65\beta_2^2\xi^4 + 34\beta_2\sigma^4\xi - 120\beta_2\sigma^2\xi^3 \\ +15\beta_2\xi^5 - 34\sigma^4\xi^2 + 20\sigma^2\xi^4 - \xi^6 \\ +\beta_1\beta_2^4 - 56\beta_1\beta_2^3\xi - 11\beta_1\beta_2^2\sigma^2 \\ +180\beta_1\beta_2^2\xi^2 + 119\beta_1\beta_2\sigma^2\xi \\ -110\beta_1\beta_2\xi^3 + 4\beta_1\sigma^4 - 75\beta_1\sigma^2\xi^2 \\ +15\beta_1\xi^4 - 25\beta_1^2\beta_2^2 + 105\beta_1^2\beta_2\xi \\ +15\beta_1^2\sigma^2 - 45\beta_1^2\xi^2 + 15\beta_1^3\end{array}\right)\frac{\tau^{6\alpha}}{6!\alpha^6},$$

$$P_7 = -\left(\begin{array}{l} \beta_2^6\xi - 63\beta_2^5\xi^2 - 57\beta_2^4\sigma^2\xi + 301\beta_2^4\xi^3 \\ +588\beta_2^3\sigma^2\xi^2 - 350\beta_2^3\xi^4 + 180\beta_2^2\sigma^4\xi \\ -896\beta_2^2\sigma^2\xi^3 + 140\beta_2^2\xi^5 - 462\beta_2\sigma^4\xi^2 \\ +350\beta_2\sigma^2\xi^4 - 21\beta_2\xi^6 - 34\sigma^6\xi \\ +154\sigma^4\xi^3 - 35\sigma^2\xi^5 + \xi^7 + \beta_1\beta_2^5 \\ -119\beta_1\beta_2^4\xi - 26\beta_1\beta_2^3\sigma^2 + 686\beta_1\beta_2^3\xi^2 \\ +602\beta_1\beta_2^2\sigma^2\xi - 770\beta_1\beta_2^2\xi^3 \\ +34\beta_1\beta_2\sigma^4 - 959\beta_1\beta_2\sigma^2\xi^2 + 245\beta_1\beta_2\xi^4 \\ -147\beta_1\sigma^4\xi + 245\beta_1\sigma^2\xi^3 - 21\beta_1\xi^5 \\ -56\beta_1^2\beta_2^3 + 490\beta_1^2\beta_2^2\xi + 119\beta_1^2\beta_2\sigma^2 \\ -525\beta_1^2\beta_2\xi^2 - 210\beta_1^2\sigma^2\xi + 105\beta_1^2\xi^3 \\ +105\beta_1^3\beta_2 - 105\beta_1^3\xi \end{array}\right)\frac{\tau^{7\alpha}}{7!\alpha^7},$$

$$P_8 = \left(\begin{array}{l} -\beta_2^7\xi + 127\beta_2^6\xi^2 + 120\beta_2^5\sigma^2\xi - 966\beta_2^5\xi^3 \\ -2136\beta_2^4\sigma^2\xi^2 + 1701\beta_2^4\xi^4 - 768\beta_2^3\sigma^4\xi \\ +5376\beta_2^3\sigma^2\xi^3 - 1050\beta_2^3\xi^5 + 3792\beta_2^2\sigma^4\xi^2 \\ -3696\beta_2^2\sigma^2\xi^4 + 266\beta_2^2\xi^6 + 496\beta_2\sigma^6\xi \\ -3024\beta_2\sigma^4\xi^3 + 840\beta_2\sigma^2\xi^5 - 28\beta_2\xi^7 \\ -496\sigma^6\xi^2 + 504\sigma^4\xi^4 - 56\sigma^2\xi^6 + \xi^8 - \beta_1\beta_2^6 \\ +246\beta_1\beta_2^5\xi + 57\beta_1\beta_2^4\sigma^2 - 2394\beta_1\beta_2^4\xi^2 \\ -2490\beta_1\beta_2^3\sigma^2\xi + 4396\beta_1\beta_2^3\xi^3 - 180\beta_1\beta_2^2\sigma^4 \\ +7518\beta_1\beta_2^2\sigma^2\xi^2 - 2450\beta_1\beta_2^2\xi^4 \\ +2064\beta_1\beta_2\sigma^4\xi - 4564\beta_1\beta_2\sigma^2\xi^3 + 476\beta_1\beta_2\xi^5 \\ +34\beta_1\sigma^6 - 1344\beta_1\sigma^4\xi^2 + 630\beta_1\sigma^2\xi^4 - 28\beta_1\xi^6 \\ +119\beta_1^2\beta_2^4 - 1918\beta_1^2\beta_2^3\xi - 602\beta_1^2\beta_2^2\sigma^2 \\ +3850\beta_1^2\beta_2^2\xi^2 + 2772\beta_1^2\beta_2\sigma^2\xi - 1820\beta_1^2\beta_2\xi^3 \\ +147\beta_1^2\sigma^4 - 1260\beta_1^2\sigma^2\xi^2 + 210\beta_1^2\xi^4 \\ -490\beta_1^3\beta_2^2 + 1260\beta_1^3\beta_2\xi + 210\beta_1^3\sigma^2 \\ -420\beta_1^3\xi^2 + 105\beta_1^4 \end{array}\right)\kappa,$$

$$\vdots$$

where $\kappa = \frac{\tau^{8\alpha}}{8!\alpha^8}$.
Therefore,

$$P(\xi,\tau) = \lim_{k\to\infty} P_k = \sum_{n=0}^{\infty} P_k\,. \tag{25}$$

Equation (25) is an approximate-analytical solution of (18) corresponding to the time-fractional one-factor Markovian model for bond pricing. We consider the financial data according to [1] w.r.t CIR model with $\sigma = 0.0894$, $\beta_1 = 0.00315$, and $\beta_2 = -0.0555$. The plots of the resulting solutions for $\alpha = 1$, $\alpha = 0.5$, and $\alpha = 0.75$ are displayed in Fig.1, Fig. 2, and Fig. 3 respectively.

4.1 Numerical Solutions

This subsection deals significantly with the approximate-analytical solutions obtained via the proposed method (CADM). In the classical setting when

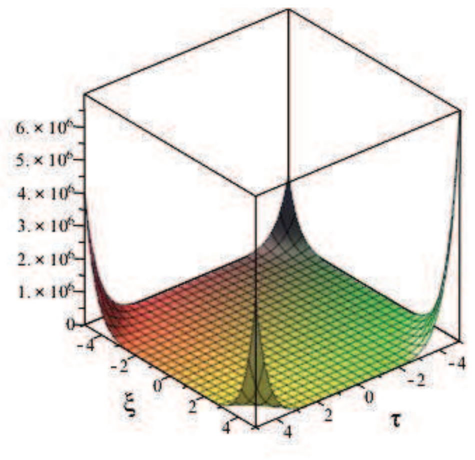

Fig. 1: Approximate analytical solution graphic for $\alpha = 1$

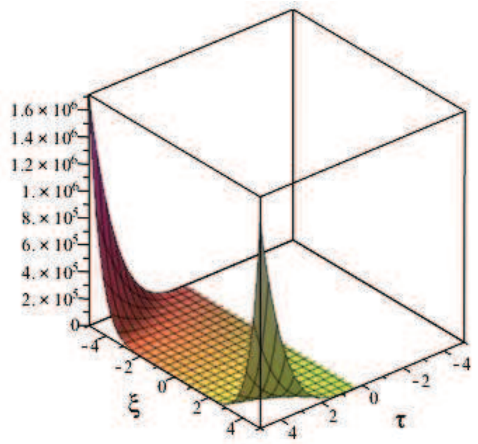

Fig. 2: Approximate analytical solution graphic for $\alpha = 0.5$

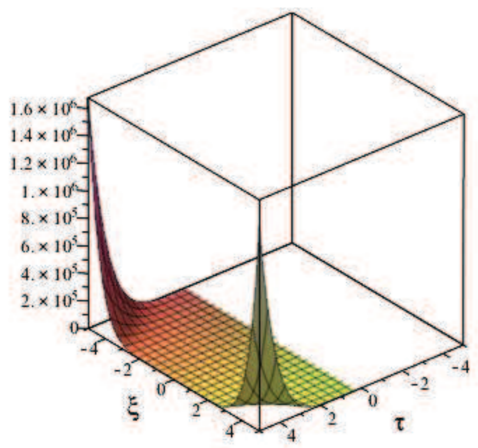

Fig. 3: Approximate analytical solution graphic for $\alpha = 0.75$

the parameter value, $\alpha \to 1$, Fig. 1 describes the geometrical behaviour of the approximate solution in line with the market (financial) data as contained in [1]. In a similar way, Fig. 2 and Fig.3 describe the geometrical behaviour of the approximate solution for the time-orders, $\alpha \to 0.5$ and $\alpha \to 0.75$ respectively. These time fractional cases $(\alpha \to 0.5$ and $\alpha \to 0.75)$ are compared with the integer case $(\alpha \to 1)$ making the result in [1] a particular case of this present work.

5 Prospectives

In this paper, approximate-analytical solution is proposed for a Time-Fractional One-Factor Markovian Model (TF1FMM) for bond pricing whose derivative is defined in conformable sense. Preference is given to the CIR model version of TF1FMM with the market parameters according to [1] in order to clarify the effectiveness of the proposed solution method. Thereafter, the solutions at different values of time-fractional order are graphically displayed. This proposed method of solution can be extended to multi-factor models formulated in terms of stochastic dynamics. In addition, the solution obtained, and the proposed method can effectively serve as benchmarks for further researches in related areas via other semi-analytical methods.

Conflict of Interests

The authors declare no conflict of interest regarding this paper.

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