Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/140417

Numerical Study of MHD-Duct Flow Using the Two-Dimensional Finite Difference Method

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Received: 4 Jan. 2020, Revised: 2 Mar. 2020, Accepted: 13 Apr. 2020
Published online: 1 Jul. 2020

Abstract: This article presents a two-dimensional finite difference method (FDM) to solve the model of magnetohydrodynamics (MHD) duct flow of an incompressible fluid in a rectangular duct under the action of an external uniform magnetic field applied transverse to the flow direction has been investigated. All the sides of the duct are electrically insulated. The coupled momentum and magnetic induction equations for velocity and induced magnetic field are first transformed into coupled non-dimensional equations by introducing non-dimensional quantities.

Keywords: Two-dimensional FDM, MHD-duct flow, Magnetic field

1 Introduction

Several researchers have recently investigated the analytical and numerical solutions of some partial differential equations that describe some physical, engineering, and other phenomena [1–5]. A study of Hartmann flow is fundamental to understand of the basic physical processes that occur in devices, such as the electromagnetic brake, the magnetohydrodynamic generator, the electromagnetic pump and the plasma accelerator [6, 7]. However, these solutions showed considerable inaccuracies, particularly for large values of the Hartmann number (applied magnetic field) [8]. The Hartmann number is the ratio of magnetic to fluid viscosity. If $M = 0$, the flow field is the classieal laminar pipe flow but if $M \geqslant 1$, the flow field is defined primarily by the $E * A$ drift. Jones and Xenophontos [9] have further revised the Kantorovich method and extended its applicability by coupling the technique with the finite element method. They have also explored MHD-duct flow and the results showed a considerable improvement in accuracy compared to the previous approaches. The two-dimensional B-spline specific elements were also applied within the B-spline specific elements method to solve the MHD channel flow problem [10]. The suggested problem was addressed in some papers [11, 12]. In this work, the two-dimensional finite-difference technique [13, 14] is proposed and implemented to solve the MHD-duct flow problem. The results are compared with the analytical solution of Shercliff [15] and the numerical solutions of Alexander [8], Jones and Xenophontos [9], L.R.T. Gardner and G.A. Gardner [10]. The present analysis is close to the analytical solution, which are consistent with the previous numerical results.
The present paper aims are to use a new numerical method to discuss the effect of the magnetic field on fluid flow within a rectangular channel.
The present paper is prepared as: In Section Two, we handle the analysis of the problem. The application of the proposed method is presented in Section Three. Section Four presents some numerical results and some figures. In Section Five, we provide an explanation of the results. Section Six, is devoted to conclusion.

2 MHD duct flow

The cross-section of an infinitely long rectangular duct is oriented with its sides parallel to the x and $y$axes and the origin of coordinates at its center. The duct width is $2a$ and the height is $2b$, the sides of the duct have equations $x = \pm a$ and $y = \pm b$. A conducting fluid flows in the z direction along the duct and is subjected to a constant applied magnetic field M acting in a direction lying in the xy-plane and making an angle ϕ with the y-axis. The equations governing the flow may be expressed in the

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normalized form [9, 16].

$$\frac{\partial P}{\partial z} = \mu\nu\nabla^2\nabla_z + \frac{A_{0x}}{\mu_0}\frac{\partial P_z}{\partial x'} + \frac{A_{0y}}{\mu_0}\frac{\partial P_z}{\partial y'}, \quad (1)$$

and the z-component of the curl of Ohm's law,

$$\nabla^2 A_z + \xi\mu_0(A_{0x}\frac{\partial U_z}{\partial x'} + A_{0y}\frac{\partial U_z}{\partial y'}) = 0, \quad (2)$$

with the boundary conditions: $U = A = 0$ at $x' = \pm\alpha, y' = \pm b$,
where ν, μ and ξ are the kinematic viscosity, density and electric conductivity of the fluid; μ_0 is the magnetic permeability in vacuum; dP/dz is the constant axial pressure gradient; B_{0x} and B_{0y} are the x' and y' components of the applied magnetic field; and U_z and A_z are the z components of velocity and induced magnetic field, respectively. Following the notation of P. C. Lu [16], who solved this problem using the Kantorovieh method, Eqs. (1) and (2) become in non-dimensionalized form,

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})U + M_x\frac{\partial A}{\partial x} + M_y\frac{\partial A}{\partial y} = -1, \quad (3)$$

and

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})A + M_x\frac{\partial U}{\partial x} + M_y\frac{\partial U}{\partial y} = -1, \quad (4)$$

with boundary conditions $U = A = 0, x = \pm\alpha, y = \pm 1$. Distance has been scaled to the duct semi-height b, so $x = x'/b$, $y = y'/b$, and $\alpha = a/b$. The following normalisations have also been used.

$$\begin{aligned}
U &= \frac{U_z}{\frac{-b^2}{\nu\mu}\frac{dP}{dz}},\\
A &= \frac{A_z}{\frac{-b^2}{\nu\mu}\frac{dP}{dz}\mu_0(\nu\mu\xi)^{\frac{1}{2}}},\\
M_x &= A_{0x'}b(\frac{\xi}{\nu\mu})^{\frac{1}{2}} = M sin(\phi),\\
M_y &= A_{0y'}b(\frac{\xi}{\nu\mu})^{\frac{1}{2}} = M cos(\phi),\\
M &= Hartmann\ no. = (M_x^2 + M_y^2)^{\frac{1}{2}} = A_0 b(\frac{\xi}{\nu\mu})^{\frac{1}{2}}.
\end{aligned} \quad (5)$$

The Hartmann number is the ratio of magnetic to fluid viscosity. If $M = 0$, the flow field is the classieal laminar pipe flow. If $M \geq 1$, the flow field is defined primarily by the $E \times A$ drift. To uncouple (3) and (4), the functions

$$H_1 = U + A, \quad (6)$$

and

$$H_2 = U - A, \quad (7)$$

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})H_1 + M_x\frac{\partial H_1}{\partial x} + M_y\frac{\partial H_1}{\partial y} = -1, \quad (8)$$

and

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})H_2 - M_x\frac{\partial H_2}{\partial x} - M_y\frac{\partial H_2}{\partial y} = -1, \quad (9)$$

with boundary conditions $H_1 = H_2 = 0, x = \pm\alpha, y = \pm 1$. Thus, if H_1 is solved as $H_1(M_x, M_y)$ from (8), then

$$H_2(M_x, M_y) = H_1(-M_x, -M_y). \quad (10)$$

Hence, the solution is completely determined when either H_1 or H_2, is known. Having determined H_1 the function H_2 is found from (10) and so the velocity field U from

$$U = \frac{1}{2}(H_1 + H_2). \quad (11)$$

3 The two-dimensional finite difference method

In this section, we take approximations for spaces x and y derivatives as:

$$\begin{aligned}
H_x &\simeq \frac{c_{i+1,j} - c_{i-1,j}}{2h},\\
H_{xx} &\simeq \frac{c_{i-1,j} + c_{i+1,j} - 2c_{i,j}}{h^2},\\
H_y &\simeq \frac{c_{i,j+1} - c_{i,j-1}}{2k},\\
H_{yy} &\simeq \frac{c_{i,j-1} + c_{i,j+1} - 2c_{i,j}}{k^2}.
\end{aligned} \quad (12)$$

Now, we assume that H_1 and H_2 are the exact solution at the grid point (x_i, y_j) and $c_{i,j}$ and $\delta_{i,j}$ are the numerical solution at the same point. Substituting (12) into (8) and (9) we get

$$\begin{aligned}
&\frac{c_{i-1,j} + c_{i+1,j} - 2c_{i,j}}{h^2} + M_x\frac{(c_{i+1,j} - c_{i-1,j})}{2h}\\
&\quad + \frac{c_{i,j-1} + c_{i,j+1} - 2c_{i,j}}{k^2}\\
&\quad + M_y\frac{(c_{i,j+1} - c_{i,j-1})}{2k} + 1 = 0,
\end{aligned} \quad (13)$$

and

$$\begin{aligned}
&\frac{\delta_{i-1,j} + \delta_{i+1,j} - 2\delta_{i,j}}{h^2} - M_x\frac{(\delta_{i+1,j} - \delta_{i-1,j})}{2h}\\
&\quad + \frac{\delta_{i,j-1} + \delta_{i,j+1} - 2\delta_{i,j}}{k^2}\\
&\quad - M_y\frac{(\delta_{i,j+1} - \delta_{i,j-1})}{2k} + 1 = 0.
\end{aligned} \quad (14)$$

Now, we can solve (13) and (14) using various methods.

4 Numerical results

In this section, we introduce some numerical results for the flow in a square duct with an applied magnetic field parallel to the $x-$axis so $M_y = 0$. To compare with the previous results [8–10, 15], we give to M the following values $M_x = 0, 2, 5$ and 8. In Table 1 we record the values of the velocity U at the duct centre with various numerical algorithms.

Table 1: U at the centre of the duct

M_x	Alexander [8]	ones and Xenophontos [9]	Bi-cubic B-spline [10]	Finite difference method	Analytic [15]
0	0.29824	0.29826	0.294685	0.294107	0.294686
2	0.26320	0.26319	0.258907	0.258625	0.258907
5	0.17430	0.17428	0.171602	0.171475	0.258907
8	0.12014	0.12013	0.118782	0.118652	0.118782

In Table 1, the results of the finite-difference method using a mesh of 20×20 were compared with those of the numerical Kantorovich technique [8], with those obtained by Jones and Xenophontos [9] using a finite element iterative adaptation of the Kantorovich technique, bi-cubic B-spline finite elements [10] and with the analytical solution of Shercliff [15]. In Figure 1, we show the velocity along the x-axis of a square duct with the magnetic field parallel to the x-axis, for Hartmann numbers 0 (top curve), to 8 (bottom curve).

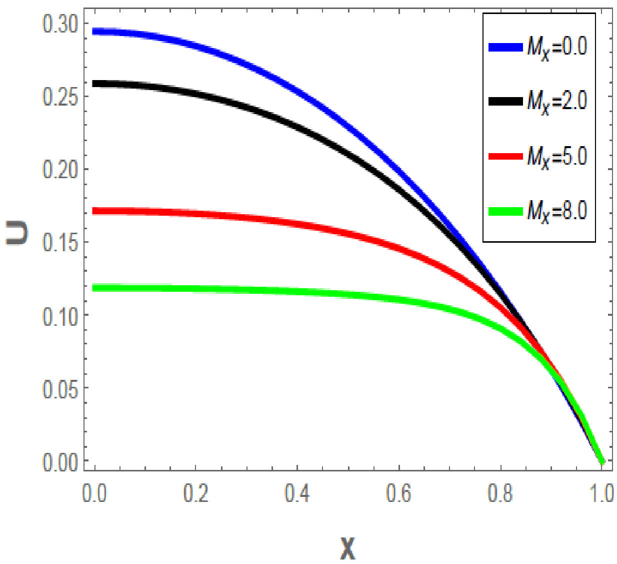

Fig. 1: Velocity profile along the x-axis of a square duct with the magnetic field parallel to the x-axis, for Hartmann numbers 0 (top curve), to 8 (bottom curve).

In Table 2, some other results are presented where the period from [-1, 1] to [-0.5, 0.5] changed with different mesh points. We also compare these results with the analytical solution of the research [15].

Table 2: U at the centre of the duct. Finite difference and analytic simulations are compared.

M_x	Finite differenc method using a mesh of 20×20	Finite differenc method using a mesh of 50×50	Analytic [15]
0	0.073526	0.073648	0.0736711
2	0.071009	0.071109	0.071128
5	0.060795	0.060838	0.060846
8	0.049334	0.049359	0.0493638

In Figure 2, we show the velocity along the x-axis of a square duct with the magnetic field parallel to the x-axis, for Hartmann numbers 0 (top curve), to 8 (bottom curve).

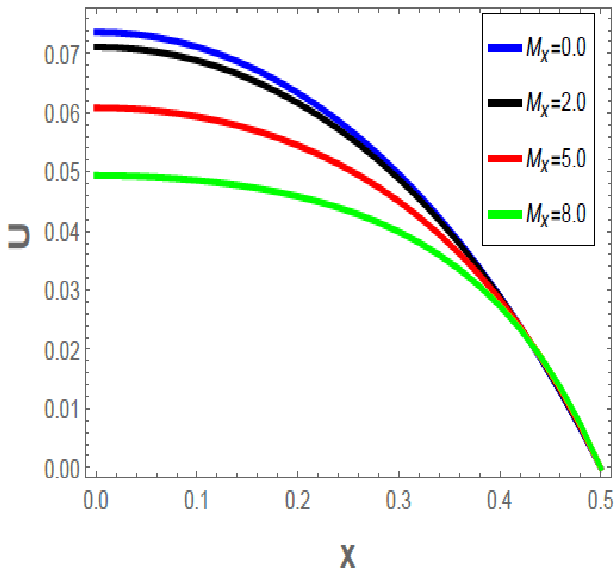

Fig. 2: Velocity profile along the x-axis of a square duct with the magnetic field parallel to the x-axis, for Hartmann numbers 0 (top curve), to 8 (bottom curve).

In Figures 1,2 and 3, the velocity U profiles along the x- and y-axes of the duct cross-section are given. Results obtained with the finite difference method are shown as a continuous curve and compared with the analytic solution [15] shown as dots. The agreement is very good for all values of the Hartmann number from 0 to 8. These graphs reveal considerable improvement in accuracy compared to those reported by Jones and Xenophontos [9].

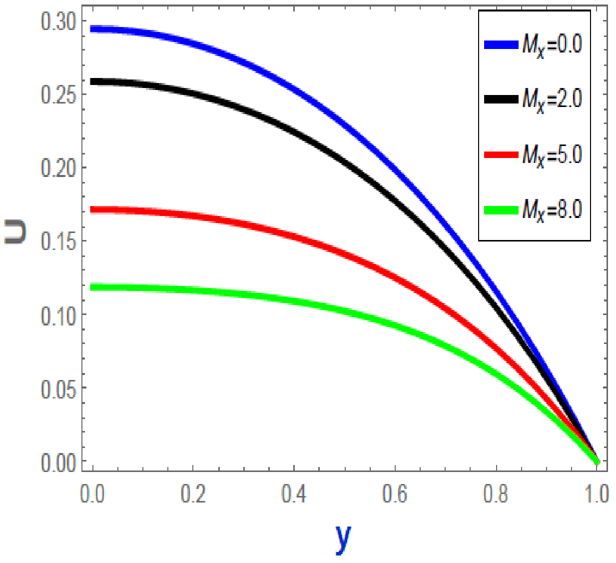

Fig. 3: Velocity profile along the y-axis of a square duct with the magnetic field parallel to the x-axis, for Hartmann numbers 0 to 8.

5 Physical explanations for the results

For various values of the Hartmann number, the solution for the velocity profile along the x and y axes is shown in figures 1, 2 and 3. As expected, increasing the magnetic field (increasing the Hartmann number) has an effect on the velocity of the fluid where the velocity decreases near the center of the channel. This apparent effect of the magnetic field intensity is already known. Accordingly, the results are compatible with the physical meaning of the effect of the magnetic field

6 Conclusion

We have investigated MHD-duct flow in a rectangular duct using a two-dimensional finite-difference approach. The numerical solution obtained for the MHD problem, in terms of to four decimal places with the analytic solution at a mesh of 50×50, exhibits higher accuracy than the other numerical methods at a mesh of 20×20 in [8, 9]. Our results are consistent with the results in [10]. It is a very useful addition to the numerical methods available for the solution of the steady-state MHD-duct flow equation and other similar equations.

Acknowledgement

We would like to thank the editor-in-chief of the journal. Also, sincere thanks go to the reviewers for their constructive and fruitful comments, which improved the paper.

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