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Isomorphism Theorem for Neutrosophic Submodules

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Abstract: In this paper, we investigate some new features of neutrosophic submodules and their properties. The neutrosophic quotient module and restriction of a neutrosophic submodule to a submodule of R-module are defined. The three fundamental theorems of module isomorphism are extended to isomorphism of neutrosophic submodules.

Keywords: Neutrosophic set, support of neutrosophic subset, neutrosophic submodule, neutrosophic quotient submodule, neutrosophic module isomorphism

1 Introduction

In classical set theory, the membership of elements in a set is assessed in binary terms 0 and 1. According to the bivalent condition, an element either belongs or does not belong to the set. As an extension, fuzzy set theory permits the gradual assessment of the membership of elements in a set. A fuzzy set A in X is characterised by a membership function which is associated with each element in X, a real number in the interval $[0,1]$. Lotfi A. Zadeh introduced fuzzy set theory [1] whose objects are sets with imprecise boundaries which allow us to represent vague concepts and contexts in natural language. Fuzzy set theory is limited to modelling a situation involving uncertainty. The intuitionistic fuzzy set theory has been introduced by Krassimir Atanassov as an extension of the fuzzy set concept, in which elements have degree of membership and non-membership.

Definition 1.1. [2,3] An intuitionistic fuzzy set A in a non empty set X is an object having the form $A = \{(x,\mu_A(x),\nu_A(x) : x \in X\}$ where $\mu : X \to [0,1]$, $\nu : X \to [0,1]$ *and* $0 \le \mu_A(x)+\nu_A(x) \le 1$. μ for membership and ν for non membership, which belongs to the real unit interval $[0,1]$ and sum belongs to the same interval.

In 1960s, Abraham Robinson developed the non-standard analysis [4], a formalization of analysis and a branch of mathematical logic that rigorously defines the infinitesimals. Informally, an infinitesimal is an infinitely small number. Let us consider the non-standard finite numbers $1^+ = 1+\varepsilon$, where 1 is its standard part and ε its non-standard part and $^-0 = 0-\varepsilon$ where 0 is its standard part and ε its non-standard part. Then, we call $(^-0,1^+)$a non-standard unit interval. Neutrosophy is a new branch of philosophy and logic introduced by Florentin Smarandache in 1980 which studies the origin and features of Neutralities in nature. Each proposition in Neutrosophic logic is approximated to have the percentage of truth (T), the percentage of indeterminacy (I) and the percentage of falsity (F). So this Neutrosophic logic is called a generalization of classical logic, conventional fuzzy logic, intuitionistic fuzzy logic and interval valued fuzzy logic. This mathematical tool is used to handle problems like imprecise, indeterminate and inconsistent data. Neutrosophic logic is a formal frame trying to measure the truth, indeterminacy and falsehood. In this paper the algebraic structure submodule and its properties are defined and neutrosophic isomorphism theorems are derived in neutosophic set framework. A module is one of the fundamental structures used in abstract algebra. The definition of a module is equivalent to that of a vector space with a ring of scalars. Vector space is a module with a field of scalars.

Definition 1.2. [5] Let R be a commutative ring with unity. A set M with a binary operation $+$ is said to be an R module or a module over the ring R if

1.$(M,+)$ is an abelian group
2.$\exists$ a map $R\times M \to M$ i.e. $(r,m) \to rm$ (an action of R on M) such that
   (a)$(r+s)m = rm+sm$

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(b)$(rs)m = r(sm)$
(c)$r(m+n) = rm+rn$
(d)$1m = m, 1 \in R, \forall r,s \in R$ and $m,n \in M$.

Definition 1.3.[6] Let M be an R module. A submodule is a subgroup N of M which is also a R module i.e, $rn \in N, \forall r \in R, n \in N$.

Remark Let I be an ideal of ring R with unity. Then the quotient (R/I) is an R module with scalar multiplication $r(x+I) = rx+I$, $\forall x \in R$.

Definition 1.4.[6] If M and N are R modules. A map $g : M \to N$ is said to be an R module homomorphism if

1.$g(x+y) = g(x)+g(y), \forall\, x,y \in M$
2.$g(rx) = rg(x), \forall\, x \in M, r \in R$.

If g is bijective, then M is isomorphic to N $(M \cong N)$.

Definition 1.5.[6] Let $g : M \to N$ an R module homomorphism,then

$$ker\ g = \{m \in M | g(M) = 0\}$$

$$Img\ g = g(M) = \{n \in N | n = g(m) forsome\ m \in M\}$$

Theorem 1.1. [7,8](First isomorphism theorem for modules) Let $g : M \to N$ be an on to R module homomorphism, then $Imgg \cong M/kerg$.

Theorem 1.2. [7,8](Second isomorphism theorem for modules) Let M be a module and S, T submodules of M, then

1.the sum, $S+T = \{s+t | s \in S, t \in S\}$is a submodule of M.
2.the intersection, $S \cap T$ is a submodule of M
3.the quotient module, $(S+T)/T \cong S/(S \cap T)$.

Theorem 1.3.[7,8](Third isomorphism theorem for modules) Let M be a module, T a submodule of M

1.If S is a submodule of M such that $T \subseteq S \subseteq M$, then S/T is a submodule of M/T.
2.If S is a submodule of M such that $T \subseteq S \subseteq M$,then the quotient module $(M/T)/(S/T) \cong M/S$.

Some authors have studied the algebraic structure in pure mathematics associated with uncertainty. In 1971, Azriel Rosenfeld [9] presented a seminal paper on fuzzy subgroup and W.J. Liu [10]developed the idea of fuzzy normal subgroup and fuzzy subring. The consolidation of the neutrosophic set hypothesis with algebraic structures is a growing trend in the mathematical research. The single-valued neutrosophic set inculcates algebraic and topological structures that lead to a paradigm shift in the research of neutrosophic algebra. One of the key developments in the neutrosophic set theory is the hybridization of the neutrosophic set with various potential algebraic structures such as bipolar set, soft set, hesitant fuzzy set, etc. The neutrosophic submodule of an R-module (classical module) by using a single-valued neutrosophic set and the basic features of neutrosophic submodules are studied by Cetkin and Olgun N [11]. The main objective of the neutrosophic set is to narrow the gap between the vague, ambiguous and imprecise real-world situations. Neutrosophic set theory gives a thorough scientific and mathematical model knowledge in which speculative and uncertain hypothetical phenomena can be managed by hierarchal membership of components "truth/indeterminacy/falsehood "[12]. The main objective of this paper is a generalization of isomorphism theorems of R modules in the neutrosophic domain. In this paper we also discussed the support, restriction and quotient of a neutrosophic submodule of R-module and derived three fundamental isomorphism theorems of neutrosophic submodules of R module.

2 Preliminaries

Definition 2.1.[11] A neutrosophic set A on the universal set X is defined as

$$A = \{x, (t_A(x), i_A(x), f_A(x)) : x \in X\}$$

where $t_A, i_A, f_A : X \to (^-0, 1^+)$. The three components t_A, i_A *and* f_A represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval $(^-0, 1^+)$.

Definition 2.2.[13] Let A and B be two neutrosophic sets on X. Then A is contained in B, denoted as $A \subseteq B$ if and only if $A(x) \leqslant B(x)$ $\forall x \in X$, this means that

$$t_A(x) \le t_B(x), i_A(x) \le i_B(x), f_A(x) \ge f_B(x)$$

Definition 2.3. [11,13] Let A and B be two neutrosophic sets on X

1.The union C of A and B is denoted by $C = A \cup B$ and defined as
$$C(x) = A(x) \vee B(x)$$
where $C(x) = (t_C(x), i_C(x), f_C(x))$ where
$$t_C(x) = t_A(x) \vee t_B(x)$$
$$i_C(x) = i_A(x) \vee i_B(x)$$
$$f_C(x) = f_A(x) \wedge f_B(x)$$
2.The intersection C of A and B is denoted by $C = A \cap B$ and is defined as
$$C(x) = A(x) \wedge B(x)$$
where $C(x) = (t_C(x), i_C(x), f_C(x))$ where
$$t_C(x) = t_A(x) \wedge t_B(x)$$
$$i_C(x) = i_A(x) \wedge i_B(x)$$
$$f_C(x) = f_A(x) \vee f_B(x)$$

Definition 2.4.[14] Let A,B be neutrosophic sets on a module M. Then their sum $A+B$ is a neutrosophic set on M, defined as follows

$$t_{A+B}(x) = \vee\{t_A(y) \wedge t_B(z) | x = y+z, y,z \in M\}$$
$$i_{A+B}(x) = \vee\{i_A(y) \wedge i_B(z) | x = y+z, y,z \in M\}$$
$$f_{A+B}(x) = \wedge\{f_A(y) \vee f_B(z) | x = y+z, y,z \in M\}$$

Definition 2.5. Let X and Y be two non empty sets and $g: X \to Y$ be a mapping. Let A and B be neutrosophic subsets of X and Y respectively. Then the image of A under the map g is denoted by $g(A)$ and is defined as $g(A)(y) = (t_{g(A)}(y), i_{g(A)}(y), f_{g(A)}(y))$ where

$$t_{g(A)}(y) = \begin{cases} \vee(t_{(A)}(x) : x \in g^{-1}(y)) & g^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$i_{g(A)}(y) = \begin{cases} \vee(i_{(A)}(x) : x \in g^{-1}(y)) & g^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$f_{g(A)}(y) = \begin{cases} \wedge(g_{(A)}(x) : x \in g^{-1}(y)) & f^{-1}(y) \neq \emptyset \\ 1 & otherwise \end{cases}$$

Definition 2.6.[11] Let M be an R module. Let $A \in U^M$ where U^Mdenotes the neutrosophic power set of R module M. Then a neutrosophic subset in M $A = \{x, t_A(x), i_A(x), f_A(x) : x \in M\}$ is called a neutrosophic sub module of M if it satisfies the following conditions

1. $t_A(0) = 1, i_A(0) = 1, f_A(0) = 0$
2. $t_A(x+y) \geq t_A(x) \wedge t_A(y), \forall x,y \in M$
   $i_A(x+y) \geq i_A(x) \wedge i_A(y), \forall x,y \in M$
   $f_A(x+y) \leq f_A(x) \vee f_A(y), \forall x,y \in M$
3. $t_A(rx) \geq t_A(x), \forall x \in M, \forall r \in R$
   $i_A(rx) \geq i_A(x), \forall x \in M, \forall r \in R$
   $f_A(rx) \leq f_A(x), \forall x \in M, \forall r \in R$

We denote the set of all neutrosophic submodules of R module M by $U(M)$.

Definition 2.7.[16] For any neutrosophic subset $A = (x, t_A(x), i_A(x), f_A(x)) : x \in X$, the support A^* of the neutrosophic set A can be defined as

$$A^* = (x \in X, t_A(x) > 0, i_A(x) > 0, f_A(x) < 1)$$

.

Theorem 2.1.[16] Let $g: X \to Y$ be a mapping. Let A and B be two neutrosophic subsets of $X$and Y respectively, then

$$g(A^*) \subseteq (g(B))^*$$

If the map g is bijective, the equality holds .

Theorem 2.2. Let $A \in U(M)$. Then A^* is an R submodule of M

Proof. Let A be a neutrosophic submodule i.e. $A \in U(M)$ and $A^* = (x \in X, t_A(x) > 0, i_A(x) > 0, f_A(x) < 1)$. Let $x,y \in A^*$ and $a,b \in R$.

$$\begin{aligned} t_A(ax+by) &\geq t_A(ax) \wedge t_A(by) \\ &\geq t_A(x) \wedge t_A(y) \\ &\geq 0 \end{aligned}$$

Similarly $i_A(ax+by) \geq 0$ and $f_A(ax+by) \leq 1$, then $ax+by \in A^*$. Hence A^* is an R submodule of M.

Definition 2.8. The restriction $A|_N$ of a neutrosophic submodule A of an R module M to the submodule N of M is an object of the form

$$A|_N = \{x, t_{A|_N}(x), i_{A|_N}(x), f_{A|_N}(x) : x \in X\} \in U(N)$$

where

$$t_{A|_N}(x) = t_A(x)$$
$$i_{A|_N}(x) = i_A(x)$$
$$f_{A|_N}(x) = f_A(x)$$

Definition 2.9. The neutrosophic quotient submodule A_N of M/N is an object of the form

$$A_N = \{x+N, t_{A_N}(x+N), i_{A_N}(x+N), f_{A_N}(x+N) : x \in M\}$$

where

$$t_{A_N}(x+N) = \vee t_A(x+n) : n \in N$$
$$i_{A_N}(x+N) = \vee i_A(x+n) : n \in N$$
$$f_{A_N}(x+N) = \wedge f_A(x+n) : n \in N, \forall x \in M$$

where M/N denotes quotient submodule of M with respect to N and $A_N \in U(M/N)$

Remark: Let $A,B \in U(M)$ such that $A \subseteq B$. Also A^* and B^* are R submodules of M. Clearly $A^* \subseteq B^*$. Thus $A^*$is a submodule of B^*. Moreover $B|_{B^*} \in U(B^*)$. We define $C \in U^{B^*/A^*}$ as follows for $x \in B^*$

$$\begin{aligned} C(x+A^*) = (&\vee(t_B(x+y)), \vee(i_B(x+y)), \\ &\wedge(f_B(x+y)) : y \in A^*) \end{aligned}$$

then $C \in U(B^*|A^*)$ is called the quotient of B with respect to A.

Definition 2.10. The quotient of B with respect to A where $A,B \in U(M)$ is a neutrosophic submodule B/A of $U(B^*/A^*)$, represented as follows

$$\begin{aligned} B/A(x+A^*) = \{&x+A^*, t_{B/A}(x+A^*), i_{B/A}(x+A^*), \\ &f_{B/A}(x+A^*) : \forall x \in B^*)\} \end{aligned}$$

where

$$t_{B/A}(x+A^*) = \vee(t_B(x+y) : y \in A^*)$$

$$i_{B/A}(x+A^*) = \vee(i_B(x+y) : y \in A^*)$$

$$f_{B/A}(x+A^*) = \wedge(f_B(x+y) : y \in A^*)$$

Theorem 2.3. Let $A,B \in U(M)$ be such that $A \subseteq B$. Then $(B|_{B^*})_{A^*} = B/A$

3 Isomorphism for Neutrosophic Sub Modules

In this section, we study the isomorphism properties of neutrosophic submodules. Let M and N be modules over the ring R and $g : M \to N$ be a module homomorphism. Also let A and B be neutrosophic submodules of M and N respectively.

Definition 3.1. A homomorphism g of M in to N is called a weak neutrosophic homomorphism of A on to B if $g(A) \subseteq B$. If g is a weak neutrosophic homomorphism of A on to B, then A is weakly homomorphic to B and we write $A \sim B$. A homomorphism g of M in to N is called a neutrosophic homomorphism of A on to B if $g(A) = B$ and we represent it as $A \approx B$.

Definition 3.2. An isomorphism g of M in to N is called a weak neutrosophic isomorphism of A on to B if $g(A) \subseteq B$ and we write $A \simeq B$. An isomorphism g of M in to N is called neutrosophic isomorphism of A on to B if $g(A) = B$ and we write $A \cong B$.

Theorem 3.1.[First isomorphism theorem] Let $A \in U(M)$, $B \in U(N)$ such that $A \approx B$, then $\exists\, C \in U(M)$ such that $C \subseteq A$ and $A/C \cong B$.

*Proof.*Let $g : M \to N$ be a module homomorphism such that $g(A) = B$. Define $C \in U^M$ as follows,

$$t_C(x) = \begin{cases} t_A(x) & x \in \ker g \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$i_C(x) = \begin{cases} i_A(x) & x \in \ker g \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$$f_C(x) = \begin{cases} f_A(x) & x \in \ker g \\ 1 & \text{otherwise} \end{cases} \quad (3)$$

Then obviously $C \subseteq A$.

Case 1 If $x \in \ker g$, $y \in \ker g$ then $x+y \in \ker g$
$\Rightarrow t_C(x) = t_A(x)$, $t_C(y) = t_A(y)$ *and* $t_C(x+y) = t_A(x+y)$
$\therefore\ t_C(x+y) = t_A(x+y) \geq t_A(x) \wedge t_A(y) = t_C(x) \wedge t_A(y)$
$\Rightarrow t_C(x+y) \geq t_C(x) \wedge t_A(y)$
Similarly $i_C(x+y) \geq i_C(x) \wedge i_A(y)$ and $f_C(x+y) \leq f_C(x) \vee f_A(y)$.

Case 2 If $x \in \ker g$, $y \notin \ker\ g$ then $x+y \notin kerg$
$\Rightarrow t_C(x) = t_A(x)$, $t_C(y) = 0$, $t_C(x+y) = 0$ *and* $t_C(x) \wedge t_C(y) = 0$
$\therefore t_C(x+y) \geq t_C(x) \wedge t_C(y)$
similarly $i_C(x+y) \geq i_C(y) \wedge i_A(x)$ and $f_C(x+y) \leq f_C(x) \vee f_A(y)$.

Case 3 If $x \notin \ker g$, $y \in \ker g$ then $x+y \notin \ker g$. Then same as case 2

Case 4 If $x \notin \ker g$, $y \notin \ker g$ then $x+y \in \ker g$ or $x+y \notin \ker g$
$\Rightarrow t_C(x) = 0$ and $t_C(y) = 0$
If $x+y \in ker\ g$ then $t_C(x+y) = t_A(x+y)$
$\therefore t_C(x+y) \geq t_C(x) \wedge t_C(y)$
similarly $i_C(x+y) \geq i_C(y) \wedge i_A(x)$ and $f_C(x+y) \leq f_C(x) \vee f_A(y)$.
If $x+y \notin ker\ g$then $t_C(x+y) = 0$
$\therefore t_C(x+y) \geq t_C(x) \wedge t_C(y)$
similarly $i_C(x+y) \geq i_C(y) \wedge i_A(x)$ and $f_C(x+y) \leq f_C(x) \vee f_A(y)$.

From the above 4 cases

$$t_C(x+y) \geq t_C(y) \wedge t_C(x)$$
$$i_C(x+y) \geq i_C(y) \wedge i_C(x)$$
$$f_C(x+y) \leq f_C(x) \vee f_C(y).$$

So C is a neutrosophic submodule of A and $A \cong B \Rightarrow g(A) = B$ and hence $(g(A))^* = B^*$ i.e. $g(A^*) = B^*$. Now consider the map $F : A^* \to B^*$ where $F = g|A^*$. This is a module homomorphism with

$$\begin{aligned} kernel\ F &= \{x \in A^* : F(x) = 0\} \\ &= \{x \in A^* : g(x) = 0\} \\ &= C^* \end{aligned}$$

So $\exists$ a module isomorphism $h : A^*/C^* \to B^*$

$$h(x+C^*) = F(x) = g(x) \ \ \forall x \in A^*.$$

Now we have

$$h(A/C)(z) = (t_{h(A/C)}(z),\ i_{h(A/C)}(z),\ f_{h(A/C)}(z))$$

where

$$\begin{aligned} t_{h(A/C)}(z) &= \vee\{t_{A/C}(x+C^*) : x \in A^*, h(x+C^*) = z\} \\ i_{h(A/C)}(z) &= \vee\{i_{A/C}(x+C^*) : x \in A^*, h(x+C^*) = z\} \\ f_{h(A/C)}(z) &= \wedge\{f_{A/C}(x+C^*) : x \in A^*, h(x+C^*) = z\} \end{aligned}$$

Now,

$$\begin{aligned} t_{h(A/C)}(z) &= \vee\{t_{A/C}(x+C^*) : x \in A^*, h(x+C^*) = z\} \\ &= \vee\{\vee t_A(x+y) : y \in C^*, x \in A^*, g(x) = z\} \\ &= \vee\{t_A(x+y) : y \in C^*, x \in A^*, g(x) = z\} \\ &= \vee\{t_A(x+y) : y \in C^*, x \in A^*, g(x+y) = z\} \\ &= \vee\{t_A(u) : u \in A^*, g(u) = z\} \\ &\quad \textit{where } u = x+y \in A^* + C^* = A^* \\ &= t_{g(A)}(z) \\ &= t_B(z) \end{aligned}$$

$$\begin{aligned} i_{h(A/C)}(z) &= \vee\{i_{A/C}(x+C^*) : x\in A^*, h(x+C^*)=z\} \\ &= \vee\{\vee i_A(x+y) : y\in C^*, x\in A^*, g(x)=z\} \\ &= \vee\{i_A(x+y) : y\in C^*, x\in A^*, g(x)=z\} \\ &= \vee\{i_A(x+y) : y\in C^*, x\in A^*, g(x+y)=z\} \\ &= \vee\{i_A(u) : u\in A^*, g(u)=z\} \\ &\quad \textit{where } u=x+y\in A^*+C^*=A^* \\ &= i_{g(A)}(z) \\ &= i_B(z) \end{aligned}$$

$$\begin{aligned} f_{h(A/C)}(z) &= \wedge\{f_{A/C}(x+C^*) : x\in A^*, h(x+C^*)=z\} \\ &= \wedge\{\wedge f_A(x+y) : y\in C^*, x\in A^*, g(x)=z\} \\ &= \wedge\{i_A(x+y) : y\in C^*, x\in A^*, g(x)=z\} \\ &= \wedge\{f_A(x+y) : y\in C^*, x\in A^*, g(x+y)=z\} \\ &= \wedge\{f_A(u) : u\in A^*, g(u)=z\} \\ &\quad \textit{where } u=x+y\in A^*+C^*=A^* \\ &= f_{g(A)}(z) \\ &= f_B(z) \end{aligned}$$

Thus $h(A/C)\cong B \therefore A/C\cong B$.

Theorem 3.2.[Second isomorphism theorem] If $A \text{ and } B\in U(M)$, then $B/(A\cap B)\backsimeq(A+B)/A$.

Proof. We know that A^* and B^* are R submodules of M. By the second isomorphism theorem for modules, there is an isomorphism

$$g : B^*/(A^*\cap B^*)\to(A+B)^*/A^*,$$

where g is given by $g(x+(A\cap B)^*)=x+A^*, \forall\, x\in B^*$. i.e.$B^*/(A^*\cap B^*)\cong(A^*+B^*)/A^*$.

$$B^*/(A^*\cap B^*)\cong(A+B)^*/A^*.$$

Now

$$\begin{aligned} t_{g(B/A\cap B)}(x+A^*) = t_{B/(A\cap B)}&(x+(A\cap B)^*) \\ &= \vee\{t_B(z) : z\in x+(A\cap B)^*\} \\ &= \vee\{t_{A+B}(z) : z\in x+(A\cap B)^*\} \\ &\leq \{t_{A+B}(z) : z\in x+A^*\} \\ &= \{t_{A+B/A}(x+A)^*, \forall x\in B^*\} \end{aligned}$$

$\therefore$

$$t_{g(B/(A\cap B))}(x+A^*)\subseteq t_{A+B/A}(x+A^*)$$

similarly we can prove that

$$i_{g(B/(A\cap B))}(x+A^*)\subseteq i_{A+B/A}(x+A^*)$$

Now consider

$$\begin{aligned} f_{g(B/(A\cap B)}(x+A^*) = f_{B/(A\cap B)}&(x+(A\cap B)^* \\ &= \wedge\{f_B(z) : z\in x+(A\cap B)^*\} \\ &= \wedge\{f_{A+B}(z) : z\in x+(A\cap B)^*\} \\ &\leq \wedge\{f_{A+B}(z) : z\in x+(A)^*\} \\ &= \{f_{A+B}(z) : z\in x+(A)^*\}, \forall x\in B^* \\ &= \{f_{A+B/A}(z) : z\in x+A^*\} \end{aligned}$$

Thus,

$$f(B/A\cap B)\subseteq(A+B/A)$$

Hence, $(B/A\cap B)\backsimeq(A+B/A)$(weak neutrosophic isomorphism).

Theorem 3.3.[Third isomorphism theorem] If $A, B \textit{ and } C\in U(M)$ with $A\subseteq B\subseteq C$, then $(C/A)/(B/A)\cong C/B$.

*Proof.*Since $A\subseteq B\subseteq C$, then A^* is an R submodule of B^* and both $A^*$and $B^*$are R submodules of C^*. Then by third isomorphism theorem for modules $\dfrac{C^*/A^*}{B^*/A^*}\cong\dfrac{C^*}{B^*}$ where the isomorphism function g is defined as $g(x+A^*+(B^*/A^*))=x+B^*, \forall x\in C^*$
Now,

$$\begin{aligned} t_{g(\frac{C/A}{B/A})}(x+B^* &= t_{\frac{C/A}{B/A}}(x+A^*+(B^*/A^*)) \\ &= \vee\{t_{C/A}(y+A^*) : y\in C^*, \\ &\quad y+A^*\in x+A^*+(B^*/A^*)\} \\ &= \vee\{\vee\{t_C(z) : z\in y+A^*\}, \\ &\quad y\in C^*, y+A^*\in x+A^*+(B^*/A^*)\} \\ &= \vee\{t_C(z) : z\in C^*, \\ &\quad z+A^*\in x+A^*+(B^*/A^*)\} \\ &= \vee\{t_C(z) : z\in C^*, t(z)\in x+B^*\} \\ &= t_{C/B}(x+B^*), \forall x\in C^* \end{aligned}$$

similarly we can prove that $i_{g(\frac{C/A}{B/A})}(x+B^*)=i_{C/A}(x+B^*), \forall x\in C^*$ and

$$\begin{aligned} f_{g(\frac{C/A}{B/A})}(x+B^*) &= f_{(\frac{C/A}{B/A})}(x+A^*+(B^*/A^*)) \\ &= \wedge\{f_{C/A}(y+A^*) : y\in C^*, \\ &\quad y+A^*\in x+A^*+(B^*/A^*)\} \\ &= \wedge\{\wedge\{f_C(z) : z\in y+A^*\}, y\in C^*, \\ &\quad y+A^*\in x+A^*+(B^*/A^*)\} \\ &= \wedge\{f_C(z) : z\in C^*, \\ &\quad z+A^*\in x+A^*+(B^*+A^*)\} \\ &= \wedge\{f_C(z) : z\in C^*, f(z)\in x+B^*\} \\ &= f_{C/B}(x+B^*), \forall\, x\in C^* \end{aligned}$$

Thus $f_{\frac{C/A}{B/A}}=C/B$. Hence$\frac{C/A}{B/A}\cong\frac{C}{B}$.

4 Application of Neutrosophic Isomorphism

Neutrosophic set theory deals with information of imprecise, indeterminate and inconsistent data in natural language. Isomorphisms preserve the structure, operation and order of objects in neutrosophic set. Structural properties are invariant under neutrosophic isomorphism, and it corresponds to the essentially same objects and relabelling of objects. Based on the above mentioned properties of isomorphism in data structure, data clustering algorithms are easily evaluated from neutrosophic isomorphisms. We can analyse the equivalence of various representations of the design of an electronic circuit and identify chemical compounds using similarity measure in neutrosophic set. The neutrosophic isomorphism properties play a key role in medical imaging and behavioural science of complex system.

5 Conclusion

Neutrosophic submodule is one of the generalizations of an algebraic structure module. In this paper we have presented the algebraic structure module in a neutrosophic environment and investigated the algebraic properties of neutrosophic submodule. Also, the notion of neutrosophic quotient sub module, the restriction of neutrosophic submodule of R-module to the submodule of R-module have been defined, and its elementary algebraic properties are derived. This paper summarizes the fundamental concepts of neutrosophic set, neutrosophic sub module and its isomorphism properties. The three fundamental theorems of neutrosophic submodule isomorphism have been established in this paper. We can extend the study of neutrosophic submodules in injective and projective neutrosophic submodules. The isomorphism properties of neutrosophic structures find varied applications in engineering and multi parameter decision analysis.

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