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Euclidean Jordan Algebras, Strongly Regular Graphs and Cauchy Schwarz Inequalities

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Abstract: Let G be a primitive strongly regular Graph of order n and A its matrix of adjacency and let $\mathscr{A}$ be the Euclidean Jordan subalgebra of the Euclidean Jordan algebra of real symmetric matrices of order n equipped with the Jordan product of matrices and with the inner product of two matrices being the usual trace of them, spanned by the identity of order n and the natural powers of A. In this paper we establish some admissibility asymptotic conditions on the parameters and on the spectra of G, and next some admissibility conditions are established recurring to the inequality of Cauchy-Schwarz and recurring to the Frobenius norm.

Keywords: Euclidean Jordan Algebras, Strongly Regular Graphs, Admissibility Conditions

1 Introduction

Several applications of Euclidean Jordan algebras were developed on the various fields of science, namely on establishing the formalism of quantum mechanics [1], on the generalization of the theory of matrices to simple Euclidean Jordan algebras, [2-7], on establishing admissibility conditions on the spectra of a strongly regular graph [8-10], on the construction of theory for interior-point methods [11-17], and on developing applications to statistics and to probability, [18-20].

In this work we first establish some asymptotic inequalities over the spectra of a strongly regular graph, and next we deduce some inequalities involving the parameters and the spectra of a strongly regular graph recurring to the inequality of Cauchy-Schwarz in the environment of Euclidean Jordan algebras.

This paper is organized as follows. In the section 2, we present the main concepts and theorems about Jordan algebras and Euclidean Jordan algebras and some examples. In section 3 we expose the main concepts and results about strongly regular graphs needed for a clear understanding of this paper. In section 4, we establish, in an algebraic asymptotic way, inequalities over the spectra of a strongly regular graph, after associating a three-dimensional real Euclidean Jordan algebra $\mathscr{A}$ to it's adjacency matrix; and finally we deduce some inequalities involving the parameters and the spectra of a strongly regular graph recurring to the Cauchy-Schwarz inequality applied to two elements of $\mathscr{A}$. Finally, in section 5 we present some conclusions.

2 Main Results on Euclidean Jordan Algebras

In this section the main concepts about Euclidean Jordan algebras are presented. Good expositions about Jordan algebras and Euclidean Jordan algebras can be found, in between other works, in the books of James D. Malley, [21] and of Kevin McCrimmom [22], in the monograph by Faraut and Korányi, [23], and in Koecher's lecture notes [24].

Now, we only present the principal results needed for a clear understanding of this paper. Let $\mathscr{A}$ be a $n-$dimensional algebra over a field $\mathbb{K}$ with a bilinear map $\bullet$. We consider the notation $u^{2\bullet} = u \bullet u$ for any element u in $\mathscr{A}$. Then, $\mathscr{A}$ is a Jordan algebra if for all $u, v \in \mathscr{A}$ we have:

i)$u \bullet v = v \bullet u$,
ii)$u \bullet (u^{2\bullet} \bullet v) = u^{2\bullet} \bullet (u \bullet v)$.

The properties $i)$ and $ii)$ can be replaced by the properties $j1)$ and $j2)$,

$j_1) \forall u, v \in \mathscr{A}, L(u)(v) = L(v)(u)$

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$j_2)\forall u,v \in \mathscr{A}, L(u)(L(u^{2\bullet})(v)) = L(u^{2\bullet})(L(u)(v))$, where the operators $L(u)$ and $L(u^{2\bullet})$ are the endomorphisms: $\begin{array}{l} L(u): \mathscr{A} \mapsto \mathscr{A} \\ \quad v \quad \mapsto u \bullet v \end{array}$ and $\begin{array}{l} L(u^{2\bullet}): \mathscr{A} \mapsto \mathscr{A} \\ \quad v \quad \mapsto u^{2\bullet} \bullet v. \end{array}$

We always assume that $\mathbb{K}$ has not characteristic 2, and we suppose in the later sections that $\mathbb{K}$ is usually $\mathbb{R}$ or $\mathbb{C}$. In the case that the field $\mathbb{K}$ is the field of real numbers we call the Jordan algebra $\mathscr{A}$ a real Jordan.

Let $\mathscr{A} = \mathrm{Sym}(n,\mathbb{R})$ be the space of real symmetric matrices of order n and let's consider on $\mathscr{A}$ the product $u \bullet v = \frac{1}{2}(uv + vu)$, where uv and vu represent the usual product of the matrix u by the matrix v, and of the matrix v by the matrix u respectively. We define the following notation for the powers of an element u in $\mathscr{A}$: $u^{0\bullet} = I_n, u^{1\bullet} = u$ and $u^{k+1\bullet} = u \bullet u^{k\bullet}$ for any natural number k. Here, we must say that the usual power of order n, u^n coincides with the power $u^{n\bullet}$ for any natural n. Then $\mathscr{A}$ with this product is a real Jordan algebra. Indeed let u and v be elements of $\mathscr{A}$, then we have $u \bullet v = \frac{uv+vu}{2} = \frac{vu+uv}{2} = v \bullet u$.

We have

$$\begin{aligned} u \bullet (u^{2\bullet} \bullet v) &= \frac{u\left(u^{2\bullet} \bullet v\right) + \left(u^{2\bullet} \bullet v\right)u}{2} \\ &= \frac{u\left(\frac{u^{2\bullet}v + vu^{2\bullet}}{2}\right) + \left(\frac{u^{2\bullet}v + vu^{2\bullet}}{2}\right)u}{2} \\ &= \frac{u\left(\frac{u^{2}v + vu^{2}}{2}\right) + \left(\frac{u^{2}v + vu^{2}}{2}\right)u}{2} \\ &= \frac{\frac{u^3v + uvu^2 + u^2vu + vu^3}{2}}{2} \\ &= \frac{\frac{u^3v + u^2vu + uvu^2 + vu^3}{2}}{2} \\ &= \frac{u^2\left(\frac{uv+vu}{2}\right) + \left(\frac{uv+vu}{2}\right)u^2}{2} \\ &= \frac{u^{2\bullet}(u \bullet v) + (u \bullet v)u^{2\bullet}}{2} \\ &= u^{2\bullet} \bullet (u \bullet v). \end{aligned}$$

Some Jordan algebras are obtained in a natural way from associative algebras. If in an associative algebra $\mathscr{A}$ we change his product $u \bullet v$ for u and v in $\mathscr{A}$ by the product $u \star v = \frac{u \bullet v + v \bullet u}{2}$ we obtain a new Jordan Algebra $\mathscr{A}^+$. We call a Jordan algebra $\mathscr{B}$ special if $\mathscr{B}$ is isomorphic to a subalgebra of an algebra $\mathscr{A}^+$. Otherwise if the Jordan algebra $\mathscr{B}$ is not isomorphic to a subalgebra of the Jordan algebra $\mathscr{A}^+$, then for any associative algebra $\mathscr{A}$ is called an exceptional Jordan algebra.

From now on, when we say let $\mathscr{A}$ be a Jordan algebra we suppose that $\mathscr{A}$ is finite dimensional real Jordan algebra and has a unit element denoted by $\mathbf{e}$. Let $\mathscr{A}$ be a n dimensional Jordan algebra. Then $\mathscr{A}$ is power associative, this is an algebra such that for any u in $\mathscr{A}$ the algebra spanned by u and $\mathbf{e}$ is associative. Therefore, we define for any element u in $\mathscr{A}$ the power of u in the following way $u^{0\bullet} = \mathbf{e}, u^{1\bullet} = u$ and $u^{(k+1)\bullet} = u \bullet u^{k\bullet}$ for any natural number k.

The rank of an element u in $\mathscr{A}$ is the least natural number l such that $\{u^{0\bullet}, u^{1\bullet}, u^{2\bullet}, \cdots, u^{l\bullet}\}$ is linearly dependent and we write $\mathrm{rank}(u) = l$. Since $\mathrm{rank}(u) \leq n$ the rank of $\mathscr{A}$ is defined as being the natural number $\mathrm{rank}(\mathscr{A}) = \max\{\mathrm{rank}(u) : u \in \mathscr{A}\}$. An element u in $\mathscr{A}$ is regular if $r = \mathrm{rank}(u) = \mathrm{rank}(\mathscr{A})$. Let u be a regular element of $\mathscr{A}$ and $r = \mathrm{rank}(u)$. Then, there exist real scalars $\alpha_1(u), \alpha_2(u), \ldots, \alpha_{r-1}(u)$ and $\alpha_r(u)$ such that

$$u^{r\bullet} - a_1(u)u^{r-1\bullet} + \cdots + (-1)^r a_r(u)\mathbf{e} = 0, \tag{1}$$

where 0 is the zero vector of $\mathscr{A}$. Taking into account (1) we conclude that the polynomial $p(u,-)$ such that

$$p(u,\lambda) = \lambda^r - \alpha_1(u)\lambda^{r-1} + \cdots + (-1)^r \alpha_r(u) \tag{2}$$

is the minimal polynomial of u. We call to the polynomial $p(u,\lambda)$ the characteristic polynomial of u. The coefficients $\alpha_i(u)$ for $i = 1, \cdots, r$ of the polynomial $p(u,-)$ are homogeneous polynomials in the coordinates of u on a fixed basis of $\mathscr{A}$. The coefficient $\alpha_1(u)$ is called the trace of u, which we denote by $\mathrm{tr}(u)$, and the coefficient $\alpha_r(u)$ is called the determinant of u, which we denote by $\det(u)$. This is natural, since when u is a regular element of $\mathscr{A}$, then in the subalgebra $\mathbb{R}[u]$ of $\mathscr{A}$, spanned by u^0 and the natural powers of u, then the trace and the determinant of the matrix of the linear application $L_0(u)$, from $\mathbb{R}[u]$ onto $\mathbb{R}[u]$, such that $L_0(u)(v) = u \bullet v$ relatively to the basis $\mathscr{B} = \{u^{0\bullet}, u^{1\bullet}, u^{2\bullet}, \cdots, u^{r-1\bullet}\}$ is respectively $\alpha_1(u)$ and $\alpha_r(u)$. Since the set of regular elements of $\mathscr{A}$ is a dense set in $\mathscr{A}$ then we extend by continuity to the non regular elements of $\mathscr{A}$ the definitions of characteristic polynomial, trace and determinant. For any u in $\mathscr{A}$, we call the roots of the characteristic polynomial $p(u,-)$ of u the eigenvalues of u.

A real Euclidean Jordan algebra $\mathscr{A}$ is a Jordan algebra with an inner product $< \cdot,\cdot >$ such that for any u, v and w in $\mathscr{A}$, $< L(u)v, w > = < v, L(u)w >$. Now, we show that the real vector space of real symmetric matrices of order n, $\mathrm{Sym}(n,\mathbb{R})$, is a real Euclidean Jordan algebra endowed with the Jordan product and the inner product defined by $< u, v > = \mathrm{trace}(u \bullet v)$, where trace denotes the usual trace of matrices of order n. The unit element in this case is the identity matrix of order n, I_n. Indeed, let u, v and $w \in \mathscr{A} = \mathrm{Sym}(n,\mathbb{R})$, then we have

$$\begin{aligned} &< L(u)v, w > = \mathrm{trace}\left((u \bullet v) \bullet w\right) \\ &= \mathrm{trace}\left(\frac{\frac{uv+vu}{2}w + w\frac{uv+vu}{2}}{2}\right) \\ &= \mathrm{trace}\left(\frac{(uv)w}{4} + \frac{(vu)w}{4} + \frac{w(uv)}{4} + \frac{w(vu)}{4}\right) \\ &= \mathrm{trace}\left(\frac{(vu)w}{4}\right) + \mathrm{trace}\left(\frac{w(uv)}{4}\right) + \mathrm{trace}\left(\frac{(uv)w}{4}\right) + \mathrm{trace}\left(\frac{w(vu)}{4}\right) \\ &= \mathrm{trace}\left(\frac{v(uw)}{4}\right) + \mathrm{trace}\left(\frac{(wu)v}{4}\right) + \mathrm{trace}\left(\frac{w(uv)}{4}\right) + \mathrm{trace}\left(\frac{(vu)w}{4}\right) \\ &= \mathrm{trace}\left(\frac{v(uw)}{4}\right) + \mathrm{trace}\left(\frac{v(wu)}{4}\right) + \mathrm{trace}\left(\frac{(wu)v}{4}\right) + \mathrm{trace}\left(\frac{v(uw)}{4}\right) \\ &= \mathrm{trace}\left(\frac{v(uw)}{4}\right) + \mathrm{trace}\left(\frac{v(wu)}{4}\right) + \mathrm{trace}\left(\frac{(wu)v}{4}\right) + \mathrm{trace}\left(\frac{(uw)v}{4}\right) \\ &= \mathrm{trace}\left(\frac{v\left(\frac{uw+wu}{2}\right)}{2}\right) + \mathrm{trace}\left(\frac{\left(\frac{uw+wu}{2}\right)v}{2}\right) \\ &= \ \mathrm{trace}\left(\frac{v(u \bullet w)}{2} + \frac{(u \bullet w)v}{2}\right) \\ &= \mathrm{trace}(v \bullet (u \bullet w)) \\ &= \mathrm{trace}(v \bullet L(u)w) \\ &= < v, L(u)w > . \end{aligned}$$

Let $\mathscr{A}$ be a real Euclidean Jordan algebra with unit element $\mathbf{e}$. An element u in $\mathscr{A}$ is an idempotent if $u^{2\bullet} = u$.

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Two idempotents u and v are orthogonal if $u \bullet v = 0$. The set $\{u_1, u_2, \ldots, u_k\}$ of nonzero idempotents is a complete system of orthogonal idempotents of $\mathscr{A}$ if and only if $u_i^{2\bullet} = u_i$, for $i = 1, \ldots, k$, $u_i \bullet u_j = 0$ if $i \neq j$ and $\sum_{i=1}^{k} u_i = \mathbf{e}$. An idempotent u is primitive if it is a nonzero idempotent of $\mathscr{A}$ and if it can't be written as a sum of two nonzero idempotents. We say that $\{u_1, u_2, \ldots, u_m\}$ is a Jordan frame of $\mathscr{A}$ if $\{u_1, u_2, \ldots, u_m\}$ is a complete system of orthogonal of idempotents of $\mathscr{A}$ such that each idempotent is primitive.

*Example 1.*Let u be a matrix of the Euclidean Jordan algebra $\mathscr{A} = \mathrm{Sym}(n, \mathbb{R})$ with l distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{l-1}$ and λ_l then the set $S = \{u_1, u_2, \cdots, u_l\}$ is a complete system of orthogonal idempotents of $\mathscr{A}$ where each idempotent u_i for $i = 1, \cdots, l$ is the projector on the eigenvector space of u associated to the eigenvalue λ_i defined by the equality

$$u_i = \prod_{j=1, j\neq i}^{l} \frac{\lambda_j I_n - u}{\lambda_j - \lambda_i}$$

for $i = 1, \cdots, l$. We must say that they are unique, and we have $u_1 + u_2 + \cdots + u_l = I_n$, $u_i \bullet u_j = 0$ for $i \neq j$ and $i, j \in \{1, \cdots, l\}$ where $\bullet$ is the Jordan product of matrices, $u_i^{2\bullet} = u_i$ for $i = 1, \cdots, l$ and we have $u = \sum_{j=1}^{l} \lambda_j u_j$.

*Example 2.*Let's consider the Euclidean Jordan Algebra $\mathscr{A} = \mathrm{Sym}(n, \mathbb{R})$ with the Jordan product $u \bullet v = \frac{uv+vu}{2}$ and the inner product $u|v = \mathrm{trace}(u \bullet v)$. Then $S = \{E_{11}, \cdots, E_{nn}\}$ where the matrix E_{ii} is such that $(E_{ii})_{ii} = 1$ and $(E_{ii})_{\alpha\beta} = 0$ if $\alpha \neq i$ or $\beta \neq i$ is a Jordan frame of $\mathscr{A}$.

Theorem 2.1([23], p. 43).
Let $\mathscr{V}$ be a real Euclidean Jordan algebra. Then for v in $\mathscr{V}$ there exist unique real numbers $\lambda_1(v), \lambda_2(v), \ldots, \lambda_k(v)$, all distinct, and a unique complete system of orthogonal idempotents $\{u_1, u_2, \ldots, u_k\}$ such that

$$v = \lambda_1(v)u_1 + \lambda_2(v)u_2 + \cdots + \lambda_k(v)u_k. \tag{3}$$

The numbers $\lambda_j(v)$'s of (3) are the eigenvalues of v and the decomposition (3) is the first spectral decomposition of v.

Theorem 2.2([23], p. 44)
Let $\mathscr{V}$ be a real Euclidean Jordan algebra with $\mathrm{rank}(\mathscr{V}) = r$. Then for each v in $\mathscr{V}$ there exists a Jordan frame $\{u_1, u_2, \cdots, u_r\}$ and real numbers $\lambda_1(v), \cdots, \lambda_{r-1}(v)$ and $\lambda_r(v)$ such that

$$v = \lambda_1(v)u_1 + \lambda_2(v)u_2 + \cdots + \lambda_r(v)u_r. \tag{4}$$

And, we have $\det(v) = \prod_{i=1}^{r} \lambda_i(v)$ and $\mathrm{tr}(v) = \sum_{i=1}^{r} \lambda_i(v)$. The decomposition (4) is called the second spectral decomposition of v. The distinct $\lambda_j(v)$s are the eigenvalues of v.

*Example 3.*In the Euclidean Jordan algebra $\mathscr{A} = \mathrm{Sym}(n, \mathbb{R})$ the second spectral decomposition of a matrix u that has an orthonormal basis $\mathscr{B} = \{u_1, u_2, \cdots, u_n\}$ of eigenvectors of u such that $uu_i = \lambda_i u_i$ for $i = 1, \cdots, n$ has the spectral decomposition $u = \sum_{i=1}^{n} \lambda_i u_i u_i^T$. We must say that $\{u_1 u_1^T, u_2 u_2^T, \cdots, u_n u_n^T\}$ is a Jordan frame of $\mathscr{A}$.

*Remark.*Herein, we must say that any Jordan frame on a finite dimensional Euclidean Jordan algebra has the same number of elements and is equal to the rank of the algebra. Let's consider an Euclidean Jordan algebra such that $\mathrm{rank}(\mathscr{A}) = r$ and u in $\mathscr{A}$ with the second spectral decomposition $u = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_r u_r$ where $\{u_1, u_2, \cdots, u_r\}$ is a Jordan frame. Then the Frobenius norm of u is $||u||_F = \sqrt{\mathrm{tr}(u \bullet u)} = \sqrt{\sum_{i=1}^{r} \lambda_i^2}$.

3 Graphs and Strongly Regular Graphs

The results presented in this section can be founded in a more detailed way on the very readable book Spectra of graphs written by A. E. Brouwer and W. H. Haemers, see [25] and in the book Algebraic graph theory written by C. Godsil and G. F. Royle, see [26].

Along this paper we consider only non-empty, simple herein called graphs. By simple graphs we mean graphs without loops and parallel edges.

We denote the set of vertices of a graph X by $V(X)$ and the set of edges of X by $E(X)$ and the number of vertices of X, $|V(X)|$ is called the order of X. An edge whose endpoints are the vertices u and v is represented by uv and, in this case, the vertices u and v are adjacent or neighbors.

A graph in which all pairs of vertices are adjacent is called a complete graph and a graph such that all pair of vertices is non adjacent is called a null graph. The number of neighbors of a vertex u in $V(X)$ is the degree of u. If all vertices of a graph X have the same degree k then X is a $k-$regular graph. The adjacency matrix of a graph X of order n is a matrix $A = [a_{ij}]$ such that $a_{ij} = 1$, if $u_i u_j \in E(X)$, otherwise $a_{ij} = 0$. We define the eigenvalues of X as being the eigenvalues of A.

We observe that a graph X is $k-$regular if and only if his adjacency matrix has an eigenvector with all the coordinates equal to 1 associated to k.

A non-null and non-complete graph X it is a $(n, k; \lambda, \mu)-$ strongly regular graph if it is a $k-$regular graph, and each pair of adjacent vertices have λ common neighbors and each pair of non-adjacent vertices have μ common neighbors.

The complement of a simple graph X denoted by $\overline{X}$ is a simple graph that has the same set of vertices of X and such that two distinct vertices of $\overline{X}$ are adjacent if and only if they are non adjacent vertices of X. A graph X is a $(n, k; \lambda, \mu)-$strongly regular graph if and only if $\overline{X}$ is a

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$(n,n-k-1;n-2-2k+\mu,n-2k+\lambda)-$strongly regular graph.

A graph X is a $(n,k;\lambda,\mu)-$strongly regular graph if and only if its adjacency matrix A verifies the inequality (5).

$$A^2 = kI_n + \lambda A + \mu(J_n - A - I_n). \tag{5}$$

The parameters of a strongly regular graph are related by the equality (6).

$$k(k-\lambda-1) = (n-k-1)\mu. \tag{6}$$

It is known, see for instance [25,26], that the eigenvalues of a $(n,k;\lambda,\mu)-$strongly regular graph X are k,θ and τ which are given by the relations (7) and (8).

$$\theta = \frac{\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} \tag{7}$$

$$\tau = \frac{\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}. \tag{8}$$

Therefore, the restricted eigenvalues θ and τ are such that the former is the positive eigenvalue and the latter is the negative eigenvalue of X. Their multiplicities f_θ and f_τ, are defined by the equalities (9) and (10).

$$f_\theta = \frac{|\tau|n+\tau-k}{\theta-\tau} \tag{9}$$

$$f_\tau = \frac{\theta n+k-\theta}{\theta-\tau}. \tag{10}$$

If the parameter set $(n,k;\lambda,\mu)$ is a parameter set of a strongly regular graph then the Krein conditions obtained in [27] that we present in the inequalities (11) and (12) hold,

$$(\theta+1)(k+\theta+2\theta\tau) \leq (k+\theta)(\tau+1)^2 \tag{11}$$

$$(\tau+1)(k+\tau+2\theta\tau) \leq (k+\tau)(\theta+1)^2. \tag{12}$$

and the Seidel's absolute bounds(see [28]) that are the relations (13) and (14).

$$n \leq \frac{f_\theta(f_\theta+3)}{3} \tag{13}$$

$$n \leq \frac{f_\tau(f_\tau+3)}{3}. \tag{14}$$

Given a graph X, we denote by a path in X between two vertices v_1 and v_{k+1} to a non null sequence of vertices(distinct) and edges(distinct) $W = v_1e_1v_2e_2v_3\cdots v_ke_kv_{k+1}$ whose terms are vertices and edges alternated and such that for $1 \leq i \leq k$ the vertices v_i and v_{i+1} define the edge e_i.

A graph X' is a subgraph of a graph X and we write $X' \sqsubseteq X$ if $V(X') \sqsubseteq V(X)$ and $E(X') \sqsubseteq E(X)$. If $X' \neq X$, we say that X' is a proper subgraph of X. We must observe that for any non empty subset V' of $V(X)$ we can construct a subgraph of X whose set of vertices is V' and such that $E(X') \sqsubseteq E(X)$, and $uv \in E(X')$ if and only if $u \in V'$ and $v \in V'$, which we call the induced subgraph of X and which we denote by $X(V')$. Two vertices v_1 and v_2 of a graph X are connected if there is a path between v_1 and v_2 in X. This relation between vertices is a relation of equivalence in the set of vertices of the graph $X, V(X)$, whereby there exists a partition of $V(X)$ in non-empty subsets $V_1, V_2, \cdots, V_l$ of $V(X)$ such that two vertices are connected if and only if they belong to the same set V_i for a given $i \in \{1,2,\cdots,l\}$. The subgraphs $X(V_1), X(V_2), \cdots, X(V_l)$ are called the connected components of X. If X has only one component then we say that the graph X is connected otherwise the graph X is a disconnected graph. A strongly regular graph X is primitive if and only if X and $\overline{X}$ are connected, otherwise the strongly regular graph X is called imprimitive.

Now, we must say, that a $(n,k;\lambda,\mu)-$strongly regular is imprimitive if and only if $\mu = 0$ or $\mu = k$. From now on, on the next sections we only consider primitive strongly regular graphs.

4 Some Asymptotic Conditions on the Parameters of a Strongly Regular Graph

Herein, we first introduce some notation needed for this section. We denote the set of real matrices of order n by $M_n(\mathbb{R})$. For two matrices $C = [c_{ij}]$ and $D = [d_{ij}]$ in $M_n(\mathbb{R})$ we define the Hadamard product of the matrices C and D as being the matrix of $M_n(\mathbb{R})$, $C \circ D = [c_{ij}d_{ij}]$ and we define the Kronecker product of the matrices C and D as being the matrix of $M_{n^2}(\mathbb{R})$, $C \otimes D = [c_{ij}D]$. We define the Hadamard powers of a matrix $Z \in M_n(\mathbb{R})$ in the following way: $Z^{0\circ} = J_n, Z^{1\circ} = X$ and $Z^{(1+j)\circ} = Z \circ Z^{j\circ}$ for any natural number j.

Let's consider a primitive $(n,k;\lambda,\mu)-$strongly regular graph X such that its adjacency matrix has three distinct eigenvalues k,θ and τ, and let's consider the Euclidean Jordan subalgebra $\mathscr{A}$ of the Jordan Euclidean Jordan algebra $\mathrm{Sym}(n,\mathbb{R})$ equipped with the Jordan product $u \bullet v = \frac{uv+vu}{2}$ and with the inner product $u|v = \mathrm{trace}(uv)$ for any symmetric matrices u and v of order n, spanned by the powers of I_n and A. We have $\mathrm{rank}(\mathscr{A}) = 3$ since A has three distinct eigenvalues. Now, consider the unique Jordan frame $\mathscr{B} = \{F_1, F_2, F_3\}$ of $\mathscr{A}$ associated to A, with $F_1 = 1/nI_n + 1/nA + 1/n(J_n - A - I_n)$, $F_2 = (|\tau|n+\tau-k)/(n(\theta-\tau))I_n + (n+\tau-k)/(n(\theta-\tau))A + (\tau-k)/(n(\theta-\tau))(J_n-A-I_n)$, and $F_3 = (\theta n+k-\theta)/(n(\theta-\tau))I_n + (-n+k-\theta)/(n(\theta-\tau))A + (k-\theta)/(n(\theta-\tau))(J_n-A-I_n)$.

Let y be a real positive number that's than one and let's consider the binomial Hadamard series $S_y = \sum_{j=0}^{+\infty}(-1)^j \binom{-y}{j} \left(\frac{A^2}{k+\mu}\right)^{j\circ}$. Now $S_y = \sum_{i=1}^{3} q_{yi}F_i$ is the second spectral decomposition of S_y respectively to the Jordan frame $\mathscr{B} = \{F_1, F_2, F_3\}$ of $\mathscr{A}$. We deduce that the eigenvalues q_{yi} of S_y are all positive. Since

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$(-1)^j\binom{-y}{j} = (-1)^j\frac{(-y)(-y-1)(-y-2)\cdots(-y-j+1)}{j!}$ then $(-1)^j\binom{-y}{j} = (-1)^{2j}\frac{(y)(y+1)(y+2)\cdots(y+j-1)}{j!} \geq 0$.

We have $S_{yn} = \sum_{j=0}^{n}(-1)^j\binom{-y}{j}\left(\frac{A^2}{k+\mu}\right)^{j\circ}$. Let's consider the second spectral decomposition of S_{yn} relatively to the Jordan frame $\mathscr{B}$, $S_{yn} = q_{yn1}F_1 + q_{yn2}F_2 + q_{yn3}F_3$.

Hence, since the eigenvalues of $\frac{A^2}{k+\mu}$ are positive and since we have that for any two real matrices of order n, V and W $\lambda_{\min}(V)\lambda_{\min}(W) \leq \lambda_{\min}(V\circ W)$ and since $\mathscr{B}$ is a Jordan frame of $\mathscr{A}$ that is a basis of $\mathscr{A}$ and $\mathscr{A}$ is closed for the Hadamard product then we deduce that the eigenvalues of S_{yn} are all positive.

Therefore, using the fact that $q_{y1} = \lim_{n\to+\infty} q_{yn1}, q_{y2} = \lim_{n\to+\infty} q_{yn2}$ and $q_{y3} = \lim_{n\to+\infty} q_{yn3}$ then we conclude that $q_{y1} \geq 0, q_{y2} \geq 0$ and $q_{y3} \geq 0$. We note that $S_yF_1 = q_{y1}F_1, S_yF_2 = q_{y2}F_2$ and $S_yF_3 = q_{y3}F_3$, and therefore we have

$$\begin{aligned} q_{y1} &= \frac{1}{(\frac{\mu}{k+\mu})^y} + \frac{1}{(\frac{k+\mu-\lambda}{k+\mu})^y}k + \\ &+ \frac{1}{(\frac{k}{k+\mu})^y}(n-k-1), \\ q_{y2} &= \frac{1}{(\frac{\mu}{k+\mu})^y} + \frac{1}{(\frac{k+\mu-\lambda}{k+\mu})^y}\theta + \\ &+ \frac{1}{(\frac{k}{k+\mu})^y}(-\theta-1) \\ q_{y3} &= \frac{1}{(\frac{\mu}{k+\mu})^y} + \frac{1}{(\frac{k+\mu-\lambda}{k+\mu})^y}\tau + \\ &+ \frac{1}{(\frac{k}{k+\mu})^y}(-\tau-1). \end{aligned}$$

Let's now consider the element $S_{y3} = F_3 \circ S_y$ of the Euclidean Jordan algebra $\mathscr{A}$. Since the eigenvalues of F_3 and of S_y are positive and since $\lambda_{\min}(F_3)\lambda_{\min}(S_y) \leq \lambda_{\min}(F_3\circ S_y)$ then the eigenvalues of $F_3\circ S_y$ are positive. Now, let's consider the second spectral decomposition $F_3\circ S_y = q_{y3}^1F_1 + q_{y3}^2F_2 + q_{y3}^3F_3$. Then, we have

$$\begin{aligned} q_{y3}^1 &= \frac{\theta n+k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{\mu}{k+\mu}\right)^y} + \frac{-n+k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k+\mu-\lambda}{k+\mu}\right)^y}k + \\ &+ \frac{k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k}{k+\mu}\right)^y}(n-k-1). \\ q_{y3}^2 &= \frac{\theta n+k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{\mu}{k+\mu}\right)^y} + \frac{-n+k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k+\mu-\lambda}{k+\mu}\right)^y}\theta \\ &+ \frac{k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k}{k+\mu}\right)^y}(-\theta-1). \end{aligned}$$

Supposing that $k < \frac{n}{2}$ and that $\lambda \geq \mu$ and by an asymptotical analysis of the spectrum of $F_3 \circ S_y$ we conclude the inequalities (15) and (23) of the Theorems **3.1.** and **3.2.**, respectively. From now on when we say let G be a $(n,k;\lambda,\mu)$-strongly regular graph we mean that G is primitive strongly regular graph.

Theorem 3.1. Let μ,λ,k and n be natural numbers such that $n-1 > k > \mu$ and let G be a $(n,k;\lambda,\mu)$-strongly regular graph such that $k < \frac{n}{2}$ and $\lambda \geq \mu$, and with the distinct eigenvalues τ,θ and k. Then

$$\left(\frac{k}{k+\mu-\lambda}\right)^k < \left(\frac{k}{\mu}\right)^{2\theta+1}. \tag{15}$$

Proof. Since $q_{y3}^1 \geq 0$ then we have

$$\begin{aligned} &\frac{\theta n+k-\theta}{n(\theta-\tau)}\cdot\frac{1}{\left(\frac{\mu}{k+\mu}\right)^y} + \frac{-n+k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k+\mu-\lambda}{k+\mu}\right)^y}k + \\ &+ \frac{k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k}{k+\mu}\right)^y}(n-k-1) \geq 0. \end{aligned} \tag{16}$$

Knowing that $\frac{\theta n+k-\theta}{n(\theta-\tau)} + \frac{-n+k-\theta}{n(\theta-\tau)}k + \frac{k-\theta}{n(\theta-\tau)}(n-k-1) = 0$ we conclude that

$$\begin{aligned} \frac{k-\theta}{n(\theta-\tau)}(n-k-1) = &-\frac{-n+k-\theta}{n(\theta-\tau)}k - \\ &-\frac{\theta n+k-\theta}{n(\theta-\tau)}. \end{aligned} \tag{17}$$

Now, recurring to the equality (17) to rewriting the inequality (16) we deduce the inequality (18).

$$\begin{aligned} &\frac{\theta n+k-\theta}{n(\theta-\tau)}\cdot\left(\frac{1}{\left(\frac{\mu}{k+\mu}\right)^y} - \frac{1}{\left(\frac{k}{k+\mu}\right)^y}\right) + \\ &+ \frac{-n+k-\theta}{n(\theta-\tau)}\left(\frac{1}{\left(\frac{k+\mu-\lambda}{k+\mu}\right)^y} - \frac{1}{\left(\frac{k}{k+\mu}\right)^y}\right)k \geq 0. \end{aligned} \tag{18}$$

After some algebraic manipulation of (18) we conclude that the equality (19) is verified.

$$\begin{aligned} 1 \geq &\frac{n-k+\theta}{\theta n+k-\theta}k\frac{\left(\frac{\mu}{k+\mu}\right)^y}{\left(\frac{k+\mu-\lambda}{k+\mu}\right)^y}\cdot \\ &\cdot\frac{\left(\frac{k}{k+\mu}\right)^y - \left(\frac{k+\mu-\lambda}{k+\mu}\right)^y}{\left(\frac{k}{k+\mu}\right)^y - \left(\frac{\mu}{k+\mu}\right)^y}. \end{aligned} \tag{19}$$

Now, since $k < \frac{n}{2}$ then we deduce that $\frac{n-k+\theta}{\theta n+k-\theta} > \frac{1}{2\theta+1}$. After making y tend to zero on the right hand side of (19) we obtain the inequality (20)

$$1 > \frac{k}{2\theta+1}\frac{\ln\left(\frac{k}{k+\mu}\right)-\ln\left(\frac{k+\mu-\lambda}{k+\mu}\right)}{\ln\left(\frac{k}{k+\mu}\right)-\ln\left(\frac{\mu}{k+\mu}\right)}. \tag{20}$$

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Then, from (20) we establish the inequality (21).

$$1 > \frac{k}{2\theta+1}\frac{\ln\left(\frac{k}{k+\mu-\lambda}\right)}{\ln(\frac{k}{\mu})}. \tag{21}$$

Hence, the inequality (22) is verified.

$$(2\theta+1)\ln\left(\frac{k}{\mu}\right) > k\ln\left(\frac{k}{k+\mu-\lambda}\right). \tag{22}$$

So, from (22) we conclude that $\left(\frac{k}{k+\mu-\lambda}\right)^k < \left(\frac{k}{\mu}\right)^{2\theta+1}$, this is (15) which is satisfied.

Theorem 3.2. Let λ,μ,k and n be natural numbers such that $n-1>k>\mu$ and let G be a $(n,k;\lambda,\mu)$-strongly regular graph such that $k<\frac{n}{2}$ and $\lambda\geq\mu$, with the distinct eigenvalues τ,θ and k. Then

$$\left(\frac{k}{\mu}\right)^{2\theta+1} < \left(\frac{k+\mu-\lambda}{k}\right)^{\theta}. \tag{23}$$

Proof. Now, since $q_{y3}^2\geq 0$ then we have the inequality (24).

$$\frac{\theta n+k-\theta}{n(\theta-\tau)}\cdot\frac{1}{\left(\frac{\mu}{k+\mu}\right)^y}+\frac{-n+k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k+\mu-\lambda}{k+\mu}\right)^y}\theta+ \\ +\frac{k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k}{k+\mu}\right)^y}(-\theta-1)\geq 0. \tag{24}$$

Since $\frac{\theta n+k-\theta}{n(\theta-\tau)}+\frac{-n+k-\theta}{n(\theta-\tau)}\theta+\frac{k-\theta}{n(\theta-\tau)}(-\theta-1)=0$ we conclude that (25) is verified.

$$\frac{k-\theta}{n(\theta-\tau)}(-\theta-1) = -\frac{-n+k-\theta}{n(\theta-\tau)}\theta- \\ -\frac{\theta n+k-\theta}{n(\theta-\tau)} \tag{25}$$

Proceeding, in the same way that we have followed in the proof of Theorem 3.1 and using the fact that $\frac{n-k+\theta}{\theta n+k-\theta}>\frac{1}{2\theta+1}$ when $k<\frac{n}{2}$ we deduce the inequality (26).

$$\ln\left(\frac{k}{\mu}\right) > \frac{\theta}{2\theta+1}\ln\left(\frac{k}{k+\mu-\lambda}\right). \tag{26}$$

Therefore we conclude that $\left(\frac{k}{k+\mu-\lambda}\right)^{\theta} < \left(\frac{k}{\mu}\right)^{2\theta+1}$, this is (23) which is established.

Now considering the norm of Frobenius $||x||_F = \sqrt{\sum_{i=1}^{3}\lambda_i^2}$ for $x=\lambda_1F_1+\lambda_2F_2+\lambda_3F_3$ and, recurring to the inequality of Cauchy Schwarz for the elements $(A^3-\frac{k^3}{n}J_n)\circ I_n$ and I_n of the Euclidean Jordan algebra $\mathscr{A}$ we establish the inequalities (27) of Theorem 3.3.

Theorem 3.3. Let λ,μ,k and n be natural numbers such that $n-1>k>\mu$ and G be a $(n,k;\lambda,\mu)$-strongly regular graph, with the distinct eigenvalues τ,θ and k. Then

$$(\lambda>\mu)\Rightarrow\left(\theta\geq\sqrt[3]{\left|\lambda k-\frac{k^3}{n}\right|}\right), \\ (\lambda<\mu)\Rightarrow\left(|\tau|\geq\sqrt[3]{\left|\lambda k-\frac{k^3}{n}\right|}\right). \tag{27}$$

Proof. Since G is a $(n,k;\lambda,\mu)$ strongly regular graph then its adjacency matrix verifies the inequality (28).

$$A^2 = kI_n+\lambda A+\mu(J_n-A-I_n). \tag{28}$$

Therefore, we conclude that

$$A^2 = (k-\mu)I_n+(\lambda-\mu)A+\mu J_n. \tag{29}$$

But, writing A^3 on the basis $\mathscr{B}_1=\{I_n,A,J_n-A-I_n\}$ of the Euclidean Jordan subalgebra $\mathscr{A}$ of the Euclidean Jordan algebra $\mathrm{Sym}(n,\mathbb{R})$ spanned by I_n and the natural powers of A we obtain the following calculations:

$$\begin{aligned}A^3 &= (k-\mu)A+(\lambda-\mu)A^2+\mu AJ_n \\ &= (k-\mu)A+(\lambda-\mu)(kI_n+\lambda A+\mu(J_n-A-I_n))+ \\ &+\mu kJ_n \\ &= (k-\mu)A+(\lambda-\mu)kI_n+(\lambda-\mu)\lambda A+ \\ &+(\lambda-\mu)\mu(J_n-A-I_n)+\mu k(J_n-A-I_n)+\mu kA+ \\ &+\mu kI_n \\ &= ((\lambda-\mu)k+\mu k)I_n+((k-\mu)+(\lambda-\mu)\lambda+\mu k)A+ \\ &+((\lambda-\mu)\mu+\mu k)(J_n-A-I_n) \\ &= \lambda kI_n+((k-\mu)+(\lambda-\mu)\lambda+\mu k)A+ \\ &+((\lambda-\mu)\mu+\mu k)(J_n-A-I_n).\end{aligned}$$

Hence, we have established that $A^3=\lambda kI_n+((k-\mu)+(\lambda-\mu)\lambda+\mu k)A+((\lambda-\mu)\mu+\mu k)(J_n-A-I_n)$.

Therefore, we conclude that

$$\begin{aligned}A^3-\frac{k^3}{n}J_n &= (\lambda k-\frac{k^3}{n})I_n+ \\ &+\left((k-\mu)+(\lambda-\mu)\lambda+\mu k-\frac{k^3}{n}\right)A+ \\ &+\left((\lambda-\mu)\mu+\mu k-\frac{k^3}{n}\right)(J_n-A-I_n).\end{aligned}$$

So, we have that $(A^3-\frac{k^3}{n}J_n)\circ I_n=(\lambda k-\frac{k^3}{n})I_n$, has the following spectral decomposition, $(A^3-\frac{k^3}{n}J_n)I_n=q_1F_1+q_2F_2+q_3F_3$. If $\lambda>\mu$ then we have that all the q_is are less or equal θ^3.

Now using the norm of Frobenius and the inner product trace on the Euclidean Jordan algebra $\mathscr{A}$ we deduce that

$$\left|\mathrm{trace}\left(\left((A^3-\frac{k^3}{n}J_n)\circ I_n\right)I_n\right)\right| = \left|\left((\lambda k-\frac{k^3}{n})3\right)\right|$$

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and since $\lambda > \mu$ by the Cauchy-Schwarz inequality we have

$$\left|(\lambda k - \frac{k^3}{n})3\right| \leq \sqrt{3(\theta^3)^2}\sqrt{\mathrm{tr}(I_n)}.$$

This is, we have

$$\left|(\lambda k - \frac{k^3}{n})3\right| \leq \theta^3\sqrt{3}\sqrt{3}.$$

Hence, we have

$$\left|\lambda k - \frac{k^3}{n}\right| \leq \theta^3.$$

And, therefore $\theta \geq \sqrt[3]{\left|\lambda k - \frac{k^3}{n}\right|}$. In the same way, if $\mu > \lambda$ we deduce that $|\tau| \geq \sqrt[3]{\left|\lambda k - \frac{k^3}{n}\right|}$.

5 Conclusions

In this paper we have established new admissibility conditions over the parameters and over the spectra of a $(n,k;\lambda,\mu)-$strongly regular graph in the environment of Euclidean Jordan algebras in an algebraic asymptotic way and recurring to the Cauchy-Schwarz inequality. Further investigations must proceed so that new admissibility conditions over the parameters of a strongly regular graph X are established, recurring to the spectral analysis of other Hadamard series of other elements of the Euclidean Jordan algebra associated to the adjacency matrix of the strongly regular graph X.

The theory developed in this paper can be extended to develop admissibility conditions over the parameters of symmetric structures like the symmetric associations schemes.

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