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A Single-Product Stochastic Inventory System Modulated by A Queueing System and Subject to A Compulsory Waiting Period for Re-Ordering

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Abstract: A continuous review of single-product stochastic inventory system is considered in which an adjustable reorder can be placed only after the expiry of a Compulsory Waiting Period (CWP) even if the inventory position of the system demands an earlier placement of a reorder. Compulsory waiting period follows an exponential distribution with mean $1/\gamma$. The maximum capacity of the inventory is M. It is assumed that replenishment is instantaneous. Customers arrive to the inventory system according to a Poisson process with rate λ. There is a single server attached with the inventory and the job of the server is to serve each customer one at a time with one item from the inventory according to First-In-First-Out (FIFO) policy. The server takes a random time to serve each customer and this service time follows an exponential distribution with mean $1/\mu$. When the server serves a customer, all other customers queue-up in a waiting room to meet out the FIFO policy. The capacity of the waiting room is assumed to be $M-1$. All arriving demands are assumed to be lost when the waiting room is full. By a numerical illustration, the steady-state joint probability distribution for the number of customers in the queueing system and the number of items in inventory is obtained and performance measures such as stationary mean number of replenishment, mean number of demand satisfied and mean number of demand lost are analysed.

Keywords: queueing system, inventory system, compulsory waiting period, replenishment

1 Introduction

Quite extensive research has been done on inventory systems modulated by queueing systems (see Schwarz et al. [1]). In such systems, inventory systems are integrated with queueing systems in the sense that a positive service time is required for satisfying a demand and while serving a demand, all other arriving demands have to wait before being served with items from the inventory. Sigman and Simchi-Levi [2] were the first to introduce positive service time in the study of inventory systems. A huge variety of research on queueing-inventory systems has emerged in the past two decades (see for example, [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]). Krishnamoorthy et al. [18] have provided a brief survey of queueing-inventory systems.

In continuous review of inventory systems, it is usually assumed that the product under consideration is always available in plenty for meeting out the demands and an order for replenishment can be made at any epoch of time. However there are situations where this assumption may not be true. When the availability of the product becomes scarce, restrictions on placement of reorders may be imposed by the supplier. For example, after the replenishment of stock in the inventory for an order, the system may have to wait compulsorily for some time because of rationing of supply of items before the placement of next order. In several organizations, the manpower management is a big concern. Wastage of employees is a stochastic phenomenon and the vacancies arising due to wastage are not filled up immediately; in addition the work is executed by recruiting casual labourers on temporary basis. The organization waits compulsorily for a random amount of time till recruitment process is initiated. As another real-time situation where CWP is encountered, we come across several big industries where the manufacturing of the main product such as truck or car requires several sub-components. The manufacturing of sub-components is let to be manufactured by external small industries in competitive market. From the view point of an outside vendor, it has

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to wait compulsorily for some period of time till getting another order by the main company even though the vendor might have completed the job-order on hand. Generally, the compulsory waiting period begins from the instant of replenishment. Hence a reorder for replenishment can be placed only after the expiry of this Compulsory Waiting Period (CWP) even if the inventory position of the system demands an earlier placement of a reorder. The compulsory waiting period is considered as a random variable. This random variable cannot be clubbed with the lead-time random variable in the sense that the lead-time random variable can be taken as the sum of the two random variable namely the compulsory waiting period and the actual lead time. This implies that the lead time commences as soon as the compulsory waiting period is over. This is not always true; suppose that in a realization the compulsory waiting period is over and the inventory position of the system at that epoch may not require the placement of reorder and hence the lead time may not commence at that epoch. Hence the two random variables are to be dealt with separately. The concept of CWP in the study of stochastic inventory systems has been introduced by Udayabaskaran et al. [19, 20] when they analysed continuous-review stochastic inventory systems subject to rationing of supply due to scarcity of commodities. They have analysed continuous review stochastic inventory systems subject to CWP together with the restriction that the replenishment is instantaneous without lead time. Recently, Yadavalli and Udayabaskaran [21] have analysed a single-product perishable inventory system with CWP for reordering and with stochastic lead time.

In the present paper, a stochastic model of an inventory-queueing system subject to CWP is studied. For this model, the stationary probability distribution of the inventory level, the stationary mean rate of (i) the demands satisfied; (ii) the demands lost; and (iii) the reorders are analysed.

The organization of the paper is as follows: In Section 2, we describe the model. Section 3 derives the governing equations for the model. In Section 4, we provide the steady-state equations. Performance measures are analysed in Section 5 by a numerical illustration.

2 Model Description

Customers arrive to a single-product inventory system according to a Poisson process with rate λ. There is a single server attached with the inventory and the job of the server is to serve each customer one at a time with one item from the inventory according to a FIFO policy. The maximum capacity of the inventory is M. The server takes a random time to serve each customer and this service time follows an exponential distribution with mean $1/\mu$. When the server serves a customer, all other arriving customers queue-up in a waiting room and follow the FIFO policy. The capacity of the waiting room is $M-1$. All arriving customers are lost when the waiting is full. A reorder can be placed only after the expiry of a CWP even if the inventory position of the system demands an earlier placement of a reorder. It is assumed that replenishment is instantaneous. Compulsory waiting period follows an exponential distribution with mean $1/\gamma$.

3 Governing equations

Let $X(t)$ be the number of items in the inventory system X and $Y(t)$ be the number customers in the queueing system at time t. We define

$$J(t) = \begin{cases} 1 & \text{if the inventory system is in CWP;} \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z(t) = (X(t), Y(t), J(t))$. Then the vector process $\{Z(t)|t \geq 0\}$ is Markov. The state space of the process is given by

$$\Omega = \{(i,j,k)|i = 0,1,2,\cdots,M; j = 0,1,2,\cdots; k = 0,1\}.$$

At time $t = 0$, the queueing-inventory process is in the state $(M,0,1)$. We define the state probabilities as follows:

$$p(i,j,k,t) = Pr\{Z(t) = (i,j,k)|Z(0) = (M,0,1)\}.$$

By the description of the model, it is easily seen that

$$p(0,j,0,t) = 0, j = 0,1,2,\cdots.$$

By using probability laws, we obtain integral equations for the other state probabilities as follows:

Case (i) State $(i,0,0)$, **where** $i = 1,2,\cdots,M-1$:

Using Fig. 1, we get

$$P(i,0,0,t) = [\mu P(i+1,1,0,t) + \gamma P(i,0,1,t)] \copyright e^{-\lambda t}. \quad (1)$$

$$i = 1,2,\cdots,M-1$$

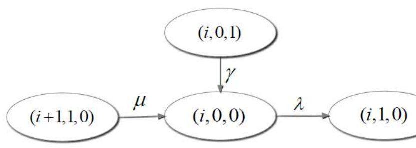

Fig. 1: Flow diagram at $(i,0,0)$

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Case (ii) State $(M,0,0)$:
Using Fig. 2, we get

$$P(M,0,0,t) = \gamma P(M,0,1,t) \copyright e^{-\lambda t}. \tag{2}$$

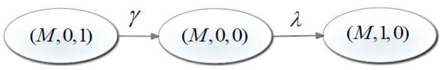

Fig. 2: Flow diagram at $(M,0,0)$

Case (iii) State $(1,j,0)$, **where** $j = 1,2,\cdots,M-1$:
Using Fig. 3, we get

$$P(1,j,0,t) = [\lambda P(1,j-1,0,t) + \mu P(2,j+1,0,t) + \gamma P(1,j,1,t)] \copyright e^{-(\lambda+\mu)t}. \tag{3}$$

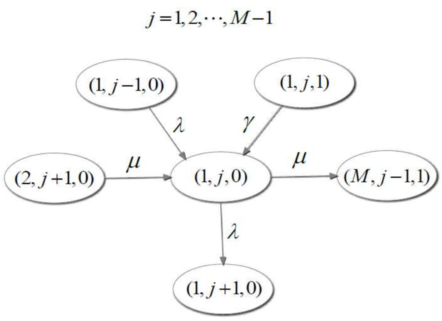

Fig. 3: Flow diagram at $(1,j,0)$

Case (iv) State $(1,M,0)$:
Using Fig. 4, we get

$$P(1,M,0,t) = [\lambda P(1,M-1,0,t) + \gamma P(1,M,1,t)] \copyright e^{-\mu t}. \tag{4}$$

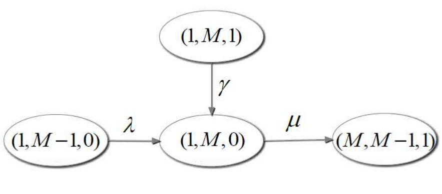

Fig. 4: Flow diagram at $(1,M,0)$

Case (v) State $(i,j,0)$,
where $i = 2,3,\cdots,M-1; j = 1,2,\cdots,M-1$:
Using Fig. 5, we get

$$P(i,j,0,t) = [\lambda P(i,j-1,0,t) + \mu P(i+1,j+1,0,t) + \gamma P(i,j,1,t)] \copyright e^{-(\lambda+\mu)t}. \tag{5}$$

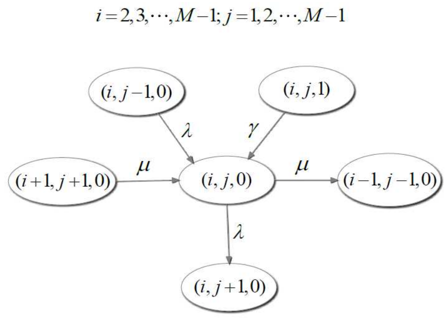

Fig. 5: Flow diagram at $(i,j,0)$

Case (vi) State $(i,M,0)$, **where** $i = 2,3,\cdots,M$:
Using Fig. 6, we get

$$P(i,M,0,t) = [\lambda P(i,M-1,0,t) + \gamma P(i,M,1,t)] \copyright e^{-\mu t}. \tag{6}$$

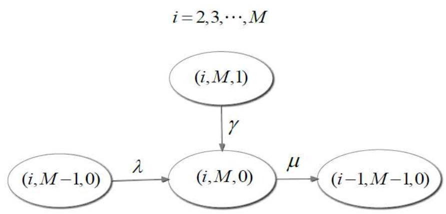

Fig. 6: Flow diagram at $(i,M,0)$

Case (vii) State $(M,j,0)$, **where** $j = 1,2,\cdots,M-1$:
Using Fig. 7, we get

$$P(M,j,0,t) = [\lambda P(M,j-1,0,t) + \gamma P(M,j,1,t)] \copyright e^{-(\lambda+\mu)t}. \tag{7}$$

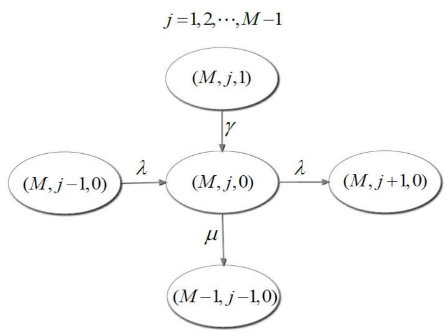

Fig. 7: Flow diagram at $(m,j,0)$

Case (viii) State $(0,0,1)$:
Using Fig. 8, we get

$$P(0,0,1,t) = \mu P(1,1,1,t) \copyright e^{-(\lambda+\gamma)t}. \tag{8}$$

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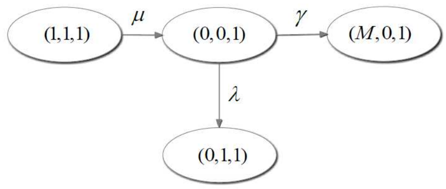

Fig. 8: Flow diagram at $(0,0,1)$

Case (ix) State $(0,j,1)$, **where** $j=1,2,\cdots,M-1$:
Using Fig. 9, we get

$$P(0,j,1,t)=[\lambda P(0,j-1,1,t)$$
$$+\mu P(1,j+1,1,t)] \copyright e^{-(\lambda+\gamma)t}. \quad (9)$$
$$j=1,2,\cdots,M-1$$

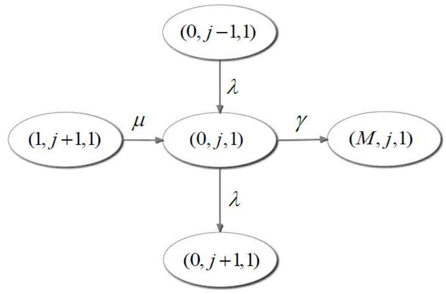

Fig. 9: Flow diagram at $(0,j,1)$

Case (x) State $(0,M,1)$:
Using Fig. 10, we get

$$P(0,M,1,t)=\lambda P(0,M-1,1,t) \copyright e^{-\gamma t}. \quad (10)$$

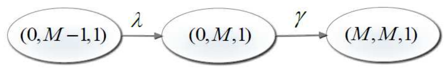

Fig. 10: Flow diagram at $(0,M,1)$

Case (xi) State $(i,0,1)$, **where** $i=1,2,\cdots,M-1$:
Using Fig. 11, we get

$$P(i,0,1,t)=\mu P(i+1,1,1,t) \copyright e^{-(\lambda+\gamma)t}. \quad (11)$$
$$i=1,2,\cdots,M-1$$

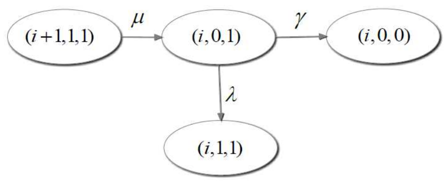

Fig. 11: Flow diagram at $(i,0,1)$

Case (xii) State $(M,0,1)$:
Using Fig. 12, we get

$$P(M,0,1,t)=[\mu P(1,1,0,t)+\gamma P(0,0,1,t)] \copyright e^{-(\lambda+\gamma)t}. \quad (12)$$

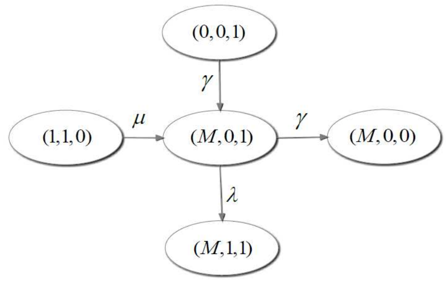

Fig. 12: Flow diagram at $(M,0,1)$

Case (xiii) State $(i,j,1)$,
where $i=1,2,\cdots,M-1, j=1,2,\cdots,M-1$:
Using Fig. 13, we get

$$P(i,j,1,t)=[\lambda P(i,j-1,1,t)$$
$$+\mu P(i+1,j+1,1,t)] \copyright e^{-(\lambda+\mu+\gamma)t}. \quad (13)$$
$$i=1,2,\cdots,M-1; j=1,2,\cdots,M-1$$

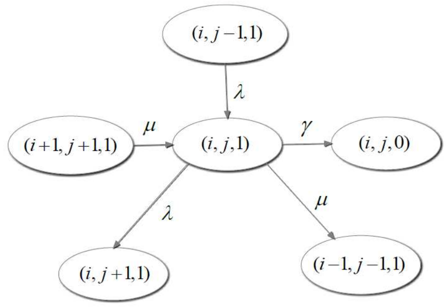

Fig. 13: Flow diagram at $(i,j,1)$

Case (xiv) State $(i,M,1)$, **where** $i=1,2,\cdots,M-1$:
Using Fig. 14, we get

$$P(i,M,1,t)=\lambda P(i,M-1,1,t) \copyright e^{-(\mu+\gamma)t}. \quad (14)$$
$$i=1,2,\cdots,M-1$$

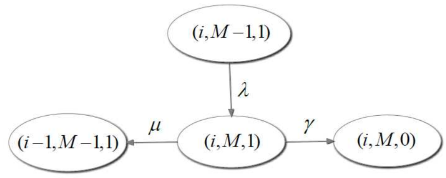

Fig. 14: Flow diagram at $(i,M,1)$

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Case (xv) State $(M, j, 1)$, where $j = 1, 2, \cdots, M-1$:
Using Fig. 15, we get

$$P(M, j, 1, t) = [\lambda P(M, j-1, 1, t) + \mu P(1, j+1, 0, t)] \copyright e^{-(\lambda+\mu+\gamma)t}. \quad (15)$$

$$j = 1, 2, \cdots, M-1$$

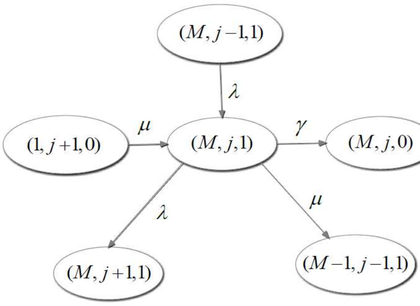

Fig. 15: Flow diagram at $(M, j, 1)$

Case (xvi) State $(M, M, 1)$:
Using Fig. 16, we get

$$P(M, M, 1, t) = [\lambda P(M, M-1, 1, t) + \gamma P(0, M, 1, t)] \copyright e^{-(\mu+\gamma)t}. \quad (16)$$

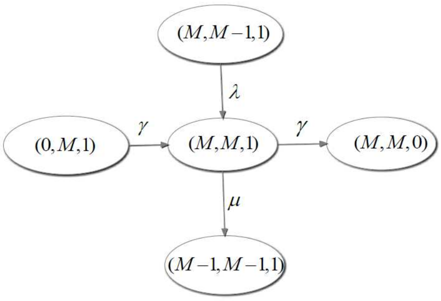

Fig. 16: Flow diagram at $(M, M, 1)$

4 Steady-state solution

We define the steady-state probabilities as follows:

$$\pi(i, j, k) = \lim_{t \longrightarrow \infty} p(i, j, k, t).$$

By applying the final value theorem of Laplace transform, we get

$$\pi(i, j, k) = \lim_{\theta \longrightarrow 0} \theta p^*(i, j, k, \theta),$$

where $p^*(i, j, k, \theta)$ is the Laplace transform of $p(i, j, k, t)$. By applying Laplace transform on both sides of (1)-(16), we get

$$(\theta+\lambda)P^*(i, 0, 0, \theta) = \mu P^*(i+1, 1, 0, \theta) + \gamma P^*(i, 0, 1, \theta), i = 1, 2, \cdots, M-1. \quad (17)$$

$$(\theta+\lambda)P^*(M, 0, 0, \theta) = \gamma P^*(M, 0, 1, \theta). \quad (18)$$

$$(\theta+\lambda+\mu)P^*(1, j, 0, \theta) = \lambda P^*(1, j-1, 0, \theta) + \mu P^*(2, j+1, 0, \theta) + \gamma P^*(1, j, 1, \theta), \quad j = 1, 2, \cdots, M-1. \quad (19)$$

$$(\theta+\mu)P^*(1, M, 0, \theta) = \lambda P^*(1, M-1, 0, \theta) + \gamma P^*(1, M, 1, \theta). \quad (20)$$

$$(\theta+\lambda+\mu)P^*(i, j, 0, \theta) = \lambda P^*(i, j-1, 0, \theta) + \mu P^*(i+1, j+1, 0, \theta) + \gamma P^*(i, j, 1, \theta), \quad i = 2, 3, \cdots, M-1; j = 1, 2, \cdots, M-1. \quad (21)$$

$$(\theta+\mu)P^*(i, M, 0, \theta) = \lambda P^*(i, M-1, 0, \theta) + \gamma P^*(i, M, 1, \theta), i = 2, 3, \cdots, M. \quad (22)$$

$$(\theta+\lambda+\mu)P^*(M, j, 0, \theta) = \lambda P^*(M, j-1, 0, \theta) + \gamma P^*(M, j, 1, \theta), j = 1, 2, \cdots, M-1. \quad (23)$$

$$(\theta+\lambda+\gamma)P^*(0, 0, 1, \theta) = \mu P^*(1, 1, 1, \theta). \quad (24)$$

$$(\theta+\lambda+\gamma)P^*(0, j, 1, \theta) = \lambda P^*(0, j-1, 1, \theta) + \mu P^*(1, j+1, 1, \theta), j = 1, 2, \cdots, M-1. \quad (25)$$

$$(\theta+\gamma)P^*(0, M, 1, \theta) = \lambda P^*(0, M-1, 1, \theta). \quad (26)$$

$$(\theta+\lambda+\gamma)P^*(i, 0, 1, \theta) = \mu P^*(i+1, 1, 1, \theta), \quad i = 1, 2, \cdots, M-1. \quad (27)$$

$$(\theta+\lambda+\gamma)P^*(M, 0, 1, \theta) = \mu P^*(1, 1, 0, \theta) + \gamma P^*(0, 0, 1, \theta). \quad (28)$$

$$(\theta+\lambda+\mu+\gamma)P^*(i, j, 1, \theta) = \lambda P^*(i, j-1, 1, \theta) + \mu P^*(i+1, j+1, 1, \theta), \quad i = 1, 2, \cdots, M-1; j = 1, 2, \cdots, M-1. \quad (29)$$

$$(\theta+\mu+\gamma)P^*(i, M, 1, \theta) = \lambda P^*(i, M-1, 1, \theta), \quad i = 1, 2, \cdots, M-1. \quad (30)$$

$$(\theta+\lambda+\mu+\gamma)P^*(M, j, 1, \theta) = \lambda P^*(M, j-1, 1, \theta) + \mu P^*(1, j+1, 0, \theta), \quad j = 1, 2, \cdots, M-1. \quad (31)$$

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$$(\theta+\mu+\gamma)P^*(M,M,1,\theta)=\lambda P^*(M,M-1,1,\theta)+\gamma P^*(0,M,1,\theta). \quad (32)$$

By applying Final value theorem with (11)-(20), we get the steady state balance equations as follows:

$$\lambda\pi(i,0,0)=\mu\pi(i+1,1,0)+\gamma\pi(i,0,1),\quad i=1,2,\cdots,M-1. \quad (33)$$

$$\lambda\pi(M,0,0)=\gamma\pi(M,0,1). \quad (34)$$

$$(\lambda+\mu)\pi(1,j,0)=\lambda\pi(1,j-1,0)+\mu\pi(2,j+1,0)+\gamma\pi(1,j,1), j=1,2,\cdots,M-1. \quad (35)$$

$$\mu\pi(1,M,0)=\lambda\pi(1,M-1,0)+\gamma\pi(1,M,1). \quad (36)$$

$$(\lambda+\mu)\pi(i,j,0)=\lambda\pi(i,j-1,0)+\mu\pi(i+1,j+1,0)+\gamma\pi(i,j,1), i=2,3,\cdots,M-1; j=1,2,\cdots,M-1. \quad (37)$$

$$\mu\pi(i,M,0)=\lambda\pi(i,M-1,0)+\gamma\pi(i,M,1), i=2,3,\cdots,M. \quad (38)$$

$$(\lambda+\mu)\pi(M,j,0)=\lambda\pi(M,j-1,0)+\gamma\pi(M,j,1),\quad j=1,2,\cdots,M-1. \quad (39)$$

$$(\lambda+\gamma)\pi(0,0,1)=\mu\pi(1,1,1). \quad (40)$$

$$(\lambda+\gamma)\pi(0,j,1)=\lambda\pi(0,j-1,1)+\mu\pi(1,j+1,1), j=1,2,\cdots,M-1. \quad (41)$$

$$\gamma\pi(0,M,1)=\lambda\pi(0,M-1,1). \quad (42)$$

$$(\lambda+\gamma)\pi(i,0,1)=\mu\pi(i+1,1,1), i=1,2,\cdots,M-1. \quad (43)$$

$$(\lambda+\gamma)\pi(M,0,1)=\mu\pi(1,1,0)+\gamma\pi(0,0,1). \quad (44)$$

$$(\lambda+\mu+\gamma)\pi(i,j,1)=\lambda\pi(i,j-1,1)+\mu\pi(i+1,j+1,1), i=1,2,\cdots,M-1; j=1,2,\cdots,M-1. \quad (45)$$

$$(\mu+\gamma)\pi(i,M,1)=\lambda\pi(i,M-1,1), i=1,2,\cdots,M-1. \quad (46)$$

$$(\lambda+\mu+\gamma)\pi(M,j,1)=\lambda\pi(M,j-1,1)+\mu\pi(1,j+1,0),\quad j=1,2,\cdots,M-1. \quad (47)$$

$$(\mu+\gamma)\pi(M,M,1)=\lambda\pi(M,M-1,1)+\gamma\pi(0,M,1). \quad (48)$$

The system of equations (33)-(48) can be solved and the steady-state distribution $\{\pi(i,j,k)\}$ can be obtained.

5 Measures of system performance

In this section, we study some performance measures of the queueing-inventory system. These performance measures manifest the effects of the CWP. We specifically study the following measures:

(i) The mean of the stock-out period per unit time;

(ii) The mean rate of lost demands;

Mean stationary rate of events:

Let E be the event that a replenishment has just occurred at an epoch and the CWP has just commenced at that epoch. Then E events constitute a renewal process. For any state ω of the system, we can obtain the proportion or fraction of mean-time that the system stays in the state ω in the long run. To achieve this, we define

$$\phi_\omega(t)=\begin{cases}1 \text{ if the system is in the state } \omega \text{ at time } t;\\ 0 \text{ otherwise.}\end{cases}$$

Let $P_\omega(t)$ denote the probability that the system is in the state ω at time t. Then we get

$$P_\omega(t)=Pr\{\phi_\omega(t)=1|E \text{ at } t=0\}.$$

The total sojourn time that the system is in the state ω in the interval $(0,t]$ is given by the stochastic integral

$$U_\omega(t)=\int_0^t \phi_\omega(u)du.$$

The expected value of $U_\omega(t)$ is given by

$$E\left[U_\omega(t)\right]=E\left[\int_0^t \phi_\omega(u)du\right]$$

$$=\int_0^t E[\phi_\omega(u)]du=\int_0^t P_\omega(u)du.$$

Then the proportion or fraction of mean sojourn time per unit time is given by

$$\frac{1}{t}\int_0^t P_\omega(u)]du.$$

Consequently the proportion or fraction of mean sojourn time per unit time in the long-run is given by

$$\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t P_\omega(u)du=\lim_{t\longrightarrow\infty}P_\omega(t)=P_\omega,$$

where P_ω is the steady-state probability that the system is in the state ω.

If ε represents the event that the system is in the stock-out period, then the proportion or fraction of mean stock-out period per unit time in the long-run is given by

$$\varepsilon=\sum_{j=0}^{M}\pi(0,j,1). \quad (49)$$

If ω represents the event that replenishment takes place at an epoch, then the stationary mean rate $E(r)$ of replenishment is given by

$$E(r) = \sum_{j=0}^{M} \pi(1, j, 0)\lambda + \sum_{j=0}^{M} \pi(0, j, 1)\gamma. \tag{50}$$

If l represents the event that a demand is lost, then the stationary mean rate $E(l)$ of lost demands is given by

$$E(l) = \sum_{i=0}^{M} \pi(0, M, 1)\lambda. \tag{51}$$

6 A Numerical Illustration

In this section, we illustrate the behaviour of the model by controlling the parameters of the system.

6.1 Steady-state probability distribution

We compute the steady-state probabilities by assuming the following values:

$$\lambda = 10.0; \mu = 20.0; \gamma = 5.0; M = 3. \tag{52}$$

The steady-state probabilities are listed in table 1.

Table 1: Steady-state probabilities $\pi(i, j, k)$

$\pi(0,0,1) = 0.037341$	$\pi(1,1,1) = 0.028006$	$\pi(2,1,0) = 0.03369$	$\pi(3,0,1) = 0.080237$
$\pi(0,1,1) = 0.044943$	$\pi(1,2,0) = 0.030324$	$\pi(2,1,1) = 0.027348$	$\pi(3,1,0) = 0.020082$
$\pi(0,2,1) = 0.037981$	$\pi(1,2,1) = 0.015037$	$\pi(2,2,0) = 0.026457$	$\pi(3,1,1) = 0.040253$
$\pi(0,3,1) = 0.075963$	$\pi(1,3,0) = 0.016666$	$\pi(2,2,1) = 0.030778$	$\pi(3,2,0) = 0.010198$
$\pi(1,0,0) = 0.085612$	$\pi(1,3,1) = 0.0060146$	$\pi(2,3,0) = 0.016306$	$\pi(3,2,1) = 0.021024$
$\pi(1,0,1) = 0.036465$	$\pi(2,0,0) = 0.066999$	$\pi(2,3,1) = 0.012311$	$\pi(3,3,0) = 0.015146$
$\pi(1,1,0) = 0.050843$	$\pi(2,0,1) = 0.053671$	$\pi(3,0,0) = 0.040119$	$\pi(3,3,1) = 0.040187$

6.2 Mean-state stationary rate of stock-out period

The mean stationary rate of stock-out period is given by $\sum_{j=0}^{M} \pi(0, j, 1)$. We analyse the behaviour of the proportion or fraction ε of mean stock-out period per unit time in the long-run by varying γ from 1.1 to 6.0. We fix the other parameters as in (52). The variation of ε as a function of γ is depicted in Fig. 17.

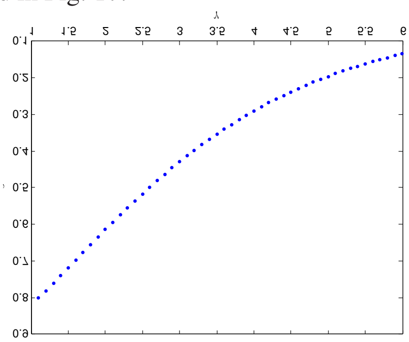

Fig. 17: Variation of ε versus γ

The values of ε are provided in table 2.

Table 2: Variation of ε versus γ

γ	ε	γ	ε	γ	ε	γ	ε	γ	ε
1.1	0.80146	2.1	0.59439	3.1	0.413	4.1	0.2792	5.1	0.1888
1.2	0.78093	2.2	0.57446	3.2	0.39741	4.2	0.2684	5.2	0.18167
1.3	0.76021	2.3	0.55484	3.3	0.38232	4.3	0.25802	5.3	0.17485
1.4	0.73936	2.4	0.53559	3.4	0.36773	4.4	0.24805	5.4	0.16831
1.5	0.71843	2.5	0.51673	3.5	0.35363	4.5	0.23849	5.5	0.16204
1.6	0.69748	2.6	0.49829	3.6	0.34004	4.6	0.22931	5.6	0.15604
1.7	0.67657	2.7	0.48029	3.7	0.32693	4.7	0.22051	5.7	0.15029
1.8	0.65575	2.8	0.46274	3.8	0.3143	4.8	0.21207	5.8	0.14477
1.9	0.63509	2.9	0.44568	3.9	0.30214	4.9	0.20398	5.9	0.13949
2.0	0.61462	3.0	0.42909	4.0	0.29045	5.0	0.19623	6.0	0.13443

We find that as the mean of CWP decreases, the mean stationary rate of stock-out period decreases. This behaviour is quite expected.

6.3 Mean-state stationary rate of replenishment

The stationary mean rate $E(r)$ of replenishment is given by $E(r) = \sum_{j=0}^{3} \pi(1, j, 0)\lambda + \sum_{j=0}^{3} \pi(0, j, 1)\gamma$.

We analyse the behaviour of $E(r)$ by varying γ from 1.1 to 6.0. We fix the other parameters as in (52). The variation of $E(r)$ as a function of γ is depicted in Fig. 18.

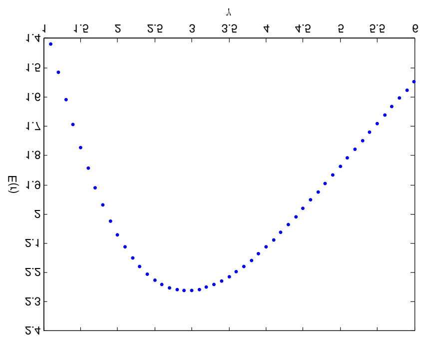

Fig. 18: Variation of $E(r)$ versus γ

The values of $E(r)$ are given in table 3.

Table 3: Variation of $E(r)$ versus γ

γ	$E(r)$	γ	$E(r)$	γ	$E(r)$	γ	$E(r)$	γ	$E(r)$
1.1	1.4189	2.1	2.1126	3.1	2.2573	4.1	2.0877	5.1	1.8069
1.2	1.5167	2.2	2.1486	3.2	2.2504	4.2	2.0622	5.2	1.7775
1.3	1.6081	2.3	2.1791	3.3	2.2406	4.3	2.0358	5.3	1.7481
1.4	1.6933	2.4	2.2043	3.4	2.2283	4.4	2.0086	5.4	1.7189
1.5	1.772	2.5	2.2247	3.5	2.2136	4.5	1.9809	5.5	1.6897
1.6	1.8444	2.6	2.2403	3.6	2.1968	4.6	1.9526	5.6	1.6608
1.7	1.9104	2.7	2.2514	3.7	2.178	4.7	1.9239	5.7	1.6321
1.8	1.9701	2.8	2.2585	3.8	2.1575	4.8	1.8949	5.8	1.6037
1.9	2.0236	2.9	2.2616	3.9	2.1356	4.9	1.8657	5.9	1.5755
2.0	2.071	3.0	2.2611	4.0	2.1122	5.0	1.8364	6.0	1.5477

We find that the stationary mean rate of replenishment increases for lower values of γ and decreases for higher values of γ. This is because of the fact that the impact of CWP becomes dominant for larger values of γ.

6.4 Stationary mean rate of lost demands

The stationary mean rate $E(l)$ of lost demands is given by $E(l) = \sum_{i=0}^{M} \pi(0, M, 1)\lambda$. We analyse the behaviour of $E(l)$ by varying γ from 1.1 to 6.0. We fix the other parameters as

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in (52). The variation of $E(l)$ as a function of γ is depicted in Fig. 19.

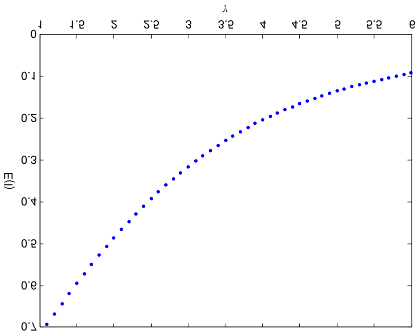

Fig. 19: Variation of $E(l)$ versus γ

The values of $E(l)$ are furnished in table 4.

Table 4: Variation of $E(l)$ versus γ

γ	$E(l)$	γ	$E(l)$	γ	$E(l)$	γ	$E(l)$	γ	$E(l)$
1.1	0.69327	2.1	0.46583	3.1	0.30182	4.1	0.19524	5.1	0.12923
1.2	0.66796	2.2	0.44651	3.2	0.28883	4.2	0.18711	5.2	0.12422
1.3	0.64316	2.3	0.42785	3.3	0.2764	4.3	0.17935	5.3	0.11946
1.4	0.61891	2.4	0.40985	3.4	0.26453	4.4	0.17197	5.4	0.11492
1.5	0.59521	2.5	0.39251	3.5	0.25318	4.5	0.16493	5.5	0.11059
1.6	0.5721	2.6	0.37581	3.6	0.24235	4.6	0.15823	5.6	0.10646
1.7	0.54959	2.7	0.35977	3.7	0.23201	4.7	0.15185	5.7	0.10253
1.8	0.52769	2.8	0.34436	3.8	0.22215	4.8	0.14578	5.8	0.098775
1.9	0.50643	2.9	0.32957	3.9	0.21275	4.9	0.13999	5.9	0.095196
2.0	0.4858	3.0	0.3154	4.0	0.20379	5.0	0.13448	6.0	0.091781

We find that the mean-stationary rate of lost demands decreases as γ increases. This is also quite expected, since as the mean $1/\gamma$ of CWP decreases, the mean rate of lost demands should decrease.

7 Conclusion

We considered a single-product inventory-queueing system subject to a compulsory waiting period for re-ordering. We assume instantaneous replenishment, Poisson arrival of demands, exponentially-distributed service time and exponentially-distributed compulsory waiting period. We wrote down the steady-state equations for the state probabilities. We've not been able to obtain analytical solution for the steady-state probabilities. However, we've studied the model by a numerical illustration. We obtain numerical results for some of the performance measures of the system. We've analysed the impact of CWP on the measures of performance and concluded that CWP vastly changes the behaviour of the stationary mean rate of demands satisfied, the lost demands and the number of replenishments.

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