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A Nonstandard Finite Difference Scheme for Water-Related Disease Mathematical Model

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Abstract: In this paper, a Nonstandard Finite Difference Scheme (NSFDS) is constructed for a water-related disease mathematical model. The properties of the resulting discrete models are analysed and compared with its corresponding deterministic model. Furthermore, we compare the numerical solutions of NSDFS, Euler method and MATLAB's ode45. It is shown that the resulting discrete model preserves essential properties of the continous model such as positivity and stability. The results are confirmed numerically. Furthermore, numerical simulations using NSFDS, Euler method and MATLAB's ode45 give similar results.

Keywords: Nonstandard finite difference scheme, mathematical model, numerical simulations.

1 Introduction

A mathematical model has been widely used to understand the complex phenomena such as population dynamics [1–3], disease transmission [4–11] and others [12–14] . A mathematical model is then solved and analysed to understand the dynamics of the studied phenomena. However, the exact solutions of the model cannot be easily derived and hence a numerical approach is used.

A number of numerical methods has been developed and largely used to solve mathematical model. However, the available methods such as Runge-Kutta and Euler sometimes fail to generate the main properties of the model such as stability, oscillation, and positivity [15–19]. This can lead to the incorrect interpretation of studied phenomena. The Nonstandard Finite Difference Scheme (NSFDS) is the nonstandard numerical scheme that can be used to simulate the solutions of mathematical model. The NSFDS has been used in simulations of biological phenomena [17, 18, 20, 21]. A review of the approach is given in [22]. It is generally found that the NSFDS overcomes the weaknesses of the traditional numerical schemes such as Runge-Kutta and Euler methods. The NSFDS is different to the standard numerical scheme where it depends on the two main rules. First, the denominator function should be replaced by $0 < \phi(h) < 1$, where $\phi(h) = h + O(h^2)$ [15, 16, 23]. Second, the nonlinear terms are approximated in a nonlocal way [16, 24, 25]. For example, the term x^2 can be approximated using $x_n x_{n+1}$. This scheme is relatively more consistent than the other traditional methods [16, 26, 27]. In this paper, we formulate a nonstandard numerical scheme for the mathematical model of the effects of the hard water consumption on kidney function.

In developing world, water quality remains the main problem in particular water with higher concentration of calcium and magnesium salts which is known as hard water. Long-term consumption of hard water can cause kidney dysfunction which may lead to various diseases such as cerebrovascular disease, diabetes and many others [28, 29]. A deterministic mathematical model for analysing the effect of hard water consumption on kidney function has been formulated by Tambaru *et al.* [30]. The model is based on the standard SIR model where the human population is divided into susceptible (S), infected (I) and recovered (R) classes and including the water compartment (W) which accounts for the level of calcium and magnesium ions in the water. The model can be extended to include other kidney-related diseases by

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adding other compartments. The model can be extended to examine the effects of water transport in the kidney on human health. However, in this paper, we focus on developing a nonstandard finite difference scheme for a mathematical model developed by Tambaru *et al.* [30]. Furthermore, although this is a simple model, the analytical solutions cannot be determined and hence a numerical approach is needed. In this paper, we present a Nonstandard Finite Difference Scheme (NSFDS) for the deterministic model proposed by Tambaru *et al.* which results in a discrete model. The properties of the discrete model are then analysed. We then compare the NSFDS, Euler method, and MATLAB's ode45 [31]. To the best of our knowledge, there is no such numerical scheme that is constructed to simulate the model of the effects of hard water consumption on kidney function.

The remainder of the paper is organised as follows. Section 2 overviews the mathematical model proposed by Tambaru *et al.* and its basic properties. Section 3 presents the nonstandard finite difference scheme for the model. Section 4 presents an analysis of the scheme properties. Finally, conclusion is presented.

2 Mathematical model

In this section, we recall the deterministic mathematical model proposed by Tambaru *et al.* [30] and several basic properties of the model.The model is based on the standard SIR model but adding water compartment to account for the level of calcium and magnesium ions in the water. The model is developed for analysing the effect of hardwater consumption on kidney function. The model comprises human and water compartments. The human population is divided into susceptible (S), infected (I), recovered (R), where the total population is $N = S+I+R$. Only one compartment is for water. Furthermore, we assume a constant population. A susceptible human becomes infected after consuming hard water with a rate $\beta\lambda(W)$. They recover at a rate γ. The human dies at a rate μ. The concentration of calcium and magnesium ions in the water increases at a rate b and their growth is limited by carrying capacity K. The concentration reduces at a rate c. The model is then governed by the following system of differential equations.

$$\begin{aligned}
\frac{dS}{dt} &= A-\beta\lambda(W)S-\mu S\\
\frac{dI}{dt} &= \beta\lambda(W)S-\gamma I-\mu I,\\
\frac{dR}{dt} &= \gamma I-\mu R,\\
\frac{dW}{dt} &= bW\left(1-\frac{W}{K}\right)-cW,
\end{aligned} \tag{1}$$

where the parameter β is the rate of ingesting calcium and magnesium from water and $\lambda(W)$ is the probability that consumptions cause kidney dysfunction. Hence, the value of $\lambda(W)$ ranges from 0 to 1. The probability that individuals have attracted kidney dysfunction is influenced by the concentration of calcium and magnesium ions in the water. Therefore, we set the equations for $\lambda(W)$ that is dependent on the concentration of calcium and magnesium ions, which is

$$\lambda(W) = \frac{W}{K+W}. \tag{2}$$

where K is the maximum concentration of calcium and magnesium ions representing the maximum solubility of each compound in the water [32]. This implies that $W > K$ does not happen.

Theorem 1.*Model* (1) *has two steady states: kidney dysfunction-free (E_1) and endemic steady states (E_2).*

$$E_1 = (A/\mu,0,0,0) \quad \textit{and} \quad E_2 = (S^*,I^*,R^*,W^*)$$

with

$$\begin{aligned}
S^* &= \frac{A(2b-c)}{b(\beta+2\mu)-c(\beta+\mu)},\\
I^* &= \frac{A\beta(b-c)}{\mu(\gamma+\mu)(b(\beta+2\mu)-c(\beta+\mu))},\\
R^* &= \frac{A\beta\gamma(b-c)}{\mu(\gamma+\mu)(b(\beta+2\mu)-c(\beta+\mu))},\\
W^* &= \frac{K(b-c)}{b},
\end{aligned} \tag{3}$$

which is physically realistic when $b > c$.

*Proof.*The proof is done by setting the right-hand side of Equations (1) to zero, doing algebraic manipulation and rearranging it.

Theorem 2.*The kidney dysfunction-free steady state E_1 is locally stable if $b < c$ and the endemic steady state E_2 is locally stable if $b > c$.*

*Proof.*We prove the local stability of E_1. First, we construct the Jacobian matrix of Model (1), and then find the characteristic equation. We then use the Routh-Hurwitz criteria to determine its stability. Details of the proof can be found in Tambaru *et al.* [30].

3 Nonstandard finite difference scheme

In this section, we present a NSFDS for Model (1). We discretise the time variable to $t_n = nh$ for $n = 0,1,2,...$ and a constant h, where $h > 0$. As stated in the introduction, there are two main rules in construction the NSFDS (see [16, 33]): (i) modification of the denominator function, (ii) discretise linear and nonlinear term in the right-hand side in a non-local way.

Applying the nonstandard finite difference scheme, we obtain the following discrete model for Model (1).

$$\begin{aligned}
\frac{S_{n+1}-S_n}{\phi(h)} &= A-\frac{\beta W_n S_{n+1}}{K+W_n}-\mu S_{n+1},\\
\frac{I_{n+1}-I_n}{\phi(h)} &= \frac{\beta W_n S_{n+1}}{K+W_n}-\gamma I_{n+1}-\mu I_{n+1},\\
\frac{R_{n+1}-R_n}{\phi(h)} &= \gamma I_{n+1}-\mu R_{n+1},\\
\frac{W_{n+1}-W_n}{\phi(h)} &= bW_n-\frac{bW_nW_{n+1}}{K}-cW_{n+1}.
\end{aligned} \tag{4}$$

Rearranging Equation (4), we obtain

$$\begin{aligned}
S_{n+1} &= \frac{S_n+\phi(h)A}{1+\phi(h)\left(\frac{\beta W_n}{K+W_n}+\mu\right)},\\
I_{n+1} &= \frac{I_n+\phi(h)\left(\beta W_n S_{n+1}/(K+W_n)\right)}{1+\phi(h)(\gamma+\mu)},\\
R_{n+1} &= \frac{R_n+\phi(h)\gamma I_{n+1}}{1+\phi(h)\mu},\\
W_{n+1} &= \frac{W_n(1+\phi(h)b)}{1+\phi(h)(bW_n/K+c)}.
\end{aligned} \tag{5}$$

The Equation (5) should be computed in sequence because the value of S_{n+1} is used for calculating the value of I_{n+1}, which is then used to calculate the value of R_{n+1} and then W_{n+1}. This process continues until the end time of interest. We choose denominator function as

$$\phi(h)=\frac{\exp\mu h-1}{\mu}. \tag{6}$$

4 Properties of the scheme

4.1 Non-negativity of the solution

The model focuses on the human population and concentration of magnesium and calcium in the water. Therefore, it should be guaranteed that the proposed numerical scheme cannot produce negative values. It can be seen that the proposed numerical scheme produces non-negative values. For the time-step $h>0$, and $S_n, I_n, R_n, W_n > 0$, numerators and denominators are positive and hence $S_{n+1}, I_{n+1}, R_{n+1}, W_{n+1} > 0$.

4.2 The equilibrium points of the numerical scheme

We determine the equilibrium points of the numerical scheme by setting $S_{n+1}=S_n$, $I_{n+1}=I_n$, $R_{n+1}=R_n$, $W_{n+1}=W_n$.

$$\begin{aligned}
S_n &= \frac{S_n+\phi(h)A}{1+\phi(h)\left(\frac{\beta W_n}{K+W_n}+\mu\right)},\\
I_n &= \frac{I_n+\phi(h)\left(\beta W_N S_{n+1}/(K+W_n)\right)}{1+\phi(h)(\gamma+\mu)},\\
R_n &= \frac{R_n+\phi(h)\gamma I_{n+1}}{1+\phi(h)\mu},\\
W_n &= \frac{W_n(1+\phi(h)b)}{1+\phi(h)(bW_n/K+c)}
\end{aligned} \tag{7}$$

Doing algebraic manipulation, we obtain two equilibrium points which are

$$P_1=(A/\mu,0,0,0) \quad \text{and} \quad P_2=(S_n^*,I_n^*,R_n^*,W_n^*)$$

with

$$\begin{aligned}
S_n^* &= \frac{A(2b-c)}{b(\beta+2\mu)-c(\beta+\mu)},\\
I_n^* &= \frac{A\beta(b-c)}{\mu(\gamma+\mu)(b(\beta+2\mu)-c(\beta+\mu))},\\
R_n^* &= \frac{A\beta\gamma(b-c)}{\mu(\gamma+\mu)(b(\beta+2\mu)-c(\beta+\mu))}\\
W_n^* &= \frac{K(b-c)}{b}
\end{aligned} \tag{8}$$

where P_1 and P_2 are disease-free and endemic equilibriums respectively. Note that the equilibrium points of the discrete model are the same as that of continous model (Model (1)) and independent of $\phi(h)$. In the next section, we conducted stability analysis for the equilibrium points.

4.3 Stability of the Equilibrium

We have used the concept of Jacobian matrix to analyse the stability of the fixed points. For the sake of simplicity, we define the following function.

$$\begin{aligned}
F_1(S,I,R,W) &= \frac{S_n+\phi(h)A}{1+\phi(h)\left(\frac{\beta W_n}{K+W_n}+\mu\right)},\\
F_2(S,I,R,W) &= \frac{I_n+\phi(h)\left(\beta W_N F_1(S,I,R,W)/(K+W_n)\right)}{1+\phi(h)(\gamma+\mu)},\\
F_3(S,I,R,W) &= \frac{R_n+\phi(h)\gamma F_2(S,I,R,W)}{1+\phi(h)\mu},\\
F_4(S,I,R,W) &= \frac{W_n(1+\phi(h)b)}{1+\phi(h)(bW_n/K+c)}.
\end{aligned} \tag{9}$$

We construct the Jacobian matrix

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix} \tag{10}$$

where

$$J_{11} = \frac{\partial F_1(S,I,R,W)}{\partial S} = \frac{1}{1+\phi(h)\left(\beta W_n/(K+W_n)+\mu\right)},$$
$$J_{12} = \frac{\partial F_1(S,I,R,W)}{\partial I} = 0, J_{13} = \frac{\partial F_1(S,I,R,W)}{\partial R} = 0,$$
$$J_{14} = \frac{\partial F_1(S,I,R,W)}{\partial W} = -\frac{O_1}{O_2}$$

with

$$O_1 = (A\phi(h)+S_n)\phi(h)\left(\beta/(K+W_n) - \beta W_n/(K+W_n)^2\right)$$
$$O_2 = (1+\phi(h)(\beta W_n/(K+W_n)+\mu))^2,$$

$$J_{21} = \frac{\partial F_2(S,I,R,W)}{\partial S} = \frac{\phi(h)\beta W_n}{O_3},$$

with

$$O_3 = (1+\phi(h)(\beta W_n/(K+W_n)+\mu))\times (K+W_n)(1+\phi(h)(\gamma+\mu)),$$
$$J_{22} = \frac{\partial F_2(S,I,R,W)}{\partial I} = \frac{1}{(1+\phi(h)(\gamma+\mu))},$$
$$J_{23} = \frac{\partial F_2(S,I,R,W)}{\partial R} = 0,$$
$$J_{24} = \frac{\partial F_2(S,I,R,W)}{\partial W} = \frac{P_1-P_2-P_3}{1+\phi(h)(\gamma+\mu)},$$

with

$$P_1 = \frac{\phi(h)\beta(\phi(h)A+S_n)}{(1+\phi(h)(\beta W_n/(K+W_n)+\mu))(K+W_n)},$$
$$P_2 = \frac{\phi(h)^2\beta W_n(A\phi(h)+S_n)\left(\frac{\beta}{(K+W_n)} - \frac{\beta W_n}{(K+W_n)^2}\right)}{(1+\phi(h)(\beta W_n/(K+W_n)+\mu))^2(K+W_n)},$$
$$P_3 = \frac{\phi(h)\beta W_n(A\phi(h)+S_n)}{(1+\phi(h)(\beta W_n/(K+W_n)+\mu))(K+W_n)^2},$$

$$J_{31} = \frac{\partial F_3(S,I,R,W)}{\partial S} = \frac{\phi(h)^2\beta W_n\gamma}{Z},$$

with

$$Z = (1+\phi(h)(\beta W_n/(K+W_n)+\mu))(K+W_n)\times (1+\phi(h)(\gamma+\mu))(\phi(h)\mu+1),$$
$$J_{32} = \frac{\partial F_3(S,I,R,W)}{\partial I} = \frac{\phi(h)\gamma}{((1+\phi(h)(\gamma+\mu))(\phi(h)\mu+1))},$$
$$J_{33} = \frac{\partial F_3(S,I,R,W)}{\partial R} = \frac{1}{(\phi(h)\mu+1)},$$

$$J_{34} = \frac{\partial F_3(S,I,R,W)}{\partial W} = \frac{\gamma\phi(h)\left(\frac{Q_1}{(R_1R_2)} - \frac{Q_2}{(R_1^2R_2)} - \frac{Q_3}{(R_1R_2^2)}\right)}{(1+\phi(h)(\gamma+\mu))(1+\phi(h)\mu)},$$

with

$$Q_1 = \phi(h)\beta(A\phi(h)+S_n),$$
$$Q_2 = \phi(h)^2\beta W_n(A\phi(h)+S_n)\left(\frac{\beta}{K+\mu} - \frac{\beta W_n}{(K+W_n)^2}\right),$$
$$Q_3 = \phi(h)\beta W_n(A\phi(h)+S_n),$$
$$R_1 = \left(1+\phi(h)\left(\frac{\beta W_n}{K+W_n}+\mu\right)\right),$$
$$R_2 = K+W_n,$$

$$J_{41} = \frac{\partial F_4(S,I,R,W)}{\partial S} = 0,$$
$$J_{42} = \frac{\partial F_4(S,I,R,W)}{\partial I} = 0,$$
$$J_{43} = \frac{\partial F_4(S,I,R,W)}{\partial R} = 0,$$
$$J_{44} = \frac{\partial F_4(S,I,R,W)}{\partial W} = \frac{b\phi(h)+1}{1+\phi(h)\left(\frac{\beta W_n}{K}+c\right)} - \frac{W_n\phi(h)b(\phi(h)b+1)}{\left(1+\phi(h)\left(\frac{\beta W_n}{K}+c\right)\right)^2 K}.$$

The NSFDS converges to the equilibrium points if the absolute of the eigenvalues is less than one.

4.3.1 Stability of disease-free equilibrium

In this section, we show the stability of disease-free equilibrium, P_1. Substituting disease-free equilibrium, P_1, to the Jacobian matrix, we have found

$$J = \begin{bmatrix} \frac{1}{\phi(h)\mu+1} & 0 & 0 & J_{14} \\ 0 & \frac{1}{1+\phi(h)(\gamma+\mu)} & 0 & J_{24} \\ 0 & \frac{\gamma\phi(h)}{(1+\phi(h)(\gamma+\mu))(\mu\phi(h)+1)} & \frac{1}{\mu\phi(h)+1} & J_{34} \\ 0 & 0 & 0 & \frac{b\phi(h)+1}{c\phi(h)+1} \end{bmatrix}.$$

with

$$J_{14} = -\frac{\phi(h)\beta(\phi(h)A + A/\mu)}{(\phi(h)\mu+1)^2 K},$$

$$J_{24} = \frac{\phi(h)\beta(\phi(h)A + A/\mu)}{(\phi(h)\mu+1)K(1+\phi(h)(\gamma+\mu))},$$

$$J_{34} = \frac{\phi(h)^2\beta(\phi(h)A + A/\mu)\gamma}{(\phi(h)\mu+1)^2 K(1+\phi(h)(\gamma+\mu))}.$$

We then determine the eigenvalues of the Jacobian matrix, J. We find that the eigenvalues are

$$\lambda_1 = \lambda_2 = \frac{1}{\mu\phi(h)+1}, \quad \lambda_3 = \frac{1}{\gamma\phi(h)+\mu\phi(h)+1},$$
$$\lambda_4 = \frac{b\phi(h)+1}{c\phi(h)+1}.$$

We can see that the absolute eigenvalues are less than unity ($|\lambda_i| < 1, i = 1,2,3,4$) if $b < c$.

4.3.2 Stability of endemic equilibrium

In this section, we prove the stability of the endemic equilibrium (P_2). Substituting the endemic equilibrium (P_2) into the Jacobian matrix and the determining the eigenvalues, we find that the eigenvalues are

$$\lambda_1 = \frac{2b-c}{(\phi(h)b\beta + 2\phi(h)b\mu - \phi(h)\beta c - \phi(h)c\mu + 2b - c)},$$

$$\lambda_2 = \frac{1}{(\phi(h)\gamma + \phi(h)\mu + 1)},$$

$$\lambda_3 = \frac{1}{\mu\phi(h)+1}, \quad \lambda_4 = \frac{c\phi(h)+1}{b\phi(h)+1}.$$

It is clear that the absolute of the eigenvalues is less than unity if $b > c$. Based on the stability analysis, the following theorem holds.

Theorem 3.*If $b < c$, the absolute eigenvalues of the disease-free equilibrium is less than unity ($|\lambda_i| < 1$ where $i = 1,2,3,4$). Therefore, the disease-free equilibrium (P_1) of the discrete system (5) is stable. In other words, the solutions of the NSFDS (5) converge to the disease-free equilibrium for any positive initial conditions of S_0, I_0, R_0, W_0. Otherwise, if $b > c$, the solutions converge to the endemic equilibrium.*

We can see that the stability condition of the equilibrium points of the discrete model is consistent with that of the continous model (Equation (1)). The discrete model preserves the main properties of the continous model (1).

5 Numerical Simulations

In this section, we present the numerical simulations of the model. The parameter values used are $\mu = 1/65$, $\gamma = 1/45$, $\beta = 0.01$, $K = 60$, $b = 0.1$, $c = 0.4$ (case $b < c$) and $b = 0.2$, $c = 0.1$ (case $b > c$) . $S_0 = 999$, $I_0 = 1$, $W_0 = 20$, $R_0 = 0$ [30]. In addition, we compare the numerical solutions of NSFDS, Euler method and MATLAB's ode45. Results are given in Figure 1 and 2.

We compare the NSFDS, Euler method and MATLAB's ode45 using $h = 0.01$ (for NSFDS and Euler method) and 'RelTol' of 10^{-8} for MATLAB's ode45. Figures 1 and 2 show that the discrete model (5) converges to the equilibriums. The results are consistent with that generated using Euler method and MATLAB's ode45.

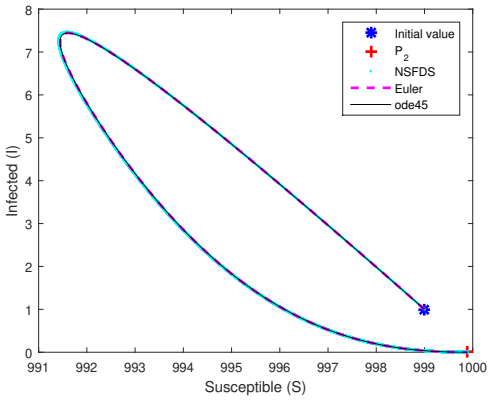

Fig. 1: Numerical simulation with $h = 0.01$ for NSFDS and Euler method and 'RelTol' of 10^{-8} for MATLAB's ode45 in S-I plane for disease-free equilibrium.

6 Conclusions

A NSFDS is developed for a mathematical model of the effects of hardwater consumption on kidney function.It is shown analytically and numerically that the discrete systems preserve the main properties of the model such as positivity and stability. The numerical simulations also confirm the results. The numerical simulations are consistent with that generated by Euler method and MATLAB's ode45. In this paper, we present a nonstandard finite difference scheme for the model proposed by Tambaru *et al.* [30]. Note that the convergence of the solution to the equilibrium points depends on the step-size h. We can use other denominator functions to improve the accuracy of the method for smaller step sizes.

The model can be extended to include the effects of kidney dysfunction on human health. The extension of the

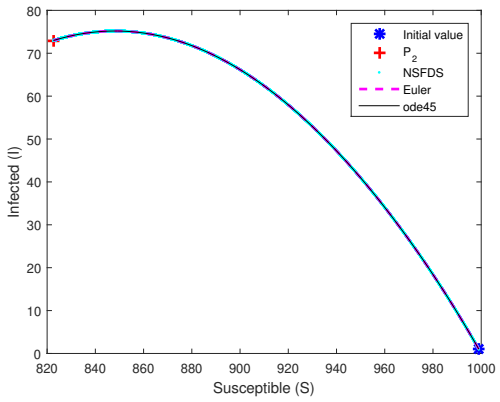

Fig. 2: Numerical simulation with $h = 0.01$ for NSFDS and Euler method and 'RelTol' of 10^{-8} for MATLAB's ode45 in S-I plane for endemic equilibrium.

model results in complex models which can affect model's properties. Therefore, the construction of a stable numerical scheme is needed to obtain correct solution. The developed scheme in this paper can be a basis for the construction of an unconditionally stable numerical scheme for complicated models.

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