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Dynamic Gompertz Model

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Abstract: After a brief introduction that includes some fundamentals of time scales, we lay the foundation for dynamic Gompertz models. We derive their unique solutions, present examples in the discrete, quantum, and mixed time scale settings, and we compare its behavior to the solution in the continuous time setting. A discussion of the results and open problems are addressed in the conclusion.

Keywords: Dynamic equations, Time scales, Gompertz equation, Exact solution, Limiting behavior

1 Introduction

In 1825, Benjamin Gompertz formulated a mathematical population model in [9] based on the assumption that with age, the mortality increases exponentially, see [13]. The model reads as

$$y'(t) = -ry(t)\log\left(\frac{y(t)}{K}\right), \tag{1}$$

where $y\colon \mathbb{R} \to \mathbb{R}_0^+$ represents the population of interest, the positive constants r and K are the growth rate and carrying capacity respectively. Other mortality rates have been discussed for the Gompertz model which led for example to the Gompertz–Makeham model [8], where an age-independent mortality component was added. Equation (1) with initial condition $y(t_0) = y_0 > 0$ has the unique solution

$$y(t) = K\exp\left\{\log\left(\frac{y_0}{K}\right)e^{-r(t-t_0)}\right\}. \tag{2}$$

The Gompertz model is still used to describe population dynamics [7], cell development [17], fermentation of chemical components [14], and microbial mineralization of growth-sustaining pesticides [12], to name some of its applications. Especially when applying the Gompertz model to experimental data, the discretization is of numerical interest. There have been formulations regarding a discrete Gompertz model, such as, in [16],

$$y(t+1) = -ry(t)\log(y(t)) \tag{3}$$

which was proposed as a discretization of (1). Another discretization, presented in [18], is

$$y(t+1) = y(t)\left(\frac{y(t)}{k}\right)^{-\delta r}. \tag{4}$$

The model we present in this work differs in the discrete space from (3) and (4), but exhibits behavior known from the continuous Gompertz model. The present Gompertz model is formulated on a general time scale and is extended by a time-dependence in the growth rate and carrying capacity.

Before we introduce the dynamic Gompertz model on time scales, we introduce some time scales fundamentals.

2 Fundamentals of Time Scales

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$.

Definition 1. *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined [4, Definition 1.1] by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

Similarly, a backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} \colon s < t\}.$$

We adopt the convention that $\inf\emptyset = \sup\mathbb{T}$ and $\sup\,\emptyset = \inf\,\mathbb{T}$. If $\sigma(t) > t$, then we say that t is

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right-scattered; if $\sigma(t) = t$, then we say that t is right-dense. Similarly, left-scattered means $\rho(t) < t$ and left-dense means $\rho(t) = t$. We define the function $f^\sigma\colon \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$. If $t \in \mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa = \mathbb{T}\setminus\{M\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2. *A function $p : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided p is continuous at t for all right-dense points t and the left-sided limit exists for all left-dense points t [5, Definition 1.24]. The set of real-valued rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{\rm rd} = C_{\rm rd}(\mathbb{T}) = C_{\rm rd}(\mathbb{T},\mathbb{R})$.*

Definition 3. *We define the graininess (or "stepsize") function $\mu\colon \mathbb{T} \to \mathbb{R}_0^+$ by $\mu(t) = \sigma(t) - t$.*

Definition 4. *A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive [4, Definition 2.25] provided*

$$1+\mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

The set of real-valued regressive and rd-continuous functions is denoted by $\mathscr{R} = \mathscr{R}(\mathbb{T}) = \mathscr{R}(\mathbb{T},\mathbb{R})$. Moreover, $p \in \mathscr{R}$ is called positively regressive, denoted by $\mathscr{R}^+$, if

$$1+\mu(t)p(t) > 0 \quad \text{for all } t \in \mathbb{T}.$$

Note that on the time scale $\mathbb{T} = \mathbb{R}$, all functions are positively regressive because the graininess μ is identically zero.

Definition 5. *Assume $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then the delta-derivative of f, denoted by f^Δ [4, Definition 1.10], is the number, such that for all $\varepsilon > 0$ there exists $\delta > 0$, such that*

$$\left| f(\sigma(t)) - f(s) - f^\Delta(\sigma(t)-s) \right| \leq \varepsilon\,|\sigma(t)-s|,$$

for all $s \in (t-\delta, t+\delta)\cap\mathbb{T}$.

Notably,

$$f^\Delta(t) = \begin{cases} f'(t), & t \text{ right-dense} \\ \dfrac{f(\sigma(t))-f(t)}{\mu(t)}, & t \text{ right-scattered.} \end{cases}$$

The delta-integral is defined so that for $t, a \in \mathbb{T}$,

$$\left(\int_a^t f(\tau)\Delta\tau\right)^\Delta = f(t), \tag{5}$$

and

$$\int_a^t f^\Delta(\tau)\Delta\tau = f(t) - f(a).$$

Also we note that for any $a \in \mathbb{T}$, $\int_a^a f(\tau)\Delta\tau = 0$. The following theorem is from [4, Theorem 2.33].

Theorem 1. *Let $p \in \mathscr{R}$ and $t_0 \in \mathbb{T}$. Then the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1$$

possesses a unique solution, called the dynamic exponential function and denoted by $e_p(\cdot,t_0)$.

Useful properties of the dynamic exponential function follow [4, Theorem 2.36].

Theorem 2. *If $p \in \mathscr{R}$ and $t,s,r \in \mathbb{T}$, then*

1. *$e_0(t,s) = 1$, and $e_p(t,t) = 1$,*
2. *$e_p(t,s) = \frac{1}{e_p(s,t)}$, and the*
3. *semigroup property holds: $e_p(t,r)e_p(r,s) = e_p(t,s)$.*

The following theorem was proven in [6].

Theorem 3. *If f is nonnegative with $-f \in \mathscr{R}^+$, then*

$$1-\int_s^t f(\tau)\Delta\tau \leq e_{-f}(t,s) \leq \exp\left(-\int_s^t f(\tau)\Delta\tau\right). \tag{6}$$

If $f \in C_{\rm rd}$ is nonnegative, then

$$1+\int_s^t f(\tau)\Delta\tau \leq e_f(t,s) \leq \exp\left(\int_s^t f(\tau)\Delta\tau\right). \tag{7}$$

We define the "circle-plus" and "circle-minus" operations which turn $(\mathscr{R},\oplus)$ into a group where the additive inverse of p is $\ominus p$ [5, p. 10].

Definition 6. *Define the "circle plus" addition on $\mathscr{R}$ for $t \in \mathbb{T}$ by*

$$(p\oplus q)(t) = p(t)+q(t)+\mu(t)p(t)q(t),$$

and the "circle minus" subtraction by

$$(p\ominus q)(t) = \frac{p(t)-q(t)}{1+\mu(t)q(t)}.$$

The following theorem can be found in [3, Theorem 3.4].

Theorem 4. *Let $\mathbb{T}$ be a time scale with $\sup\mathbb{T} = \infty$. If $r > 0$, then $\lim\limits_{t\to\infty} e_{\ominus r}(t,t_0) = 0$.*

It is not difficult to show the following identities [4].

Corollary 1. *If $p,q \in \mathscr{R}$, then for all $t,s \in \mathbb{T}$,*

a) *$e_{p\oplus q}(t,s) = e_p(t,s)e_q(t,s)$,*
b) *$e_{\ominus p}(t,s) = e_p(s,t) = \frac{1}{e_p(t,s)}$, and*
c) *if $p \in \mathscr{R}^+$, then $e_p(t,s) > 0$.*

A variation of constants formula was shown in [5, Theorem 2.1]:

Theorem 5. *Suppose $p \in \mathscr{R}$ and $f \in C_{\rm rd}$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the IVP*

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0,$$

is given by

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^t e_p(t,\sigma(s))f(s)\Delta s. \tag{8}$$

We also need the definition of a logarithm on time scales. Some different logarithms have been defined on time scales (e.g. [11], [2], [15]), but we focus on [2] and generalize it in the following way.

Definition 7. *Let $a \in \mathbb{R}$ and let p be a nonvanishing delta-differentiable function, then we define the logarithm of p for $a \in \mathbb{R}$ by*

$$L_p(t,t_0;a) = a + \int_{t_0}^{t} \frac{p^{\Delta}(\tau)}{p(\tau)} \Delta\tau.$$

Time scale	$L_p(t,t_0;a)$
$\mathbb{R}$	$a + \ln\left(\left\lvert\frac{p(t)}{p(t_0)}\right\rvert\right)$
$\mathbb{Z}$	$a + \sum_{k=t_0}^{t-1}\left(-1 + \frac{p(k+1)}{p(k)}\right)$
$q^{\mathbb{N}_0}, q>1$	$a + \sum_{k=\log_q(t_0)}^{\log_q(t)-1}\left(-1 + \frac{p(q^{k+1})}{p(q^k)}\right)$

Lemma 1. *If $p \in \mathscr{R}$ and $c \in \mathbb{R}\setminus\{0\}$, then*

$$L_{ce_p(\cdot,t_0)}(t,t_0;a) = a + \int_{t_0}^{t} p(\tau)\Delta\tau.$$

Proof. Calculate

$$\begin{aligned} L_{ce_p(\cdot,t_0)}(t,t_0;a) &= a + \int_{t_0}^{t} \frac{p(\tau)ce_p(\tau,t_0)}{ce_p(\tau,t_0)}\Delta\tau \\ &= a + \int_{t_0}^{t} p(\tau)\Delta\tau, \end{aligned}$$

as was to be shown.

The next lemma follows immediately from the fundamental theorem of calculus.

Lemma 2. *If $p \in \mathscr{R}$ is delta-differentiable and nonvanishing, then for all $t_0 \in \mathbb{T}$ and $a \in \mathbb{R}$,*

$$L_p^{\Delta}(t,t_0;a) = \frac{p^{\Delta}(t)}{p(t)}.$$

3 Gompertz model on time scales

3.1 Dynamic Gompertz model

We now consider a time scale analogue of (1),

$$y' = -r(t)y\ln\left(\frac{y}{K(t)}\right) = -r(t)y\left(\ln(y) - \tilde{K}(t)\right),$$

with $y(t_0) = y_0$, namely

$$y^{\Delta} = (\ominus r)(t)y\left(L_y(t,t_0;a) - \tilde{K}(t)\right) \tag{9}$$

with $y(t_0) = y_0 > 0$ where $r, \tilde{K} : \mathbb{T} \to \mathbb{R}$, $r \in \mathscr{R}$, $\tilde{K} \in C_{\mathrm{rd}}$, and $a \in \mathbb{R}$.

Theorem 6. *If $a \in \mathbb{R}$, $r, p \in \mathscr{R}$, and $\tilde{K} \in C_{\mathrm{rd}}$, then the unique solution to (9) is given by*

$$y(t) = e_p(t,t_0)y_0,$$

where

$$p(t) = (\ominus r)(t)\Bigg(e_{\ominus r}(t,t_0)a - \int_{t_0}^{t}(\ominus r)(s)e_{\ominus r}(t,\sigma(s))\tilde{K}(s)\Delta s - \tilde{K}(t)\Bigg).$$

Proof. We define $z(t) = L_y(t,t_0;a)$, then (9) is given by

$$z^{\Delta} = (\ominus r)(t)(z - \tilde{K}(t)), \quad z(t_0) = a,$$

which is a nonhomogeneous initial value problem, and Theorem 5 shows its unique solution is given by

$$z(t) = e_{\ominus r}(t,t_0)a - \int_{t_0}^{t}(\ominus r)(s)e_{\ominus r}(t,\sigma(s))\tilde{K}(s)\Delta s.$$

Therefore

$$\frac{y^{\Delta}}{y} = z^{\Delta} = (\ominus r)(t)\Bigg[e_{\ominus r}(t,t_0)a - \int_{t_0}^{t}(\ominus r)(s)e_{\ominus r}(t,\sigma(s))\tilde{K}(s)\Delta s - \tilde{K}(t)\Bigg],$$

i.e.,

$$y^{\Delta} = p(t)y, \quad y(t_0) = y_0,$$

where

$$p(t) = (\ominus r)(t)\Bigg(e_{\ominus r}(t,t_0)a - \int_{t_0}^{t}(\ominus r)(s)e_{\ominus r}(t,\sigma(s))\tilde{K}(s)\Delta s - \tilde{K}(t)\Bigg).$$

The unique solution of this first-order initial value problem, for $p \in \mathscr{R}$, is given by

$$y(t) = e_p(t,t_0)y_0. \tag{10}$$

Conversely, (10) solves (9), since

$$\begin{aligned}
&\int_{t_0}^{t} p(s)\Delta s \\
&= a\int_{t_0}^{t} (\ominus r)(s)e_{\ominus r}(s,t_0)\Delta s - \int_{t_0}^{t}(\ominus r)(s)\tilde{K}(s)\Delta s \\
&\quad - \int_{t_0}^{t}(\ominus r)(s)\int_{t_0}^{s}(\ominus r)(\tau)e_{\ominus r}(s,\sigma(\tau))\tilde{K}(\tau)\Delta\tau\Delta s \\
&= ae_{\ominus r}(t,t_0) - a - \int_{t_0}^{t}(\ominus r)(s)\tilde{K}(s)\Delta s \\
&\quad - \int_{t_0}^{t}(\ominus r)(\tau)\tilde{K}(\tau)\int_{\sigma(\tau)}^{t}(\ominus r)(s)e_{\ominus r}(s,\sigma(\tau))\Delta s\Delta\tau \\
&= ae_{\ominus r}(t,t_0) - a - \int_{t_0}^{t}(\ominus r)(s)\tilde{K}(s)\Delta s \\
&\quad - \int_{t_0}^{t}(\ominus r)(\tau)\tilde{K}(\tau)\left[e_{\ominus r}(t,\sigma(\tau)) - 1\right]\Delta\tau \\
&= ae_{\ominus r}(t,t_0) - a - \int_{t_0}^{t}(\ominus r)(\tau)\tilde{K}(\tau)e_{\ominus r}(t,\sigma(\tau))\Delta\tau,
\end{aligned}$$

and we have

$$\begin{aligned}
&(\ominus r)(t)y(t)\left(L_y(t,t_0;a) - \tilde{K}(t)\right) \\
&= (\ominus r)(t)y(t)\left(a + \int_{t_0}^{t}\frac{y^{\Delta}(s)}{y(s)}\Delta s - \tilde{K}(t)\right) \\
&= (\ominus r)(t)y(t)\left(a + \int_{t_0}^{t}p(s)\Delta s - \tilde{K}(t)\right) \\
&= (\ominus r)(t)y(t)\Bigg(ae_{\ominus r}(t,t_0) \\
&\qquad - \int_{t_0}^{t}(\ominus r)(\tau)\tilde{K}(\tau)e_{\ominus r}(t,\sigma(\tau))\Delta\tau - \tilde{K}(t)\Bigg) \\
&= y(t)p(t),
\end{aligned}$$

which is equal to y^{Δ} and the proof is complete.

Example 1. If $\mathbb{T}=\mathbb{Z}$, $r\in\mathbb{R}\setminus\{-1\}$, $\tilde{K},a\in\mathbb{R}$, $y_0>0$, $p\in\mathscr{R}$, and $t_0=0$, then (9) becomes

$$\Delta y(t) = \frac{-r}{1+r}y(t) \times \left[a + \sum_{k=0}^{t-1}\left(-1+\frac{y(k+1)}{y(k)}\right) - \tilde{K}\right], \quad y(0)=y_0,$$

and has, by Theorem 6, the solution

$$y(t) = y_0\prod_{k=0}^{t-1}(1+p(k)),$$

where

$$p(k) = \frac{r(\tilde{K}-a)}{(1+r)^{k+1}}.$$

The same technique can be applied to a different version of the Gompertz model, that is

$$y^{\Delta} = -r(t)y\left(L_y(t,t_0;a) - \tilde{K}(t)\right), \quad y(t_0)=y_0>0, \quad (11)$$

where $r,K:\mathbb{T}\to\mathbb{R}$ and $a\in\mathbb{R}$.

Theorem 7. *If $-r,p\in\mathscr{R}$, then the unique solution to (11) is given by*

$$y(t) = e_p(t,t_0)y_0,$$

where

$$p(t) = r(t)\tilde{K}(t) - r(t)e_{-r}(t,t_0)a - r(t)\int_{t_0}^{t}e_{-r}(t,\sigma(s))r(s)\tilde{K}(s)\Delta s.$$

3.2 Normalized Gompertz dynamic equation

Let $a\in\mathbb{R}$. We define the normalized dynamic Gompertz model for $r\in\mathscr{R}$ by the initial value problem

$$y^{\Delta} = (\ominus r)(t)yL_y(t,t_0;a), \quad y(t_0)=y_0>0. \quad (12)$$

This should be considered as the time scales analogue to

$$y' = -r(t)y\ln(y), \quad y(t_0)=y_0>0,$$

which is a special case of (1) with $K=1$.

Theorem 8. *If $r,p\in\mathscr{R}$, $a\in\mathbb{R}$, then the unique solution to (12) is given by*

$$y(t) = y_0e_p(t,t_0), \quad p(t) = a(\ominus r)(t)e_{\ominus r}(t,t_0). \quad (13)$$

The proof of Theorem 8 follows from Theorem 6 with $\tilde{K}=0$.

Example 2. If $\mathbb{T}=\mathbb{Z}$, $a\in\mathbb{R}$, $r\in\mathbb{R}\setminus\{-1\}$, and $t_0=0$, then (12) reads as

$$\Delta y(t) = \frac{-r}{1+r}y(t)\left[a + \sum_{k=0}^{t-1}\left(-1+\frac{y(k+1)}{y(k)}\right)\right],$$

with initial condition $y(0)=y_0>0$. By Theorem 8, the solution is

$$y(t) = y_0\prod_{j=0}^{t-1}\left[1 - \frac{ra}{(1+r)^{j+1}}\right],$$

see Figure 1 for its behavior.

Example 3. For $\mathbb{T}=q^{\mathbb{N}_0}$, $a,r\in\mathbb{R}$ such that for all $t\in\mathbb{T}$, $1+(q-1)tr\neq 0$, and $t_0=q^0=1$, (12) reads as

$$\frac{y(qt)-y(t)}{t(q-1)} = \frac{-r}{1+(q-1)tr}y(t)\left[a + \sum_{k=0}^{\log_q(t)-1}\left(-1+\frac{y(q^{k+1})}{y(q^k)}\right)\right], \quad (14)$$

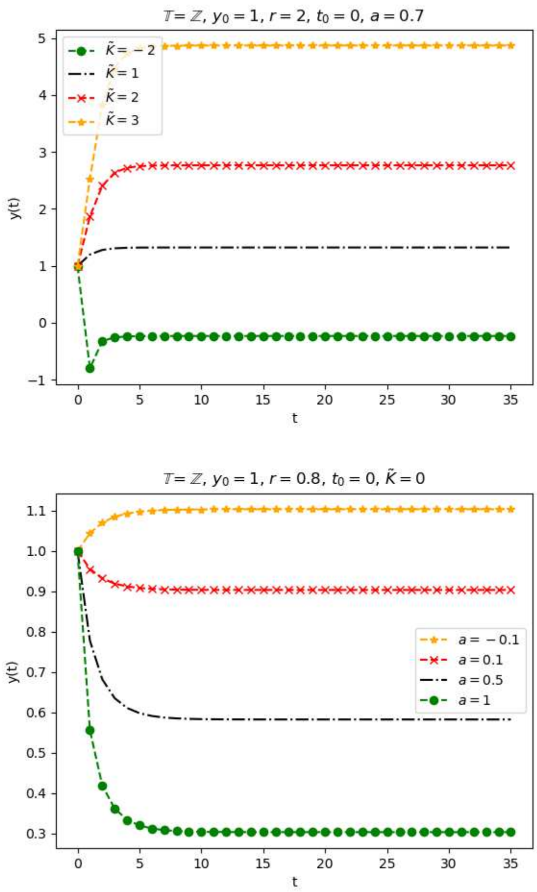

Fig. 1: Solutions of (9) for $\mathbb{T}=\mathbb{Z}$.

with initial condition $y(1)=y_0>0$. By Theorem 8, the solution is

$$y(t)=y_0\prod_{i=0}^{\log_q(t)-1}\left[1-a\frac{(q-1)q^ir}{\prod_{k=0}^{i}(1+(q-1)q^kr)}\right].$$

Example 4. If $\mathbb{T}=\{0\}\cup\bigcup_{k=1}^{\infty}\left[\frac{1}{2k},\frac{1}{2k-1}\right]$, then

$$\mu(t)=\begin{cases}\dfrac{1}{(2k-1)(2k-2)}, & t=\dfrac{1}{2k-1},k\in\{2,3,\dots\}\\ 0, & \text{otherwise.}\end{cases}$$

If $a,r\in\mathbb{R}$ such that for all $t\in\mathbb{T}$, $1+\frac{r}{(2k-1)(2k-2)}\neq 0$ for $k\in\{1,2,3,\dots\}$ and if $t_0=0$, then for $t\in\{0\}\cup\bigcup_{k=1}^{\infty}\left[\frac{1}{2k},\frac{1}{2k-1}\right)$, (12) reads as

$$y'(t)=-ry(t)\left[a+\int_0^t\frac{y^\Delta(\tau)}{y(\tau)}\Delta\tau\right],$$

while for $t=\frac{1}{2k-1},k\in\{1,2,3,\dots\}$, (12) reads as

$$\frac{y\left(\frac{1}{2(k-1)}\right)-y\left(\frac{1}{2k-1}\right)}{\frac{1}{(2k-1)(2k-2)}}$$
$$=\frac{-r}{1+\frac{r}{(2k-1)(2k-2)}}y(t)\left[a+\int_0^t\frac{y^\Delta(\tau)}{y(\tau)}\Delta\tau\right],$$

both with initial condition $y(0)=y_0>0$. By Theorem 8, the solution is

$$y(t)=y_0\left[\exp\left(a\int_{[0,t]\cap\mathbb{T}}\frac{\mathrm{d}}{\mathrm{d}t}e^{-rt}\,\mathrm{d}t\right)\right.$$
$$\left.\times\prod_{0\le\frac{1}{2k-1}<t}\left(\frac{1+\frac{r}{(2k-1)(2k-2)}(1-ae_{\ominus r}(t,0))}{1+\frac{r}{(2k-1)(2k-2)}}\right)\right].$$

We now discuss the limiting behavior of the solution for constant r.

Theorem 9. *If y is given by (13), $r>0$, $a\ge 0$, and $p\in\mathscr{R}^+$, then*

$$y_0(1-a)\le\lim_{t\to\infty}y(t)\le y_0e^{-a},\quad t\ge t_0.$$

Proof. Let $t\ge t_0$. From $r>0$, we may conclude $r\in\mathscr{R}^+$, so $e_{\ominus r}(t,t_0)>0$. Since $a\ge 0$, we see that

$$p(t)=a(\ominus r)(t)e_{\ominus r}(t,t_0)=-\frac{ar}{1+\mu(t)r}e_{\ominus r}(t,t_0)$$

is always negative. Therefore $-p$ is nonnegative and $-(-p)=p\in\mathscr{R}^+$. Hence by (6),

$$1+\int_{t_0}^t p(\tau)\Delta\tau\le e_p(t,t_0)\le\exp\left(\int_{t_0}^t p(\tau)\Delta\tau\right),$$

and thus

$$1+a(e_{\ominus r}(t,t_0)-1)\le e_p(t,t_0)\le\exp\left(ae_{\ominus r}(t,t_0)-a\right).$$

Multiplying by y_0, taking the limit as $t\to\infty$, and applying Theorem 4 yields

$$y_0(1-a)\le\lim_{t\to\infty}y(t)\le y_0e^{-a},$$

as was to be shown.

Corollary 2. *If y is given by (13), $r>0$ and $0\le a<1$, then*

$$0<\lim_{t\to\infty}y(t)\le y_0e^{-a}.$$

Note that $y_0>0$ as defined in (12).

Theorem 10. *If y is defined by (13), $r>0$, $a<0$, and $p\in\mathscr{R}$, then*

$$y_0(1+|a|)\le\lim_{t\to\infty}y(t)\le y_0\exp(|a|).$$

Proof. Since $r > 0$ (hence $r \in \mathscr{R}^+$) and $a < 0$, we see that $p(t) = \dfrac{(-a)r}{1+\mu(t)r} e_{\ominus r}(t,t_0)$ is nonnegative. Since also $p \in \mathscr{R}$, (7) implies

$$1 + \int_{t_0}^{t} p(\tau)\Delta\tau \le e_p(t,t_0) \le \exp\left(\int_{t_0}^{t} p(\tau)\Delta\tau\right).$$

Thus multiplying by y_0, taking the limit as $t \to \infty$, and applying Theorem 4, we may conclude

$$y_0(1-a) \le \lim_{t\to\infty} y(t) \le y_0 e^{-a},$$

or equivalently,

$$y_0(1+|a|) \le \lim_{t\to\infty} y(t) \le y_0 e^{|a|},$$

as was to be shown.

Figures 2 and 3 illustrate the similarities and differences to discretizations (3) and (4) of the classical Gompertz model.

In Figure 2, we see that for $\mathbb{T} = \mathbb{Z}$, the solution to (12) with $a = -0.3, r = 0.9, y_0 = 0.8$, and $\delta = 0.1$ behaves similarly as the solutions to (3) and (4). However, when changing $y_0 = 0.2$, $a = -0.3$, and $\delta = -0.1$, (3) converges to a positive population concave down, (12) converges to a positive population concave up, and (4) goes extinct.

The similarities of (4) and (12) do not hold, if we choose $y_0 = 0.9$ as shown in Figure 3: with a positive growth parameter, equations (1), (12), and (4) behave similarly while (3) goes extinct. Choosing a negative growth parameter causes (3) to disappear after finitely many steps because the logarithm becomes undefined. The behavior of (4) remains mostly unchanged, but the behavior of both (12) and (1) goes extinct.

As before, we obtain another form of the normalized Gompertz model by considering (11) with $\tilde{K} = 0$.

$$y^{\Delta}(t) = -r(t)y(t)L_y(t,t_0;a), \quad y(t_0) = y_0 > 0. \qquad (15)$$

Theorem 11. *If $-r, p \in \mathscr{R}$, $a \in \mathbb{R}$, $y_0 > 0$, then the unique solution to (15) is given by*

$$y(t) = y_0 e_p(t,t_0), \quad p(t) = -r(t)ae_{-r}(t,t_0). \qquad (16)$$

Example 5. If $\mathbb{T} = \mathbb{Z}$, $r \in \mathbb{R}\setminus\{1\}$, $a \in \mathbb{R}$, $t_0 = 0$, and $y_0 > 0$, then by Theorem 11, the solution of (15) is

$$y(t) = y_0 e_p(t,0) = y_0 \prod_{k=0}^{t-1}\left(1 - ar(1-r)^k\right).$$

Lemma 3. *Let $\mathbb{T}$ be unbounded above. If $r > 0$ with $-r \in \mathscr{R}^+$, then*

$$\lim_{t\to\infty} e_{-r}(t,t_0) = 0.$$

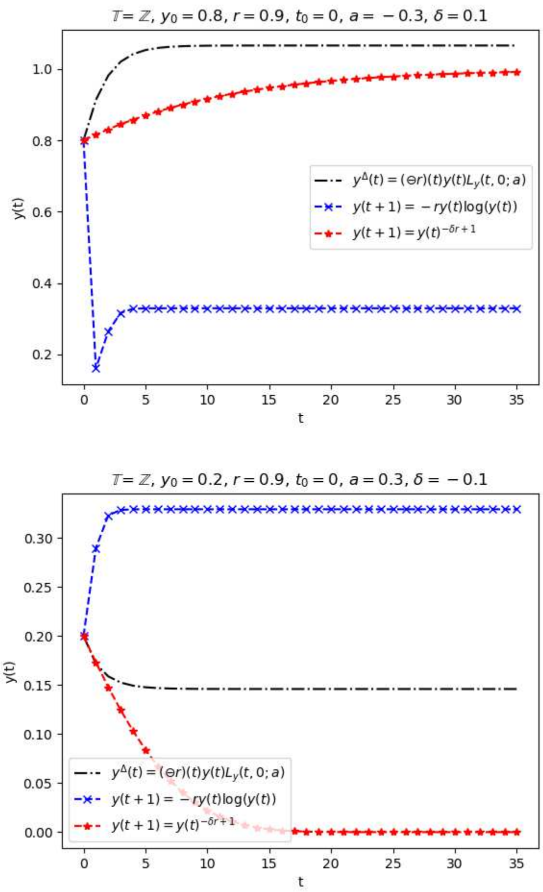

Fig. 2: Comparing previous discrete models (3) and (4) with the generalized time scales model (12) on $\mathbb{T} = \mathbb{Z}$.

Proof. Since $-r \in \mathscr{R}^+$, we know that $e_{-r}(t,t_0) \ge 0$. Now apply (6) with $f = r$ to get

$$0 \le e_{-r}(t,t_0) \le \exp(-r(t-t_0)).$$

The limit as $t \to \infty$ of the right-hand side of this inequality is zero, completing the proof.

Similarly to the proofs of Theorem 9 and Theorem 10, we can prove the following theorems using Lemma 3.

Theorem 12. *Let $\mathbb{T}$ be unbounded above, let $0 < r < \dfrac{1}{\mu(t)}$ for all $t \in \mathbb{T}$, let $a \ge 0$, and let $p \in \mathscr{R}^+$. If y is the solution to (15) given by (16), then*

$$y_0(1-a) \le \lim_{t\to\infty} y(t) \le y_0 e^{-a}.$$

Corollary 3. *Let $\mathbb{T}$ be unbounded above, let $0 < r < \dfrac{1}{\mu(t)}$ for all $t \in \mathbb{T}$, let $0 \le a < 1$, and let $p \in \mathscr{R}^+$. If y is the solution to (15) given by (16), then*

$$0 < \lim_{t\to\infty} y(t) \le y_0 e^{-a}.$$

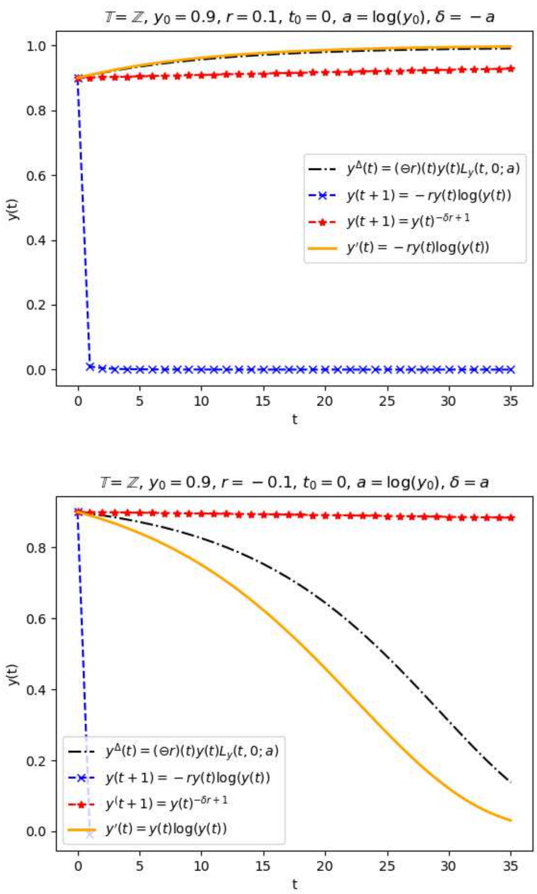

Fig. 3: Comparing previous discrete models (3) and (4) and the previous continuous model (1) with the generalized time scales model (12) on the time scale $\mathbb{T}=\mathbb{Z}$.

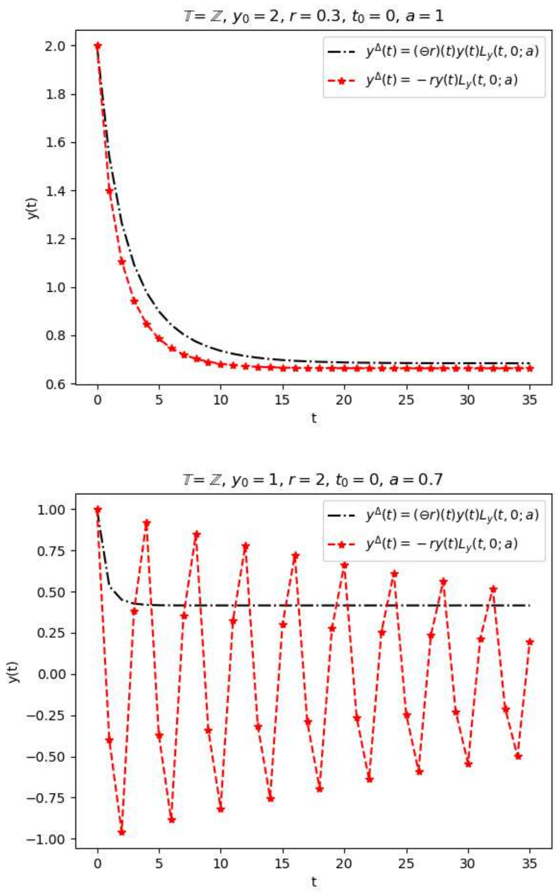

Fig. 4: Comparing solutions of (12) to solutions of (15).

Theorem 13. *Let $\mathbb{T}$ be unbounded above, let $0<r<\dfrac{1}{\mu(t)}$, let $a<0$, and let $p\in\mathscr{R}$. If y is the solution to (15) given by (16), then*

$$y_0(1+|a|)\leq\lim_{t\to\infty}e_p(t,t_0)\leq y_0e^{|a|}.$$

Figure 4 compares (12) to (15). We see that the solutions are similar under some parameters, but (15) undergoes damped oscillation.

3.3 Data fitting

The authors in [19] used the solution to the continuous Gompertz model

$$H(d)=\gamma_c\exp\left(\beta_c e^{-\alpha_c d}\right), \tag{17}$$

to fit data of a tree's height H to its diameter d. Using the same data as in [19], provided in [20], the nonlinear optimizer function "optim" in R's "{stats}" package was used with 50 randomly chosen initial conditions to obtain the parameters α_c,β_c and γ_c that reduce the least-square error for (17).

To fit the data to the introduced model for $\mathbb{T}=\mathbb{Z}$, the tree heights and diameters in the data set were multiplied by 10 to ensure that they take integer values. Using the same method, parameters α_z,β_z and γ_z were calculated to reduce the least square error of the data to the model (see Example 2),

$$H(d)=\gamma_z\prod_{j=1}^{d}\left[1-\frac{\alpha_z\beta_z}{(1+\beta_z)^j}\right]. \tag{18}$$

Not surprisingly, the values $\alpha_c,\beta_c,\gamma_c$ were not identical to $\alpha_z,\beta_z,\gamma_z$. The least square errors however were nearly the same with $255,072\cdot10^3$ for the continuous model and $255,113\cdot10^3$ for the discrete model. This example shows that not only can we obtain a closed form solution to our proposed generalization of the Gompertz model, as in the continuous case, which even exhibits the same limiting behavior as the continuous solution, but the model also

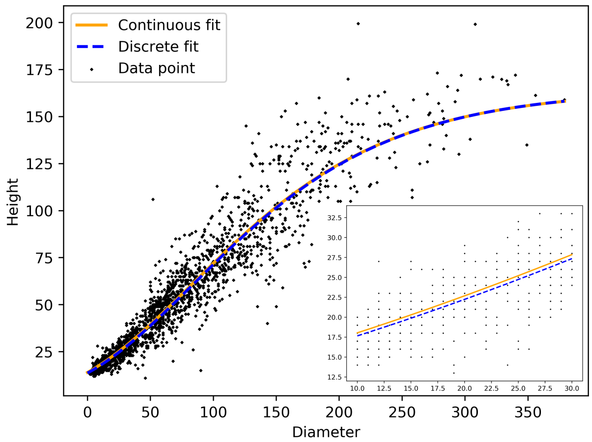

Fig. 5: Fitting data from [20] to the $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ Gompertz functions.

provides a similar fit to some data. This completes our claim of having formulated a generalization of the continuous Gompertz model on time scales.

4 Conclusion

We have presented generalizations of the classical Gompertz differential equation to time scales with time-dependent growth rate and carrying capacity. Results for the long-time stability of the normalized models have been obtained. We believe that the stability results can be sharpened and formulated for the general model (9). An analysis on precisely how $\tilde{K}$ affects the limiting behavior of the solution of (9) is warranted. It appears that the limit grows proportionally to $\tilde{K}$. This work lies the foundation to investigate systems of competing species on time scales, each following a Gompertz model [21], as well as fractional Gompertz dynamic equations, see [1] and [10] for discrete versions.

References

[1] F.M. Atıcı and S. Şengül. *Modeling with fractional difference equations*. J. Math. Anal. Appl., 369, 1–9, (2010).

[2] M. Bohner. *The logarithm on time scales*. J. Difference Equ. Appl. 11, 1305–1306, (2005).

[3] M. Bohner, G. Guseinov, and B. Karpuz. *Properties of the Laplace transform on time scales with arbitrary graininess*. Integral Transforms Spec. Funct. 22, 785–800, (2011).

[4] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser Boston Inc., Boston, MA, (2001).

[5] M. Bohner and A. Peterson. *Advances in Dynamic Equations on Time Scales*. Birkhäuser Boston, Inc., Boston, MA, (2003).

[6] M. Bohner. *Some oscillation criteria for first order delay dynamic equations*. Far East J. Appl. Math. 18, 289–304, (2005).

[7] D. Easton. *Complementary Gompertz survival models: Decreasing alive versus increasing dead*. J. Gerontol. 64A, 550–555, (2009).

[8] X. Feng, G. He, and A. Abduwali. *Estimation of parameters of the Makeham distribution using the least squares method*. Math. Comput. Simul. 77, 34–44, (2008).

[9] B. Gompertz. *On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies*. Phil. Trans. R. Soc. Lond. 115, 513–583, (1825).

[10] C. Goodrich and A. Peterson. *Discrete fractional calculus*. Cham: Springer, (2015).

[11] B. Jackson. *The time scale logarithm*. Appl. Math. Lett. 21, 215–221, (2008).

[12] A. Johnsen, P. Binning, J. Aamand, N. Badawi, and A. Rosenbom. *The Gompertz function can coherently describe microbial mineralization of growth-sustaining pesticides*. Environ. Sci. Technol. 47, 8508–8514, (2013).

[13] T. Kirkwood. *Deciphering death: a commentary on Gompertz (1825) 'on the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies'*. Philos Trans R Soc Lond B Biol Sci., 370, (2015).

[14] A. Lavrenčič, C. Mills, and B. Stefanon. *Application of the Gompertz model to describe the fermentation characteristics of chemical components in forages*. Animal Science 66, 155–161, (1998).

[15] D. Mozyrska and D. Torres. *The natural logarithm on time scales*. J. Dyn. Syst. Geom. Theor. 7, 41–48, (2009).

[16] A. Nobile, L. Ricciardi, and L. Sacerdote. *On Gompertz growth model and related difference equations*. Biol. Cybern. 42, 221–229, (1982).

[17] S. Rossi, A. Deslauriers, and H. Morin. *Application of the Gompertz equation for the study of xylem cell development*. Dendrochronologia, 21, 33–39, (2003).

[18] D. Satoh. *A discrete Gompertz equation and a software reliability growth model*. IEICE Trans. E83, 1508–1513, (2000).

[19] Z.T. Wood, D.R. Peart, P.A. Palmiotto, L. Kong, and N.V. Peart. *Asymptotic allometry and transition to the canopy in abies balsamea*. Journal of Ecology 103, 1658–1666, (2015).

[20] Z.T. Wood, D.R. Peart, P.A. Palmiotto, L. Kong, and N.V. Peart. *Data from: Asymptotic allometry and transition to the canopy in abies balsamea* (2015). https://doi.org/10.5061/dryad.r3645.

[21] Y. Yu, W. Wang, and Z. Lu. *Global stability of Gompertz model of three competing populations*. J. Math. Anal. Appl. 334, 333–348, (2007).

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