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On Generalized Strongly Convex Functions Involving Bifunction

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Abstract: In this paper, we define and introduce some new concepts of the generalized strongly convex functions and generalized strongly monotone operators with respect to an arbitrary non-negative bifunction. We establish some new relationships among various concepts of generalized strongly convex functions. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

Keywords: Convex functions, monotone operators, strongly convex functions
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1 Introduction

In recent years, several extensions and generalizations for classical convexity have been considered. Strongly convex functions were introduced and studied by Polyak [1], which play an important and significant part in the optimization theory, variational inequalities and related areas. For the applications, properties and other aspects of the strongly convex functions, see [2,3,4,5,6,7,8,9,10, 11,12,13,14,15,16,17,18,19,20,21,22,23]. Adamek [2] introduced another class of convex functions with respect to an arbitrary non-negative function $F(.-.)$, which is called the relative strongly convex functions. With appropriate choice of non-negative function $F(.-.)$, one can obtain various known classes of convex functions as special cases.

Inspired by the work of Adamek [2], Awan et. al[3-6] and Nikodem et al. [16], we introduce and consider a new class of convex functions with respect to an arbitrary non-negative bifunction $G.,.)$. This class of convex functions is called the generalized strongly convex functions. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. As special cases, on can obtain various new and refined versions of known results. The ideas and techniques of this paper may inspire further research.

2 Formulations and basic facts

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm, respectively. Let $F: K \to R$ be a continuous function. Let $G(.,): [0,\infty)\times[0,\infty)\to R$ be a non-negative bifunction.

Definition 1.[15,23] *The set K in H is said to be a convex set, if*

$$u+t(v-u)\in K, \qquad \forall u,v\in K, t\in[0,1].$$

We now introduce a new class of strongly convex functions and strongly affine convex functions with respect to an arbitrary bifunction $G(,.,)$.

Definition 2. *A function F on the convex set K is said to be generalized strongly convex with respect to an arbitrary non-negative bifunction $G(.,.)$, if there exists a constant $\mu > 0$, such that*

$$F(u+t(v-u)) \le (1-t)F(u)+tF(v) -\mu t(1-t)G(v,u), \forall u,v\in K, t\in[0,1]. \quad (1)$$

A function F is said to be generalized strongly concave, if and only if, $-F$ is generalized strongly convex.
If $t=\frac{1}{2}$ and $\mu=1$, then

$$F\left(\frac{u+v}{2}\right) \le \frac{F(u)+F(v)}{2}$$

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$$-\frac{1}{4}G(v,u), \forall u,v \in K, t \in [0,1]. \quad (2)$$

The function F is said to be generalized strongly J-convex function.

Definition 3. *A function F on the convex set K is said to be generalized strongly affine convex with respect to an arbitrary non-negative bifunction $G(.,.)$, if there exists a constant $\mu > 0$, such that*

$$F(u+t(v-u)) = (1-t)F(u)+tF(v)$$
$$-\mu t(1-t)G(v,u), \forall u,v \in K, t \in [0,1]. \quad (3)$$

Note that if the function is both generalized strongly convex and generalized strongly concave, then it is generalized strongly affine convex function.

A function F is called generalized strongly quadratic equation, if there exists a constant $\mu > 0$, such that

$$F\left(\frac{u+v}{2}\right) = \frac{F(u)+F(v)}{2} - \frac{1}{4}G(v,u), \forall u,v \in K, t \in [0,1]. \quad (4)$$

This function F is also called generalized strongly affine J-convex function.

We now discuss some special cases.

I. If $G(v,u) = \| v-u \|^2$, then the generalized strongly convex function becomes strongly convex functions, that is,

$$F(u+t(v-u)) \leq (1-t)F(u)+tF(v)$$
$$-\mu t(1-t) \| v-u \|^2, \forall u,v \in K, t \in [0,1],$$

see [4,11,14,19,20]. **II.** If $G(v,u) = F(v-u)$ then the generalized strongly convex function becomes strongly convex function, which were introduced and studied by Adamek [2], that is,

$$F(u+t(v-u)) \leq (1-t)F(u)+tF(v)$$
$$-\mu t(1-t)F(v-u), \forall u,v \in K, t \in [0,1].$$

For the properties of the generalized strongly convex functions in variational inequalities and equilibrium problems, see Noor [14,17,18].

Definition 4. *A function F on the convex set K is said to be generalized strongly quasi convex with respect to an arbitrary non-negative bifunction $G(.,)$, if there exists a constant $\mu > 0$ such that*

$$F(u+t(v-u) \leq \max\{F(u),F(v)\}$$
$$-\mu t(1-t)G(v,u), \forall u,v \in K, t \in [0,1].$$

Definition 5. *A function F on the convex set K is said to be generalized strongly log-convex with respect to the bifunction $G(v,u)$, if there exists a constant $\mu > 0$ such that*

$$F(u+t(v-u)) \leq (F(u))^{1-t}(F(v))^t$$
$$-\mu t(1-t)G(v,u), \forall u,v \in K, t \in [0,1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$F(u+t(v-u)) \leq (F(u))^{1-t}, (F(v))^t - \mu t(1-t)G(v,u)$$
$$\leq (1-t)F(u)+tF(v) - \mu t(1-t)G(v,u)$$
$$\leq \max\{F(u),F(v)\} - \mu t(1-t)G(v,u)$$

This shows that every generalized strongly log-convex function is a generalized strongly convex function and every generalized strongly convex function is a generalized quasi-convex function. However, the converse is not true.

For appropriate and suitable choice of the arbitrary bifurcation $G(.,.)$, one can obtain several new and known classes of strongly convex functions and their variant forms as special cases of generalized strongly convex functions. This shows that the class of generalized strongly convex functions is quite broad and a unifying one.

Definition 6. *An operator $T: K \rightarrow H$ is said to be:*

1. *generalized strongly G-monotone, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, u-v \rangle \leq \alpha\{G(v,u)+G(u,v)\}, \forall u,v \in K.$$

2. *generalized strongly G-pseudomonotone, if and only if, there exists a constant $\nu > 0$ such that*

$$\langle Tu, v-u \rangle + \nu G(v,u) \geq 0$$
$$\Rightarrow$$
$$\langle Tv, v-u \rangle \geq 0, \forall u,v \in K.$$

3. *generalized strongly relaxed G-pseudomonotone, if and only if, there exists a constant $\mu > 0$ such that*

$$\langle Tu, v-u \rangle \geq 0$$
$$\Rightarrow$$
$$-\langle Tv, u-v \rangle + \mu G(u,v) \geq 0, \forall u,v \in K.$$

Definition 7. *A differentiable function F on the convex set K is said to be generalized strongly pseudo G-convex function, if and only if, if there exists a constant $\mu > 0$ such that*

$$\langle F'(u), v-u \rangle + \mu G(u,v) \geq 0 \Rightarrow F(v) - F(u) \geq 0, \forall u,v \in K.$$

We also need the following assumptions regarding the bifunction $G(.,.)$.

Condition N. Let $G(.,.)$ satisfy the assumptions

$$G(u, u+t(v-u)) = -t^2 G(v,u)$$
$$G(v, u+t(v-u)) = (1-t)^2 G(v,u), \quad \forall u,v \in K, t \in [0,1].$$

Clearly for $t = 0$, we have $G(u,u) = 0$. Thus, it is clear that $G(u,v) = 0$, if and only if, $u = v, \forall u,v \in K$. One can easily show that $G(u+t(v-u),u) = t^2 G(v,u), \forall u,v \in K$.

3 Main results

In this section, we consider some basic properties of generalized strongly convex functions.

Theorem 1. *Let F be a differentiable function on the convex set K and Condition N hold. Then the function F is generalized strongly convex function with respect to the non-negative bifunction $G(.,.)$, if and only if,*

$$F(v)-F(u)\geq\langle F'(u),v-u\rangle+\mu G(v,u),\forall v,u\in K. \quad (5)$$

Proof. Let F be a generalized strongly convex function on the convex set K. Then

$$F(u+t(v-u))\leq(1-t)F(u)+tF(v)$$
$$-t(1-t)\mu G(v,u),\quad \forall u,v\in K,$$

which can be written as

$$F(v)-F(u)\geq\frac{F(u+t(v-u)-F(u)}{t}+(1-t)\mu G(v,u).$$

Taking the limit in the above inequality as $t\to 0$, we have

$$F(v)-F(u)\geq\langle F'(u),v-u)\rangle+\mu G(v,u),$$

which is (5).

Conversely, let (5) hold. Then $\forall u,v\in K,t\in[0,1]$, $v_t=u+t(v-u)\in K$ and using Condition N, we have

$$F(v)-F(v_t)\geq\langle F'(v_t),v-v_t)\rangle+\mu G(v,v_t)$$
$$=(1-t)F'(v_t),v-u\rangle+\mu(1-t)^2G(v,u)(6)$$

In a similar way, we have

$$F(u)-F(v_t)\geq\langle F'(v_t),u-v_t)\rangle+\mu G(u,v_t)$$
$$=-tF'(v_t),v-u\rangle+\mu t^2G(v,u). \quad (7)$$

Multiplying (6) by t and (7) by $(1-t)$ and adding the resultant, we have

$$F(u+t\eta(v,u))\leq(1-t)F(u)+tF(v)-t(1-t)\mu G(v,u),$$

showing that F is a generalized strongly convex function. □

Theorem 2. *Let F be differentiable on the convex set K and let Condition N hold. Then, (5) holds, if and only if, F' is generalized strongly G-monotone.*

*Proof.*Let F be a generalized strongly convex function on the convex set K. Then, from Theorem 3.1, we have

$$F(v)-F(u)\geq\langle F'(u),v-u\rangle+\mu G(v,u)\quad \forall u,v\in K. \quad (8)$$

Changing the role of u and v in (8), we have

$$F(u)-F(v)\geq\langle F'(v),u-v)\rangle+\mu G(u,v)\quad \forall u,v\in K. \quad (9)$$

Adding (8) and (9), we have

$$\langle F'(u)-F'(v),u-v\rangle\geq\mu\{G(v,u)+G(u,v)\}, \quad (10)$$

which shows that F' is generalized strongly G-monotone.

Conversely, let F' be generalized strongly G-monotone. From (10), we have

$$\langle F'(v),u-v\rangle\leq\langle F'(u),u-v)\rangle$$
$$-\mu\{G(v,u))+G(u,v)\}, \quad (11)$$

Since K is a convex set, $\forall u,v\in K$, $t\in[0,1]$ $v_t=u+t(v-u)\in K$. Taking $v=v_t$ in (11) and using Condition N, we have

$$\langle F'(v_t),u-v_t\rangle\leq\langle F'(u),u-v_t\rangle-\mu\{G(v_t,u)+G(u,v_t)\}$$
$$=-t\langle F'(u),v-u\rangle-2t^2\mu G(v,u)),$$

which implies that

$$\langle F'(v_t),v-u\rangle\geq\langle F'(u),v-u\rangle+2t\mu G(v,u). \quad (12)$$

Consider the auxiliary function

$$g(t)=F(u+t(v-u),$$

from which, we have

$$g(1)=F(v),\quad g(0)=F(u).$$

Then, from (12), we have

$$g'(t)=\langle F'(v_t,v-u\rangle\geq\langle F'(u),v-u\rangle+2\mu tG(v,u). \quad (13)$$

Integrating (13) between 0 and 1, we have

$$g(1)-g(0\,)=\int_0^1 g'(t)dt$$
$$\geq\langle F'(u),v-u\rangle+\mu G(v,u).$$

Thus it follows that

$$F(v)-F(u)\geq\langle F'(u),v-u\rangle+\mu G(v,u),$$

which is the required (5). □

We now give a necessary condition for generalized strongly G-pseudo-convex function.

Theorem 3. *Let F' be generalized strongly relaxed G-pseudomonotone and Condition N hold. Then F is a generalized strongly G-pseudo-invex function.*

Proof. Let F' be a generalized strongly relaxed G-pseudomonotone. Then, $\forall u,v\in K$,

$$\langle F'(u),v-u\rangle\geq 0.$$

implies that

$$\langle F'(v),v-u\rangle\geq\mu G(u,v). \quad (14)$$

Since K is a convex set, $\forall u,v\in K,\quad t\in[0,1]$, $v_t=u+t(v-u)\in K$. Taking $v=v_t$ in (14) and using condition Condition N, we have

$$\langle F'(v_t),v-u\rangle\geq t\mu G(v,u)). \quad (15)$$

Consider the auxiliary function

$$g(t)=F(u+t(v-u))=F(v_t), \quad \forall u,v\in K, t\in[0,1],$$

which is differentiable, since F is differentiable function. Then, using (15), we have

$$g'(t)=\langle F'(v_t),v-u\rangle\geq t\mu G(v,u)).$$

Integrating the above relation between 0 to 1, we have

$$g(1)-g(0)=\int_0^1 g'(t)dt\geq\frac{\mu}{2}G(v,u),$$

that is,

$$F(v)-F(u)\geq\frac{\mu}{2}G(v,u),$$

showing that F is a generalized strongly G-pseudo-convex function. □

Definition 8. *A function F is said to be sharply generalized strongly pseudo convex, if there exists a constant $\mu>0$ such that*

$$\langle F'(u),v-u\rangle\geq 0$$
$$\Rightarrow$$
$$F(v)\geq F(v+t(u-v))+\mu t(1-t)G(v,u.$$

Theorem 4. *Let F be a sharply generalized strongly pseudo convex function on K with a constant $\mu>0$. Then*

$$\langle F'(v),v-u\rangle\geq\mu G(v,u),\forall u,v\in K.$$

Proof. Let F be a sharply generalized strongly pesudo convex function on K. Then

$$F(v)\geq F(v+t(u-v))+t(1-t)\mu G(v,u),\forall u,v\in K, t\in[0,1],$$

from which, we have

$$\frac{F(v+t(u-v)-F(v)}{t}+\mu(1-t)G(v,u)\leq 0.$$

Taking limit in the above inequality, as $t\to 0$, we have

$$\langle F'(v),v-u\rangle\geq\mu G(v,u),$$

which is the required result.

Definition 9. *A function F is said to be a pseudo convex function, if there exists a strictly positive bifunction $b(.,.)$, such that*

$$F(v)<F(u)$$
$$\Rightarrow$$
$$F(u+t\,l(v,u))<F(u)+t(t-1)b(v,u),\forall u,v\in K, t\in[0,1].$$

Theorem 5. *If the function F) is generalized strongly convex function such that $F(v)<F(u)$, then the function F is a generalized strongly pseudo convex.*

Proof. Since $F(v)<F(u)$ and F is generalized strongly convex function, then $\forall u,v\in K,\quad t\in[0,1]$, we have

$$\begin{aligned} & F(u+t\,l(v,u))\leq F(u)+t(F(v)-F(u)) \\ & -\mu t(1-t)G(v,u) \\ < & F(u)+t(a-t)(F(v)-F(u))-\mu t(1-t)G(v,u) \\ = & F(u)+t(t-1)(F(u)-F(v))-\mu t(1-t)G(v,u) \\ < & F(u)+t(t-1)b(u,v)-\mu t(1-t)G(v,u), \end{aligned}$$

where $b(u,v)=F(u)-F(v)>0$, which is the required result.

We now discuss the optimality for the differentiable generalized strongly convex functions, which is the main motivation of our next result.

Theorem 6. *Let F be a differentiable generalized strongly convex function with modulus $\mu>0$. If $u\in K$ is the minimum of the function F, then*

$$F(v)-F(u)\geq\mu G(v,u), \quad \forall u,v\in K. \tag{16}$$

Proof. Let $u\in K$ be a minimum of the function F. Then

$$F(u)\leq F(v),\forall v\in K. \tag{17}$$

Since K is a convex set, so, $\forall u,v\in K,\quad t\in[0,1]$,

$$v_t=(1-t)u+tv\in K.$$

Taking $v=v_t$ in (17), we have

$$0\leq\lim_{t\to 0}\frac{F(u+t(v-u))-F(u)}{t}=\langle F'(u),v-u\rangle. \tag{18}$$

Since F is differentiable generalized strongly convex function, so

$$\begin{aligned} F(u+t(v-u)) \leq & F(u)+t(F(v)-F(u)) \\ & -\mu t(1-t)G(v,u), \quad u,v\in K, t\in[0,1], \end{aligned}$$

from which, using (18), we have

$$\begin{aligned} F(v)-F(u) & \geq\lim_{t\to 0}\frac{F(u+t(v-u))-F(u)}{t}+\mu G(v,u). \\ & =\langle F'(u),v-u\rangle+\mu G(v,u) \end{aligned}$$

which is the required result (16).

Remark: We would like to mention that, if

$$\langle F'(u),v-u\rangle+\mu G(v,u)\geq 0, \quad \forall u,v\in K,$$

then $u\in K$ is a minimum of the function F.

It is well known that each strongly convex functions is of the form $f+\|.\|^2$, where f is a convex function. Similar result is proved for the generalized strongly convex functions. In this direction, we have:

Theorem 7. *Let f be a generalized strongly affine function with respective to an arbitrary bifunction $G(.,.)$. Then F is a generalized strongly convex function with respect to the same arbitrary bifurcation $G(.,.)$, if and only if, $g = F - f$ is a convex function.*

Proof. Let f be generalized strongly affine function with respect to the arbitrary bifunction $G(.,.)$. Then

$$f((1-t)u+tv) = (1-t)f(u)+tf(v)-\mu t(1-t)G(v,u). \quad (19)$$

From the generalized strongly convexity of F, we have

$$F((1-t)u+tv) \leq (1-t)F(u)+tF(v)-\mu t(1-t)G(v,u). \quad (20)$$

From (19) and (20), we have

$$F((1-t)u+tv) - f((1-t)f(u)+tf(v) \leq (1-t)(F(u)-f(u))+t(F(v)-f(v)), \quad (21)$$

from which it follows that

$$g((1-t)u+tv) = F((1-t)u+tv)-f((1-t)f(u)+tf(v) \leq (1-t)(F(u)-f(u))+t(F(v)-f(v)),$$

which shows that $g = F - f$ is a convex function.
The inverse implication is obvious.

We would like to remark that one can show that a function F is a generalized strongly convex function, if and only if, F is generalized strongly affine function essentially using the technique of Adamek [2] and Noor et al. [21]. It can be shown that the generalized strongly convex functions involving the arbitrary bifunction is a Wright generalized strongly convex functions involving the arbitrary bifunction.

4 Conclusion

In this paper, we have introduced and studied a new class of convex functions with respect to any arbitrary bifunction. It is shown that several new classes of strongly convex functions can be obtained as special cases of these generalized strongly convex functions for suitable and appropriate choice of the arbitrary bifunction $G(.,.)$. We have studied the basic properties of these functions. The interested readers may explore the applications and other properties of the generalized strongly convex functions in various fields of pure and applied sciences.

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