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Dynamics of a Delayed Rumor Propagation Model with Consideration of Psychological Factors and Forgetting Mechanism

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Abstract: A delayed rumor propagation model with psychological factors and forgetting mechanism is formulated. The local stability of the rumor-free equilibrium and the rumor-prevailing equilibrium is discussed by analyzing the corresponding characteristic equations. Using Lyapunov functional, we prove that the rumor-free equilibrium is globally asymptotically stable when the basic reproduction number $R_0 \leq 1$. Also, a sufficient condition is obtained for the global asymptotic stability of the rumor-prevailing equilibrium when $R_0 > 1$. Numerical examples are presented to illustrate the theoretical results.

Keywords: Rumor spreading, local stability, global stability, time delay

1 Introduction

Rumors are a part of our everyday life, and affects the community as well as the individual. Mathematical modeling has played an important role in describing rumors. The first classical rumor spreading model, which was proposed in the 1960s by Daley and Kendall [1, 2], was called DK model. In this model, people are divided into three classes: ignorants (those who do not know rumors), spreaders (those who know and spread rumors) and stiflers (those who know rumors, but do not spread them). Afterwards, Maki and Thomson [3] developed another classical MK model in 1973. It focused on the analysis of rumor prevalence based on mathematical theory via direct contact between spreaders and others. Later, many scholars paid attention to the spread of rumors. Bettencourt et al. [4] have addressed spread process. Kawachi et al. [5] presented a rumor transmission model with various contact interactions. Al-Tuwairqi et al. [6] considered a rumor transmission model with incubation. Deng et al. [7] proposed a rumor propagation model with forget-remember mechanism.

Most previous rumor spread models mainly consider that rumor diffusion process meets the bilinear incidence rate. Rumor propagation is closely related to personal psychological quality. Thus, the bilinear incidence rate in real rumor spread is inappropriate. Recently, the nonlinear incidence rate in rumor propagation process has been suggested by several authors, see for example [8, 9].

Time delay is a common and inevitable phenomenon in nature, which is viewed as a latent period and immune period in epidemics (see, e.g. [10–16]). Similarly, in rumor spread mechanism, time delay occurs when the spreaders contact the ignorants because individuals may not timely respond to rumors. Laarabi et al. [17] introduced a rumor spreading model incorporating latent period. Li established a rumor model with time delay considering forgetting effect [18]. Zhu et al. [19] proposed time delay state feedback controller and described authorities actions in reaction-diffusion rumor spreading model. Li in [20] introduced a time delay rumor propagation model with a saturated control function in emergencies. Further works of rumor propagation models with time delay are found in [21–23].

In this work, we propose the following delayed rumor propagation model with psychological factors and

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forgetting mechanism :

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu_1 S(t) - \frac{\beta S(t)I(t-\tau)}{1+\alpha I(t-\tau)}, \\ \frac{dI(t)}{dt} = \frac{\beta S(t)I(t-\tau)}{1+\alpha I(t-\tau)} - \gamma I(t)(I(t)+R(t)) - \eta I(t) - \mu_2 I(t), \\ \frac{dR(t)}{dt} = \gamma I(t)(I(t)+R(t)) + \eta I(t) - \mu_3 R(t), \end{cases} \quad (1)$$

where $S(t)$, $I(t)$ and $R(t)$ denote the proportions of ignorants, spreaders and stiflers at time t, respectively. Λ is the recruitment rate of ignorant, η is the transfer rate from spreader to stifler due to forgetting mechanism, β is the rumor propagation rate from ignorant to spreader, and α is the saturation factors measuring the psychological or inhibitory effect of the general public toward rumor. γ is the transfer rate from spreader to stifler due to a spreader's contacts with another spreader or a stifler, with only the initiating spreader turns into a stifler. μ_1, μ_2 and μ_3 describe the removal rate of ignorant, spreader and stifler from the system when they lose interest in rumors. $\tau \geq 0$ is the average infectious delay of the rumor, i.e. the time when an individual infected with the rumor will become infectious. The parameters Λ, β, γ, μ_1, μ_2 and μ_3 are positive constants and parameters η and α are nonnegative constants.

The initial condition for system (1) takes the form

$$S(0) > 0,\ I(\theta) = \phi(\theta) \geq 0,\ \theta \in [-\tau, 0],\ \phi(0) > 0, \\ R(0) > 0, \quad (2)$$

where $\phi \in \mathscr{C}([-\tau,0],\mathbb{R}_+)$, the Banach space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}_+$.

From the fundamental theory of functional differential equations [24], system (1) has a unique solution $(S(t),I(t),R(t))$ satisfying the initial condition (2). It is easy to show that the solution $(S(t),I(t),R(t))$ of system (1) with initial condition (2) is positive for all $t > 0$.

Summing all equations of system (1) we find that the total population size $N(t) = S(t)+I(t)+R(t)$ satisfies the inequality

$$\frac{dN(t)}{dt} = \Lambda - \mu_1 S(t) - \mu_2 I(t) - \mu_3 R(t) \leq \Lambda - \mu N(t),$$

where $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. It follows that

$$\limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu},$$

so the meaningful feasible region of (1) is

$$\Delta = \left\{(S,I,R) \in \mathbb{R}_+^3 : S+I+R \leq \frac{\Lambda}{\mu}\right\}.$$

Lemma 1. The compact set

$$\Delta = \left\{(S,I,R) \in \mathbb{R}_+^3 : S+I+R \leq \frac{\Lambda}{\mu}\right\}$$

is positively invariant with respect to system (1), where $\mu = \min\{\mu_1, \mu_2, \mu_3\}$.

The rest of this paper is organized as follows: In the next section, we discuss the existence and the local stability of the equilibria using linearization method. In Section 3, by constructing suitable Lyapunov functionals, the global stability of equilibria is investigated. In Section 4, we present numerical examples to illustrate our results. Conclusion is presented in Section 5.

2 Local stability

In this section, we discuss the existence and the local asymptotic stability of the equilibria of system (1).

We define the basic reproduction number of model (1) as follows

$$R_0 = \frac{\beta\Lambda}{\mu_1(\eta+\mu_2)},$$

which represents the average number of secondary transmissions of the rumor [25].

It is easy to verify that model (1) always has a rumor-free equilibrium $E_0 = (\frac{\Lambda}{\mu_1}, 0, 0)$. Next, we obtain the following lemma which ensures the unique existence of a rumor-prevailing equilibrium E^* for $R_0 > 1$.

Lemma 2. If $R_0 > 1$, system (1) has a unique positive rumor-prevailing equilibrium $E^* = (S^*, I^*, R^*)$ with $I^* \in \left(0, \frac{\mu_3}{\gamma}\right)$.

Proof. Assume that $R_0 > 1$. From the second and third equations of (1), it follows that

$$S^* = \frac{1+\alpha I^*}{\beta}\left(\frac{\gamma(\eta+\mu_3)I^*}{\mu_3 - \gamma I^*} + \eta + \mu_2\right), \quad (3)$$

$$R^* = \frac{\gamma (I^*)^2 + \eta I^*}{\mu_3 - \gamma I^*}.$$

We have $R^* \geq 0$ implies that $I^* \in \left[0, \frac{\mu_3}{\gamma}\right)$. Hence, no positive equilibrium exists if $I^* \geq \frac{\mu_3}{\gamma}$.

Substituting equation (3) into the first equation of (1), we obtain that

$$F(I^*) = 0,$$

where

$$F(I) = \Lambda - \left(\frac{\mu_1(1+\alpha I)}{\beta} + I\right)\left(\frac{\gamma(\eta+\mu_3)I}{\mu_3 - \gamma I} + \eta + \mu_2\right), \\ I \in \left[0, \frac{\mu_3}{\gamma}\right).$$

We have

$$F(0) = \frac{\mu_1(\eta+\mu_2)}{\beta}(R_0 - 1) > 0,\ \lim_{\substack{I\to\frac{\mu_3}{\gamma} \\ I<\frac{\mu_3}{\gamma}}} F(I) = -\infty,$$

and

$$F'(I) = -\left(\frac{\alpha\mu_1}{\beta}+1\right)\left(\frac{\gamma(\eta+\mu_3)I}{\mu_3-\gamma I}+\eta+\mu_2\right) - \left(\frac{\mu_1(1+\alpha I)}{\beta}+I\right)\frac{\gamma\mu_3(\eta+\mu_3)}{(\mu_3-\gamma I)^2} < 0.$$

Hence, there exists unique $I^* \in \left(0, \frac{\mu_3}{\gamma}\right)$ such that $F(I^*) = 0$ if $R_0 > 1$. This shows that model (1) has a unique positive rumor-prevailing equilibrium $E^* = (S^*, I^*, R^*)$ when $R_0 > 1$. □

Now, we discuss the local behavior of the rumor-free equilibrium E_0. Let $x(t) = S(t) - \frac{\Lambda}{\mu_1}$, $y(t) = I(t)$ and $z(t) = R(t)$. Then, the linearized system of (1) around E_0 takes the following form

$$\begin{cases} \frac{dx(t)}{dt} = -\mu_1 x(t) - \frac{\beta\Lambda}{\mu_1} y(t-\tau), \\ \frac{dy(t)}{dt} = \frac{\beta\Lambda}{\mu_1} y(t-\tau) - (\eta+\mu_2)\, y(t), \\ \frac{dz(t)}{dt} = \eta y(t) - \mu_3 z(t). \end{cases} \quad (4)$$

The associated characteristic equation of system (4) can be described as

$$(\lambda+\mu_1)(\lambda+\mu_3)\left[\lambda+(\eta+\mu_2)\left(1-R_0 e^{-\lambda\tau}\right)\right] = 0. \quad (5)$$

Theorem 1. If $R_0 < 1$, the rumor-free equilibrium E_0 is locally asymptotically stable for all $\tau \geq 0$. E_0 is unstable if $R_0 > 1$.

Proof. If no delay exists, i.e. $\tau = 0$, Eq. (5) becomes

$$(\lambda+\mu_1)(\lambda+\mu_3)\left[\lambda+(\eta+\mu_2)(1-R_0)\right] = 0. \quad (6)$$

It is clear that (6) has three roots $\lambda_1 = -\mu_1 < 0$, $\lambda_2 = -\mu_3 < 0$ and $\lambda_3 = (\eta+\mu_2)(R_0-1)$. Hence, the equilibrium E_0 will be locally asymptotically stable if $R_0 < 1$ for $\tau = 0$.

Now, we address the circumstances of delay. Assume that (5) has a purely imaginary root $\lambda = i\omega$, with $\omega > 0$. Then, separating real and imaginary parts gives

$$\begin{cases} (\eta+\mu_2) R_0 \cos\omega\tau = (\eta+\mu_2), \\ (\eta+\mu_2) R_0 \cos\omega\tau = -\omega. \end{cases}$$

Hence

$$\omega^2 = (\eta+\mu_2)^2\left(R_0^2-1\right),$$

which ensures that Eq. (5) has no purely imaginary roots if $R_0 < 1$. Then, the equilibrium E_0 is locally asymptotically stable for any delay $\tau \geq 0$ if $R_0 < 1$.

Let $G(\lambda) = \lambda + (\eta+\mu_2)(1-R_0e^{-\lambda\tau})$. We have $G(0) = -(\eta+\mu_2)(R_0-1) < 0$ if $R_0 > 1$ and $\lim_{\lambda\to+\infty} G(\lambda) = +\infty$. Hence, $G(\lambda) = 0$ has a positive real root. Therefore, if $R_0 > 1$, the rumor-free equilibrium E_0 is unstable. The proof is completed.

Next, we focus on the local stability of the rumor-prevailing equilibrium E^* by assuming that $R_0 > 1$. Let $x(t) = S(t) - S^*$, $y(t) = I(t) - I^*$ and $z(t) = R(t) - R^*$. Then, by linearizing system (1) around E^*, we get the following system

$$\begin{cases} \frac{dx(t)}{dt} = -m_1 x(t) - m_2 y(t-\tau), \\ \frac{dy(t)}{dt} = m_3 x(t) + m_2 y(t-\tau) - m_4 y(t) - m_5 z(t), \\ \frac{dz(t)}{dt} = m_6 y(t) - m_7 z(t), \end{cases} \quad (7)$$

where

$$\begin{aligned} m_1 &= \mu_1 + \frac{\beta I^*}{1+\alpha I^*} > 0, \\ m_2 &= \frac{\beta S^*}{(1+\alpha I^*)^2} > 0, \\ m_3 &= \frac{\beta I^*}{1+\alpha I^*} > 0, \\ m_4 &= 2\gamma I^* + \gamma R^* + \eta + \mu_2 > 0, \\ m_5 &= \gamma I^* > 0, \\ m_6 &= 2\gamma I^* + \gamma R^* + \eta > 0, \\ m_7 &= \mu_3 - \gamma I^* > 0. \end{aligned}$$

Characteristic equation, which is associated with system (7), is given by

$$\det\begin{pmatrix} \lambda+m_1 & m_2e^{-\lambda\tau} & 0 \\ -m_3 & \lambda - m_2e^{-\lambda\tau}+m_4 & m_5 \\ 0 & -m_6 & \lambda+m_7 \end{pmatrix} = 0,$$

which is equivalent to

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 - (q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau} = 0, \quad (8)$$

where

$$\begin{aligned} p_2 &= m_1 + m_4 + m_7 > 0, \\ p_1 &= m_4(m_1+m_7) + m_1m_7 + m_5m_6 > 0, \\ p_0 &= m_1\left(m_4m_7 + m_5m_6\right) > 0, \\ q_2 &= m_2 > 0, \\ q_1 &= m_2(m_1 - m_3 + m_7) = m_2(\mu_1 + m_7) > 0, \\ q_0 &= m_2m_7(m_1 - m_3) = \mu_1 m_2 m_7 > 0. \end{aligned}$$

In the absence of delay, we have the following result.

Theorem 1. If $R_0 > 1$, then, when $\tau = 0$, the rumor-prevailing equilibrium E^* is locally asymptotically stable.

Proof. When $\tau = 0$, the characteristic equation (8) becomes

$$\lambda^3 + (p_2-q_2)\lambda^2 + (p_1-q_1)\lambda + p_0 - q_0 = 0, \quad (9)$$

where

$$\begin{aligned} p_2 - q_2 &= m_1 + m_7 + (m_4 - m_2), \\ p_1 - q_1 &= (\mu_1 + m_7)(m_4 - m_2) + m_1m_7 + m_3m_4 + m_5m_6, \\ p_0 - q_0 &= \mu_1 m_7(m_4 - m_2) + m_3m_4m_7 + m_1m_5m_6. \end{aligned}$$

Note that

$$m_4 - m_2 = \gamma I^* + \frac{\alpha\beta S^* I^*}{(1+\alpha I^*)^2} > 0, \qquad (10)$$

then it is easy to show that $p_2 - q_2 > 0$, $p_1 - q_1 > 0$, $p_0 - q_0 > 0$ and $(p_2 - q_2)(p_1 - q_1) > p_0 - q_0$. Thus, by the Routh-Hurwitz criterion, all roots λ_i $(i = 1,2,3)$ of (9) have negative real part. Then, we can conclude that when E^* exists (i.e., $R_0 > 1$), it is locally asymptotically stable for $\tau = 0$. The proof is completed.

Now, we handle the case of positive delay $\tau > 0$. We have the following theorem.

Theorem 3. If $R_0 > 1$ and $\mu_3 \leq 2\mu_2$, the rumor-prevailing equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.

Proof. According to Theorem 2, E^* is locally asymptotically stable for $\tau = 0$ if $R_0 > 1$. For $\tau > 0$, let $\lambda = i\omega$ be a root of Eq. (8), with $\omega > 0$. Then,

$$\begin{cases} \left(q_2\omega^2 - q_0\right)\cos w\tau - q_1\omega\sin w\tau = p_2\omega^2 - p_0, \\ \left(q_2\omega^2 - q_0\right)\sin w\tau + q_1\omega\cos w\tau = -\omega^3 + p_1\omega. \end{cases} \qquad (11)$$

Squaring and adding the two equations in (11), we obtain

$$\omega^6 + \left(p_2^2 - q_2^2 - 2p_1\right)\omega^4 + \left(p_1^2 - q_1^2 - 2p_2p_0 + 2q_2q_0\right)\omega^2 + p_0^2 - q_0^2 = 0. \qquad (12)$$

We have

$$\begin{aligned} & p_2^2 - q_2^2 - 2p_1 \\ &= m_1^2 + m_7^2 + m_4^2 - m_2^2 - 2m_5m_6 \\ &= \left(\mu_1 + \frac{\beta I^*}{1+\alpha I^*}\right)^2 + (\mu_3 - \gamma I^*)^2 + (2\gamma I^* + \gamma R^* + \eta + \mu_2)^2 \\ &\quad - \left(\frac{\beta S^*}{(1+\alpha I^*)^2}\right)^2 - 2\gamma I^*(2\gamma I^* + \gamma R^* + \eta) \\ &= \left(\mu_1 + \frac{\beta I^*}{1+\alpha I^*}\right)^2 + (\mu_3 - \gamma I^*)^2 + (\gamma I^*)^2 \\ &\quad + 2\gamma I^*(\gamma I^* + \gamma R^* + \eta + \mu_2) + (\gamma I^* + \gamma R^* + \eta + \mu_2)^2 \\ &\quad - \left(\frac{\beta S^*}{(1+\alpha I^*)^2}\right)^2 - 2\gamma I^*(2\gamma I^* + \gamma R^* + \eta) \\ &= \left(\mu_1 + \frac{\beta I^*}{1+\alpha I^*}\right)^2 + \left(\frac{\beta S^*}{1+\alpha I^*} + \frac{\beta S^*}{(1+\alpha I^*)^2}\right)\frac{\alpha\beta S^* I^*}{(1+\alpha I^*)^2} \\ &\quad + \mu_3(\mu_3 - \gamma I^*) + \gamma I^*(2\mu_2 - \mu_3), \end{aligned}$$

then $p_2^2 - q_2^2 - 2p_1 > 0$. Furthermore, we have

$$\begin{aligned} & p_1^2 - q_1^2 - 2p_2p_0 + 2q_2q_0 \\ &= m_1^2m_4^2 + m_1^2m_7^2 + \left(m_4^2 - m_2^2\right)m_7^2 + 2m_4m_5m_6m_7 \\ &\quad + m_5^2m_6^2 - \mu_1^2m_2^2 - 2m_1^2m_5m_6. \end{aligned}$$

Note that

$$\begin{aligned} & m_1^2m_4^2 - \mu_1^2m_2^2 - 2m_1^2m_5m_6 \\ &= \left(\mu_1 + \frac{\beta I^*}{1+\alpha I^*}\right)^2 (2\gamma I^* + \gamma R^* + \eta + \mu_2)^2 \\ &\quad - \mu_1^2\left(\frac{\beta S^*}{(1+\alpha I^*)^2}\right)^2 \\ &\quad - 2\left(\mu_1 + \frac{\beta I^*}{1+\alpha I^*}\right)^2 \gamma I^*(2\gamma I^* + \gamma R^* + \eta) \\ &= \gamma I^*\left(\mu_1 + \frac{\beta I^*}{1+\alpha (I^*)^2}\right)^2 (2\mu_2 - \mu_3 + \mu_3 - \gamma I^*) \\ &\quad + \left(\frac{2\mu_1\beta I^*}{1+\alpha I^*} + \left(\frac{\beta I^*}{1+\alpha I^*}\right)^2\right)\left(\frac{\beta S^*}{1+\alpha I^*}\right)^2 \\ &\quad + \mu_1^2\left(\frac{\beta S^*}{1+\alpha I^*} + \frac{\beta S^*}{(1+\alpha I^*)^2}\right)\frac{\alpha\beta S^* I^*}{(1+\alpha I^*)^2}, \end{aligned}$$

then $m_1^2m_4^2 - \mu_1^2m_2^2 - 2m_1^2m_5m_6 > 0$, and from (10), we have $m_4^2 - m_2^2 > 0$ since $m_2, m_4 > 0$. Then, $p_1^2 - q_1^2 - 2p_2p_0 + 2q_2q_0 > 0$. In addition, we have

$$p_0^2 - q_0^2 = (p_0 - q_0)(p_0 + q_0) > 0,$$

since $p_0 - q_0 > 0$ and $p_0, q_0 > 0$.

Thus, Eq. (12) has no positive real roots, which ensures that Eq. (8) has no purely imaginary roots. Hence, the equilibrium E^* is asymptotically stable for any delay $\tau \geq 0$ if $R_0 > 1$ and $\mu_3 \leq 2\mu_2$. The proof is completed.

3 Global stability

We use the convention that $S = S(t)$, $I = I(t)$, $R = R(t)$ and $I_\tau = I(t-\tau)$ to simplify the following calculations.

First, we consider the global stability of model (1) at the rumor-free equilibrium E_0.

Theorem 4. If $R_0 \leq 1$, the rumor-free equilibrium E_0 is globally asymptotically stable for all $\tau \geq 0$.

Proof. Let U be the Lyapunov functional defined as

$$U(t) = \frac{1}{2}\left(S - \frac{\Lambda}{\mu_1}\right)^2 + \frac{\Lambda}{\mu_1}I + \frac{\Lambda}{\mu_1}(\eta + \mu_2)\int_0^\tau I(t-u)du.$$

Differentiating U along the solutions of system (1), we have

$$
\begin{aligned}
\frac{dU}{dt} &= (S-\frac{\Lambda}{\mu_1})\left(\Lambda-\mu_1 S-\frac{\beta SI_\tau}{1+\alpha I_\tau}\right)+\frac{\Lambda}{\mu_1}\frac{\beta SI_\tau}{1+\alpha I_\tau} \\
&\quad -\frac{\Lambda}{\mu_1}\gamma I(I+R)-\frac{\Lambda}{\mu_1}(\eta+\mu_2)I+\frac{\Lambda}{\mu_1}(\eta+\mu_2)I \\
&\quad -\frac{\Lambda}{\mu_1}(\eta+\mu_2)I_\tau \\
&= -\mu_1(S-\frac{\Lambda}{\mu_1})^2-(S-\frac{\Lambda}{\mu_1})\frac{\beta SI_\tau}{1+\alpha I_\tau}+\frac{\Lambda}{\mu_1}\frac{\beta SI_\tau}{1+\alpha I_\tau} \\
&\quad -\frac{\Lambda}{\mu_1}\gamma I(I+R)-\frac{\Lambda}{\mu_1}(\eta+\mu_2)I_\tau \\
&= -\mu_1(S-\frac{\Lambda}{\mu_1})^2-(S-\frac{\Lambda}{\mu_1})^2\frac{\beta I_\tau}{1+\alpha I_\tau} \\
&\quad -\frac{\Lambda}{\mu_1}(S-\frac{\Lambda}{\mu_1})\frac{\beta I_\tau}{1+\alpha I_\tau}+\frac{\Lambda}{\mu_1}\frac{\beta SI_\tau}{1+\alpha I_\tau} \\
&\quad -\frac{\Lambda}{\mu_1}\gamma I(I+R)-\frac{\Lambda}{\mu_1}(\eta+\mu_2)I_\tau \\
&= -(S-\frac{\Lambda}{\mu_1})^2\left(\mu_1+\frac{\beta I_\tau}{1+\alpha I_\tau}\right)-\frac{\Lambda}{\mu_1}\gamma I(I+R) \\
&\quad +\left(\frac{\Lambda}{\mu_1}\right)^2\frac{\beta I_\tau}{1+\alpha I_\tau}-\frac{\Lambda}{\mu_1}(\eta+\mu_2)I_\tau \\
&= -(S-\frac{\Lambda}{\mu_1})^2\left(\mu_1+\frac{\beta I_\tau}{1+\alpha I_\tau}\right)-\frac{\Lambda}{\mu_1}\gamma I(I+R) \\
&\quad +\frac{\Lambda}{\mu_1}\left[\frac{\beta\Lambda}{\mu_1}\frac{1}{1+\alpha I_\tau}-(\eta+\mu_2)\right]I_\tau \\
&\le -(S-\frac{\Lambda}{\mu_1})^2\left(\mu_1+\frac{\beta I_\tau}{1+\alpha I_\tau}\right)-\frac{\Lambda}{\mu_1}\gamma I(I+R) \\
&\quad +\frac{\Lambda}{\mu_1}(R_0-1)I_\tau.
\end{aligned}
$$

Therefore, $R_0\le 1$ ensures that $\frac{dU}{dt}\le 0$. Furthermore, it is easy to verify that the singleton $\{E_0\}$ is the largest compact invariant set in $\{(S,I,R)\in\mathbb{R}^3_+ : \frac{dU}{dt}=0\}$. Adopting the LaSalle's invariance principle [26], we conclude that E_0 is globally asymptotically stable if $R_0\le 1$. □

For the global stability of rumor-prevailing equilibrium E^* of model (1), we have the following result.

Theorem 5. Assume that $R_0>1$. If $\mu\mu_3\ge\gamma\Lambda$, the rumor-prevailing equilibrium E^* is globally asymptotically stable.

Proof.

Consider the Lyapunov functional

$$V(t)=V_1(t)+V_2(t)+\beta S^* f(I^*)V_3(t)+\omega V_4(t),$$

where

$$
\begin{aligned}
V_1(t) &= S^* g\left(\frac{S}{S^*}\right), \\
V_2(t) &= I^* g\left(\frac{I}{I^*}\right), \\
V_3(t) &= \int_0^\tau g\left(\frac{I(t-u)}{I^*}\right)du, \\
V_4(t) &= \frac{1}{2}(R-R^*)^2, \\
f(x) &= \frac{x}{1+\alpha x}, \\
g(x) &= x-1-\ln x\ge g(1)=0 \ \text{ for any } x>0,
\end{aligned}
$$

and ω is a positive constant which will be defined later.

First, we calculate the derivative of V_1 along the solutions of system (1).

$$
\begin{aligned}
\frac{dV_1}{dt} &= \left(1-\frac{S^*}{S}\right)\frac{dS}{dt} \\
&= \left(1-\frac{S^*}{S}\right)(\Lambda-\mu_1 S-\beta Sf(I_\tau)) \\
&= \left(1-\frac{S^*}{S}\right)(-\mu_1(S-S^*)+\beta S^* f(I^*)-\beta Sf(I_\tau)) \\
&= -\mu_1\frac{(S-S^*)^2}{S}+\beta S^* f(I^*)\left(1-\frac{S^*}{S}\right)\left(1-\frac{Sf(I_\tau)}{S^* f(I^*)}\right) \\
&= -\mu_1\frac{(S-S^*)^2}{S}+\beta S^* f(I^*)\left(1-\frac{S^*}{S}-\frac{Sf(I_\tau)}{S^* f(I^*)}+\frac{f(I_\tau)}{f(I^*)}\right).
\end{aligned}
$$

Then, calculating the derivative of V_2 along the solutions of (1) gives

$$
\begin{aligned}
\frac{dV_2}{dt} &= \left(1-\frac{I^*}{I}\right)\frac{dI}{dt} \\
&= \left(1-\frac{I^*}{I}\right)(\beta Sf(I_\tau)-\gamma I(I+R)-(\eta+\mu_2)I) \\
&= \left(1-\frac{I^*}{I}\right)\Big(\beta Sf(I_\tau)-\gamma I(I+R)-\beta S^* f(I^*)\frac{I}{I^*} \\
&\quad +\gamma I(I^*+R^*)\Big) \\
&= \left(1-\frac{I^*}{I}\right)\left(\beta Sf(I_\tau)-\beta S^* f(I^*)\frac{I}{I^*}\right)-\gamma(I-I^*)^2 \\
&\quad -\gamma(I-I^*)(R-R^*) \\
&= \beta S^* f(I^*)\left(1-\frac{I^*}{I}\right)\left(\frac{Sf(I_\tau)}{S^* f(I^*)}-\frac{I}{I^*}\right)-\gamma(I-I^*)^2 \\
&\quad -\gamma(I-I^*)(R-R^*) \\
&= \beta S^* f(I^*)\left(\frac{Sf(I_\tau)}{S^* f(I^*)}-\frac{I}{I^*}-\frac{I^* Sf(I_\tau)}{IS^* f(I^*)}+1\right)-\gamma(I-I^*)^2 \\
&\quad -\gamma(I-I^*)(R-R^*).
\end{aligned}
$$

Now, we calculate the derivative of V_3.

$$
\begin{aligned}
\frac{dV_3}{dt} &= g\left(\frac{I}{I^*}\right)-g\left(\frac{I_\tau}{I^*}\right) \\
&= \frac{I}{I^*}-\frac{I_\tau}{I^*}+\ln\left(\frac{I_\tau}{I^*}\right)-\ln\left(\frac{I}{I^*}\right) \\
&= \frac{I}{I^*}-\frac{I_\tau}{I^*}+\ln\left(\frac{I_\tau}{I}\right).
\end{aligned}
$$

Differentiating V_4 along the solutions of system (1), we have

$$\begin{aligned}\frac{dV_4}{dt} &= (R-R^*)\frac{dR}{dt}\\ &= (R-R^*)(\gamma I(I+R)+\eta I-\mu_3 R)\\ &= (R-R^*)(\gamma I(I+R)+\eta I-\mu_3(R-R^*)-\mu_3 R^*)\\ &= (R-R^*)(\gamma I(I+R)+\eta I-\mu_3(R-R^*)\\ &\quad -\gamma I^*(I^*+R^*)-\eta I^*)\\ &= (R-R^*)(\gamma(I-I^*)(I+I^*+R^*)+\gamma I(R-R^*)\\ &\quad +\eta(I-I^*)-\mu_3(R-R^*))\\ &= (\gamma(I+I^*+R^*)+\eta)(I-I^*)(R-R^*)\\ &\quad -(\mu_3-\gamma I)(R-R^*)^2.\end{aligned}$$

Then,

$$\begin{aligned}\frac{dV}{dt} &= -\mu_1\frac{(S-S^*)^2}{S}+\beta S^* f(I^*)\left(1-\frac{S^*}{S}-\frac{Sf(I_\tau)}{S^*f(I^*)}+\frac{f(I_\tau)}{f(I^*)}\right)\\ &\quad +\beta S^* f(I^*)\left(\frac{Sf(I_\tau)}{S^*f(I^*)}-\frac{I}{I^*}-\frac{I^*Sf(I_\tau)}{IS^*f(I^*)}+1\right)\\ &\quad -\gamma(I-I^*)^2-\gamma(I-I^*)(R-R^*)\\ &\quad +\beta S^* f(I^*)\left(\frac{I}{I^*}-\frac{I_\tau}{I^*}+\ln\left(\frac{I_\tau}{I}\right)\right)\\ &\quad +\omega(\gamma(I+I^*+R^*)+\eta)(I-I^*)(R-R^*)\\ &\quad -\omega(\mu_3-\gamma I)(R-R^*)^2.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{dV}{dt} &= -\mu_1\frac{(S-S^*)^2}{S}+\beta S^* f(I^*)\left(2-\frac{S^*}{S}+\frac{f(I_\tau)}{f(I^*)}\right.\\ &\quad \left.-\frac{I^*Sf(I_\tau)}{IS^*f(I^*)}-\frac{I_\tau}{I^*}+\ln\left(\frac{I_\tau}{I}\right)\right)-\gamma(I-I^*)^2\\ &\quad +(\omega(\gamma(I+I^*+R^*)+\eta)-\gamma)(I-I^*)(R-R^*)\\ &\quad -\omega(\mu_3-\gamma I)(R-R^*)^2\\ &= -\mu_1\frac{(S-S^*)^2}{S}-\beta S^* f(I^*)\left\{\frac{S^*}{S}-1-\ln\left(\frac{S^*}{S}\right)\right.\\ &\quad +\frac{I^*Sf(I_\tau)}{IS^*f(I^*)}-1-\ln\left(\frac{I^*Sf(I_\tau)}{IS^*f(I^*)}\right)\\ &\quad \left.+\frac{I_\tau f(I^*)}{I^*f(I_\tau)}-1-\ln\left(\frac{I_\tau f(I^*)}{I^*f(I_\tau)}\right)\right\}\\ &\quad -\beta S^* f(I^*)\left(1-\frac{I_\tau f(I^*)}{I^*f(I_\tau)}-\frac{f(I_\tau)}{f(I^*)}+\frac{I_\tau}{I^*}\right)-\gamma(I-I^*)^2\\ &\quad +(\omega(\gamma(I+I^*+R^*)+\eta)-\gamma)(I-I^*)(R-R^*)\\ &\quad -\omega(\mu_3-\gamma I)(R-R^*)^2.\end{aligned}$$

Then,

$$\begin{aligned}\frac{dV}{dt} &= -\mu_1\frac{(S-S^*)^2}{S}-\beta S^* f(I^*)\left(g\left(\frac{S^*}{S}\right)+g\left(\frac{I^*Sf(I_\tau)}{IS^*f(I^*)}\right)\right.\\ &\quad \left.+g\left(\frac{I_\tau f(I^*)}{I^*f(I_\tau)}\right)\right)\\ &\quad -\beta S^* f(I^*)\left(1-\frac{I_\tau f(I^*)}{I^*f(I_\tau)}-\frac{f(I_\tau)}{f(I^*)}+\frac{I_\tau}{I^*}\right)\\ &\quad +\frac{1}{2}(\omega(\gamma(I+I^*+R^*)+\eta)-\gamma)(I-I^*+R-R^*)^2\\ &\quad -\frac{1}{2}(\omega(\gamma(I+I^*+R^*)+\eta)+\gamma)(I-I^*)^2\\ &\quad +\frac{1}{2}(\gamma-\omega(\gamma(I^*+R^*)+\eta)+\omega\gamma I-2\omega\mu_3)(R-R^*)^2.\end{aligned}$$

Note that

$$1-\frac{I_\tau f(I^*)}{I^*f(I_\tau)}-\frac{f(I_\tau)}{f(I^*)}+\frac{I_\tau}{I^*}=\frac{\alpha(I_\tau-I^*)^2}{I^*(1+\alpha I^*)(1+\alpha I_\tau)}.$$

Then,

$$\begin{aligned}\frac{dV}{dt} &= -\mu_1\frac{(S-S^*)^2}{S}-\beta S^* f(I^*)\left(g\left(\frac{S^*}{S}\right)\right.\\ &\quad \left.+g\left(\frac{I^*Sf(I_\tau)}{IS^*f(I^*)}\right)+g\left(\frac{I_\tau f(I^*)}{I^*f(I_\tau)}\right)\right)\\ &\quad -\beta S^* f(I^*)\frac{\alpha(I_\tau-I^*)^2}{I^*(1+\alpha I^*)(1+\alpha I_\tau)}\\ &\quad +\frac{1}{2}(\omega(\gamma(I+I^*+R^*)+\eta)-\gamma)(I-I^*+R-R^*)^2\\ &\quad -\frac{1}{2}(\omega(\gamma(I+I^*+R^*)+\eta)+\gamma)(I-I^*)^2\\ &\quad +\frac{1}{2}(\gamma-\omega(\gamma(I^*+R^*)+\eta)+\omega\gamma I-2\omega\mu_3)(R-R^*)^2.\end{aligned}$$

By Lemma 1, we have $I\le\frac{\Lambda}{\mu}$. Hence,

$$\begin{aligned}\frac{dV}{dt} &\le -\mu_1\frac{(S-S^*)^2}{S}-\beta S^* f(I^*)\left(g\left(\frac{S^*}{S}\right)\right.\\ &\quad \left.+g\left(\frac{I^*Sf(I_\tau)}{IS^*f(I^*)}\right)+g\left(\frac{I_\tau f(I^*)}{I^*f(I_\tau)}\right)\right)\\ &\quad -\beta S^* f(I^*)\frac{\alpha(I_\tau-I^*)^2}{I^*(1+\alpha I^*)(1+\alpha I_\tau)}\\ &\quad +\frac{1}{2}\left(\omega\left(\gamma\left(\frac{\Lambda}{\mu}+I^*+R^*\right)+\eta\right)-\gamma\right)\\ &\quad \times(I-I^*+R-R^*)^2\\ &\quad -\frac{1}{2}(\omega(\gamma(I+I^*+R^*)+\eta)+\gamma)(I-I^*)^2\\ &\quad +\frac{1}{2}\left(\gamma-\omega(\gamma(I^*+R^*)+\eta)+\omega\gamma\frac{\Lambda}{\mu}-2\omega\mu_3\right)\\ &\quad \times(R-R^*)^2.\end{aligned}$$

Choose $\omega = \frac{\gamma}{\gamma\left(\frac{\Lambda}{\mu}+I^*+R^*\right)+\eta}$, then

$$\frac{dV}{dt} \leq -\mu_1\frac{(S-S^*)^2}{S} - \beta S^* f(I^*)\left(g\left(\frac{S^*}{S}\right)\right.$$
$$\left. +g\left(\frac{I^* S f(I_\tau)}{I S^* f(I^*)}\right) + g\left(\frac{I_\tau f(I^*)}{I^* f(I_\tau)}\right)\right)$$
$$-\beta S^* f(I^*)\frac{\alpha(I_\tau - I^*)^2}{I^*(1+\alpha I^*)(1+\alpha I_\tau)}$$
$$-\frac{1}{2}\left(\omega\left(\gamma(I+I^*+R^*)+\eta\right)+\gamma\right)(I-I^*)^2$$
$$+\omega\left(\gamma\frac{\Lambda}{\mu}-\mu_3\right)(R-R^*)^2.$$

Hence, $\mu\mu_3 \geq \gamma\Lambda$ ensures that $\frac{dV}{dt} \leq 0$, and the equality occurs only at E^*. Applying the LaSalle's invariance principle, we can obtain that the rumor-prevailing equilibrium E^* of model (1) is globally asymptotically stable if $R_0 > 1$ and $\mu\mu_3 \geq \gamma\Lambda$.

4 Numerical examples

Example 1. Consider the following parameters $\Lambda = 0.25$, $\beta = 0.4$, $\gamma = 0.4$, $\eta = 0.1$, $\alpha = 0.7$, $\mu_1 = 0.4$, $\mu_2 = 0.3$, $\mu_3 = 0.7$. By calculation, we obtain $R_0 = 0.625$. Hence, according to Theorem 4, the rumor-free equilibrium $E_0 = (0.625, 0, 0)$ is globally asymptotically stable for different delays suggesting that the rumor disappears.

Example 2. We keep all the system (1) parameters the same as in Example 1 except that μ_2 reduced from 0.3 to 0.1. Then, $R_0 = 1.25 > 1$, and we can conclude, by Theorem 1, that the rumor-free equilibrium $E_0 = (0.625, 0, 0)$ is unstable.

Example 3. Consider the following parameters $\Lambda = 0.15$, $\beta = 0.45$, $\gamma = 0.25$, $\eta = 0.08$, $\alpha = 0.9$, $\mu_1 = 0.2$, $\mu_2 = 0.15$, $\mu_3 = 0.5$. Then, $R_0 = 1.4673 > 1$ and $\mu\mu_3 = 0.075 \geq \gamma\Lambda = 0.0375$. From Theorem 5, the rumor-prevailing equilibrium E^* is globally asymptotically stable for different delays, which means that rumor persists.

5 Conclusion

In this paper, we have presented a delayed rumor propagation model with psychological factors and forgetting mechanism. From the model we get the equilibria and the basic reproduction number R_0. We have shown that if $R_0 \leq 1$, the rumor-free equilibrium is globally asymptotically stable, which means that the rumor disappears. Moreover, we have proved that if $R_0 > 1$, the rumor-prevailing equilibrium is globally asymptotically stable provided that $\mu\mu_3 \geq \gamma\Lambda$, so rumor will persist at the unique positive equilibrium.

Competing interests

The authors declare that they have no competing interests.

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