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Fuzzy Soft Inner Product Spaces

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Abstract: In this paper, we investigate the structure of a fuzzy soft inner product on fuzzy soft linear spaces and invoke a definition in terms of fuzzy soft points. We examine some properties and examples of fuzzy soft inner product spaces as well as fuzzy soft Cauchy-Schwartz inequality. It is shown that fuzzy soft orthogonality and fuzzy soft Hilbert spaces can be used as new tools to understand the most complex problems.

Keywords: Fuzzy set, Fuzzy soft inner product space, Fuzzy soft linear space, Fuzzy soft set, Soft set

1 Introduction

In real world, complexity generally arises from uncertainty represented the form of ambiguity. Thus, we always have various complicated problems in areas, such as economics, engineering, medical science, environmental science, sociology, business management and many other fields. We cannot successfully use classical mathematical methods to overcome difficulties of uncertainties in those problems. In 1965, Zadeh [1] extended the set theory when introducing the theory of fuzzy sets to deal with uncertainty. Just as a crisp set on a universal set X is defined by its characteristic function from X to $\{0,1\}$, a fuzzy set on a domain X is defined by its membership (characteristic) function from X to $[0,1]$.

In 1999, Molodtsov [2] introduced an extension of the set theory namely soft set theory to overcome uncertainties and solve complicated problems which cannot be handled using classical methods in many areas such as Riemann integration, environmental science, decision making, physics, engineering, computer science, medicine, economics and many other fields. The soft set is a mathematical tool for modeling uncertainty by associating a set with a set of parameters, i.e. it is a parameterized family of subsets of the universal set. After that, several authors introduced new extended concepts based on soft sets, gave examples for them and investigated their properties, including soft point [3], soft metric spaces [4], soft normed spaces [5], soft inner product spaces [6] and soft Hilbert spaces [7], etc. Despite this progress in real life problems and situations, we still have inexact information on our considered objects. Hence, to improve those two concepts; fuzzy set and soft set, Maji et al. [8] combined them together in one concept denoted as fuzzy soft set. This new concept widened the soft sets approach from crisp (ordinary) cases to fuzzy cases which are more general than the others. In recent years, numerous authors applied this notion and gave some concepts such as fuzzy soft point [9], fuzzy soft real number [10], fuzzy soft metric spaces [11] and fuzzy soft normed spaces [12].

In this paper, we surpass these previous studies by introducing the fuzzy soft inner product on fuzzy soft vector spaces, studying its properties and giving fuzzy soft Hilbert space definition. The rest of the paper is organized as follows: Section(2) introduces the basic concepts and definitions for fuzzy set, soft set and fuzzy soft set. In addition, it presents fuzzy soft metric space, fuzzy soft normed space and their properties. Section(3) covers the fuzzy soft inner product spaces and some related properties. Furthermore, fuzzy soft Cauchy-Schwartz inequality, fuzzy soft orthogonality and fuzzy soft Hilbert space are investigated. Section(4) is devoted to conclusion and further research.

2 Definitions and Preliminaries

This section lists some notations, definitions and preliminaries of fuzzy set, soft set and fuzzy soft set

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which shall be needed in the following discussion. In addition, it presents the fuzzy soft point definition including our present modification. Furthermore, it introduces the definitions of fuzzy soft metric space and fuzzy soft normed space as well as their properties which have been studied before and can be used in the new results.

Definition 2.1.(Fuzzy set)[1] Let U be a universal set (space of points or objects). A fuzzy set (class) X over U is a set characterized by a function $f_X : U \to [0,1]$. f_X is called the membership, characteristic or indicator function of the fuzzy set X and the value $f_X(u)$ is called the grade of membership of $u \in U$ in X.

Definition 2.2.(Soft set)([2],[13]) Let U be a universal set, E be a set of parameters and $A \subseteq E$. The power set of U is defined by $P(U) = 2^U$. A pair (F,A) is called a soft set over U and is defined as a set of ordered pairs $F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in P(U)\}$, where F is a mapping given by $F : A \to P(U)$. A is called the support of F_A and we have $F_A(e) \neq \phi$ for all $e \in A$ and $F_A(e) = \phi$ for all $e \notin A$. In other words, a soft set (F,A) over U is a parameterized family of the set U, denoted by $\tilde{U}$. Clearly, a soft set is not a set.

Definition 2.3.(Fuzzy soft set)[8] Let U be a universal set, E be a set of parameters and $A \subseteq E$. A pair (G,A) is called a fuzzy soft set over U, where G is a mapping given by $G : A \to \mathscr{F}(U)$, $\mathscr{F}(U)$ is the family of all fuzzy subsets of U (the power set of fuzzy sets on U) and the fuzzy subset of U is defined as a map f from U to $[0,1]$.

Definition 2.4.(Absolute or complete fuzzy soft set)[8] A fuzzy soft set (G,A) over a universal set U is said to be an absolute (or complete) fuzzy soft set, denoted by $\widetilde{C}_A$, if $f_{G_A(a)} = 1$ for all $a \in A$.

Definition 2.5.(Null fuzzy soft set)[8] A fuzzy soft set (G,A) over a universal set U is said to be a null fuzzy soft set, denoted by $\widetilde{\Phi}$, if for all $a \in A$, we have $f_{G_A(a)} = 0$.

Definition 2.6.(Fuzzy soft containment)[8] Let (G,A) and (H,B) be two fuzzy soft sets over a common universal set U. (G,A) is said to be a fuzzy soft subset of (H,B) if $A \subseteq B$, and $G(a) \subseteq H(a)$ for all $a \in A$. We write $(G,A)\widetilde{\subseteq}(H,B)$. In this case, (H,B) is said to be a fuzzy soft superset of (G,A), denoted by $(H,B)\widetilde{\supseteq}(G,A)$.

Definition 2.7.(Equality of two fuzzy soft sets)[8] Two fuzzy soft sets (G,A) and (H,B) over a common universal set U are said to be fuzzy soft equal, denoted by $(G,A)\widetilde{=}(H,B)$, if they are fuzzy soft subsets of each other, i.e. (G,A) is a fuzzy soft subset of (H,B) and (H,B) is a fuzzy soft subset of (G,A).

Definition 2.8.(The complement of a fuzzy soft set)[8] The complement of a fuzzy soft set (G,A) is defined by $(G,A)^C = (G^C,A)$, where $G^C : A \to \mathscr{F}(U)$ is a mapping given by $f_{G^C(a)} = 1 - f_{G(a)}$ for all $a \in A$.

It should be noted that $1 - f_{G(a)}$ denotes the fuzzy complement of $f_{G(a)}$.

Definition 2.9.(Union of two fuzzy soft sets)[8] The union of two fuzzy soft sets (G,A) and (H,B) over a common universal set U is a fuzzy soft set (S,C), written as $(S,C) = (G,A)\widetilde{\cup}(H,B)$, where $C = A \cup B$ and for all $c \in C$,

$$f_{S(c)}(u) = \begin{cases} f_{G(c)}(u),\ if\ c \in A - B, u \in U \\ f_{H(c)}(u),\ if\ c \in B - A, u \in U \\ \max[f_{G(c)}(u), f_{H(c)}(u)],\ if\ c \in A \cap B, u \in U. \end{cases}$$

Definition 2.10.(Intersection of two fuzzy soft sets)[8] The intersection of two fuzzy soft sets (G,A) and (H,B) over a common universal set U is a fuzzy soft set (S,C), written as $(S,C) = (G,A)\widetilde{\cap}(H,B)$, where $C = A \cup B$ and for all $c \in C$,

$$f_{S(c)}(u) = \begin{cases} f_{G(c)}(u),\ if\ c \in A - B, u \in U \\ f_{H(c)}(u),\ if\ c \in B - A, u \in U \\ \min[f_{G(c)}(u), f_{H(c)}(u)],\ if\ c \in A \cap B, u \in U. \end{cases}$$

To discuss efficiently, we consider only all fuzzy soft sets (G,A) over a universal set U in which all the parameter sets A are the same. The family of all these fuzzy soft sets is denoted by $FSS(U)_A = FSS(\tilde{U})$. The following definition and its consequent related definitions take their present formula according to our modification as follows:

Definition 2.11.(Fuzzy soft point)([9],[11]) The fuzzy soft set $(G,A) \in FSS(\tilde{U})$ is called a fuzzy soft point over U, denoted by $(u_{f_{G(e)}}, A)$ (briefly denoted by $\tilde{u}_{f_{G(e)}}$), if for the element $e \in A$ and $u \in U$,

$$f_{G(e)}(u) = \begin{cases} \alpha\ ,\ if\ u = u_0 \in U\ and\ e = e_0 \in A, \\ 0\ ,\ if\ u \in U - \{u_0\}\ or\ e \in A - \{e_0\} \end{cases},$$

where $\alpha \in (0,1]$ is the value of the characteristic function (membership degree). The fuzzy soft point can be considered the quadruple (u_0, e_0, G, α).

Definition 2.12.(The complement of a fuzzy soft point)[14] The fuzzy soft point $\tilde{u}_{f_{G^C(e)}}$ is called the fuzzy soft complement of a fuzzy soft point $\tilde{u}_{f_{G(e)}}$, denoted by $(\tilde{u}_{f_{G(e)}})^C$, if for the element $e \in A$ and $u \in U$,

$$f_{G^C(e)}(u) = \begin{cases} 1 - f_{G(e)}(u),\ if\ u = u_0 \in U\ and\ e = e_0 \in A, \\ 0\ ,\ if\ u \in U - \{u_0\}\ or\ e \in A - \{e_0\}. \end{cases}$$

Definition 2.13.[9] The fuzzy soft point $\tilde{u}_{f_{G(e)}}$ is said to be in the fuzzy soft set (H,A) (belongs to it), denoted by $\tilde{u}_{f_{G(e)}} \tilde{\in} (H,A)$, if for the element $e \in A$, we have $G(e) \subseteq H(e)$.

Definition 2.14.(Equal fuzzy soft points)([11],[15]) Two fuzzy soft points $(u^1_{f_{1_{G(e_1)}}}, A)$ and $(u^2_{f_{2_{G(e_2)}}}, A)$ over a common universal set U are said to be equal fuzzy soft points if $u^1 = u^2$, $e_1 = e_2$ and $f_1 = f_2$.

Definition 2.15.(Different fuzzy soft points)([11],[15]) Two fuzzy soft points $(u^1_{f_{1_{G(e_1)}}}, A)$ and $(u^2_{f_{2_{G(e_2)}}}, A)$ over a common universal set U are said to be different fuzzy soft points if $u^1 \neq u^2$, $e_1 \neq e_2$ or $f_1 \neq f_2$.

The collection of all fuzzy soft points over U is denoted by $FSP(U)_A = FSP(\tilde{U})$.

Proposition 2.1.[11] Let (G,A) be a fuzzy soft set over a universal set U. Then (G,A) can be expressed as a union of all its fuzzy soft points (belonging to it). Conversely, any set of fuzzy soft points can be considered as a fuzzy soft set, i.e., $G_A = \bigcup_{\tilde{u}_{f_{G(e)}} \in FSP(\tilde{U})} \tilde{u}_{f_{G(e)}}$.

Definition 2.16.(Fuzzy soft real set)([10],[15]) Let $\mathbb{R}$ be the set of all real numbers, E be a set of parameters, $A \subseteq E$ and $\mathscr{F}^{\mathbb{B}(\mathbb{R})}$ be the collection of all non-empty fuzzy bounded subsets of $\mathbb{R}$. A pair $(\mathfrak{R},A)$ is called a fuzzy soft real set over $\mathbb{R}$ and is defined as a set of ordered pairs $\mathfrak{R}_A = \{(e,\mathfrak{R}_A(e)) : e \in A, \mathfrak{R}_A(e) \in \mathscr{F}^{\mathbb{B}(\mathbb{R})}\}$, where $\mathfrak{R}$ is a mapping given by $\mathfrak{R} : A \to \mathscr{F}^{\mathbb{B}(\mathbb{R})}$. A is called the support of $\mathfrak{R}_A$.

Definition 2.17.(Fuzzy soft real number)([10],[15]) The fuzzy soft real set $(\mathfrak{R},A)$ is called a fuzzy soft real number in $\mathbb{R}$, denoted by (r,A) (briefly denoted by $\tilde{r}$), if it is a singleton fuzzy soft real set. $\mathbb{R}(A)$ denotes the set of all fuzzy soft real numbers and $\mathbb{R}^+(A)$ denotes the set of all non-negative fuzzy soft real numbers.

Definition 2.18.(Fuzzy soft complex set)[16] Let $\mathbb{C}$ be the set of all complex numbers, E be a set of parameters, $A \subseteq E$ and $\mathscr{F}^{\mathbb{B}(\mathbb{C})}$ be the collection of all non-empty fuzzy bounded subsets of $\mathbb{C}$. A pair $(\mathfrak{C},A)$ is called a fuzzy soft complex set over $\mathbb{C}$ and is defined as a set of ordered pairs $\mathfrak{C}_A = \{(e,\mathfrak{C}_A(e)) : e \in A, \mathfrak{C}_A(e) \in \mathscr{F}^{\mathbb{B}(\mathbb{C})}\}$, where $\mathfrak{C}$ is a mapping given by $\mathfrak{C} : A \to \mathscr{F}^{\mathbb{B}(\mathbb{C})}$. A is called the support of $\mathfrak{C}_A$.

Definition 2.19.(Fuzzy soft complex number)[16] The fuzzy soft complex set $(\mathfrak{C},A)$ is called a fuzzy soft complex number in $\mathbb{C}$, denoted by (c,A) (briefly denoted by $\tilde{c}$), if it is a singleton fuzzy soft complex set. $\mathbb{C}(A)$ denotes the set of all fuzzy soft complex numbers.

Definition 2.20.(Fuzzy soft metric space)[11] A fuzzy soft metric space $(\tilde{U},\tilde{d})$ is a fuzzy soft set $\tilde{U}$ with a fuzzy soft real-valued function $\tilde{d} : FSP(\tilde{U}) \times FSP(\tilde{U}) \to \mathbb{R}^+(A)$ satisfying the fuzzy soft metric conditions as the following:

(FSM1) $\tilde{d}(\tilde{u}^1_{f_{1_{G(e_1)}}}, \tilde{u}^2_{f_{2_{G(e_2)}}}) \tilde{\geq} \tilde{0}$, for all $\tilde{u}^1_{f_{1_{G(e_1)}}}, \tilde{u}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSP(\tilde{U})$,
and $\tilde{d}(\tilde{u}^1_{f_{1_{G(e_1)}}}, \tilde{u}^2_{f_{2_{G(e_2)}}}) \tilde{=} \tilde{0} \Leftrightarrow \tilde{u}^1_{f_{1_{G(e_1)}}} \tilde{=} \tilde{u}^2_{f_{2_{G(e_2)}}}$.

(FSM2) $\tilde{d}(\tilde{u}^1_{f_{1_{G(e_1)}}}, \tilde{u}^2_{f_{2_{G(e_2)}}}) \tilde{=} \tilde{d}(\tilde{u}^2_{f_{2_{G(e_2)}}}, \tilde{u}^1_{f_{1_{G(e_1)}}})$, for all $\tilde{u}^1_{f_{1_{G(e_1)}}}, \tilde{u}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSP(\tilde{U})$.

(FSM3) $\tilde{d}(\tilde{u}^1_{f_{1_{G(e_1)}}}, \tilde{u}^2_{f_{2_{G(e_2)}}}) \tilde{\leq} \tilde{d}(\tilde{u}^1_{f_{1_{G(e_1)}}}, \tilde{u}^3_{f_{3_{G(e_3)}}}) + \tilde{d}(\tilde{u}^3_{f_{3_{G(e_3)}}}, \tilde{u}^2_{f_{2_{G(e_2)}}})$,
for all $\tilde{u}^1_{f_{1_{G(e_1)}}}, \tilde{u}^2_{f_{2_{G(e_2)}}}, \tilde{u}^3_{f_{3_{G(e_3)}}} \tilde{\in} FSP(\tilde{U})$.

Let U be a vector space over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$) and the parameter set E be the set of all real numbers $\mathbb{R}$ and $A \subseteq E$.

Definition 2.21.(Fuzzy soft vector)([12],[17]) The fuzzy soft set $(G,A) \in FSS(\tilde{U})$ is called a fuzzy soft vector over U, denoted by $(v_{f_{G(e)}},A)$ (briefly denoted by $\tilde{v}_{f_{G(e)}}$), if there is exactly one $e \in A$ such that $f_{G(e)}(v) = \alpha$ for some $v \in U$ and $f_{G(e')}(v) = 0$ for all $e' \in A - \{e\}$ ($\alpha \in (0,1]$ is the value of the membership degree).

The set of all fuzzy soft vectors over U is denoted by $FSV(U)_A = FSV(\tilde{U})$.

Proposition 2.2.([12],[17]) The set $FSV(\tilde{U})$ is a fuzzy soft vector (fuzzy soft linear) space according to the following two operations:

1. $\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}} = (\widetilde{v^1+v^2})_{(f_{1_{G(e_1)}}+f_{2_{G(e_2)}})}$, for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$.
2. $\tilde{r} \cdot \tilde{v}_{f_{G(e)}} = (\widetilde{rv})_{f_{G(re)}}$, for all $\tilde{v}_{f_{G(e)}} \tilde{\in} FSV(\tilde{U})$ and for all $\tilde{r} \tilde{\in} \mathbb{R}(A)$.

Definition 2.22.(Fuzzy soft normed space)([12],[17]) Let $FSV(\tilde{U})$ be a fuzzy soft vector space. Then, a mapping $\widetilde{\|\cdot\|} : FSV(\tilde{U}) \to \mathbb{R}^+(A)$ is said to be a fuzzy soft norm on $FSV(\tilde{U})$ if $\widetilde{\|\cdot\|}$ satisfies the following conditions:

(FSN1) $\widetilde{\|\tilde{v}_{f_{G(e)}}\|} \tilde{\geq} \tilde{0}$, for all $\tilde{v}_{f_{G(e)}} \tilde{\in} FSV(\tilde{U})$, and $\widetilde{\|\tilde{v}_{f_{G(e)}}\|} \tilde{=} \tilde{0} \Leftrightarrow \tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\theta}$.

(FSN2) $\widetilde{\|\tilde{r} \cdot \tilde{v}_{f_{G(e)}}\|} \tilde{=} |\tilde{r}| \widetilde{\|\tilde{v}_{f_{G(e)}}\|}$, for all $\tilde{v}_{f_{G(e)}} \tilde{\in} FSV(\tilde{U})$ and for all fuzzy soft scalar $\tilde{r}$.

(FSN3) $\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}\|} \tilde{\leq} \widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}}\|} + \widetilde{\|\tilde{v}^2_{f_{2_{G(e_2)}}}\|}$, for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$.

The fuzzy soft vector space $FSV(\tilde{U})$ with a fuzzy soft norm $\widetilde{\|\cdot\|}$ is said to be a fuzzy soft normed linear space and is denoted by $(\tilde{U},\widetilde{\|\cdot\|})$.

Definition 2.23.(Fuzzy soft convergence)[15] A sequence of fuzzy soft vectors $\{\tilde{v}^n_{f_{n_{G(e_n)}}}\}$ in a fuzzy soft normed space $(\tilde{U},\widetilde{\|\cdot\|})$ is said to be fuzzy soft convergent and converges to $\tilde{v}^0_{f_{0_{G(e_0)}}}$, if $\lim_{n\to\infty} \widetilde{\|\tilde{v}^n_{f_{n_{G(e_n)}}} - \tilde{v}^0_{f_{0_{G(e_0)}}}\|} \tilde{=} \tilde{0}$, i.e., $\forall \tilde{\varepsilon} \tilde{>} \tilde{0}, \exists n_0 \in \mathbb{N}$ *such that* $\widetilde{\|\tilde{v}^n_{f_{n_{G(en)}}} - \tilde{v}^0_{f_{0_{G(e_0)}}}\|} \tilde{<} \tilde{\varepsilon}$, $\forall n \geq n_0$. It is denoted by $\lim_{n\to\infty} \tilde{v}^n_{f_{n_{G(en)}}} \tilde{=} \tilde{v}^0_{f_{0_{G(e_0)}}}$, or, briefly, $\tilde{v}^n_{f_{n_{G(en)}}} \xrightarrow[n\to\infty]{\sim} \tilde{v}^0_{f_{0_{G(e_0)}}}$.

Definition 2.24.(Fuzzy soft Cauchy sequence)[15] A sequence of fuzzy soft vectors $\{\tilde{v}^n_{f_{n_{G(en)}}}\}$ in a fuzzy soft normed space $(\tilde{U},\widetilde{\|\cdot\|})$ is said to be a fuzzy soft Cauchy sequence, if $\forall \tilde{\varepsilon} \tilde{>} \tilde{0}, \exists n_0 \in \mathbb{N}$ *such that* $\widetilde{\|\tilde{v}^n_{f_{n_{G(en)}}} - \tilde{v}^m_{f_{m_{G(em)}}}\|} \tilde{<} \tilde{\varepsilon}, \forall n,m \geq n_0, n > m$, that is to say that $\widetilde{\|\tilde{v}^n_{f_{n_{G(en)}}} - \tilde{v}^m_{f_{m_{G(em)}}}\|} \xrightarrow[n,m\to\infty]{\sim} \tilde{0}$.

Proposition 2.3.[15] Every fuzzy soft convergent sequence is a fuzzy soft Cauchy sequence.

Definition 2.25.(Fuzzy soft completeness)[15] A fuzzy soft normed space $(\tilde{U}, \widetilde{\|\cdot\|})$ is called fuzzy soft complete if every fuzzy soft Cauchy sequence is fuzzy soft convergent sequence in it.

Definition 2.26.(Fuzzy soft Banach space)[15] Every fuzzy soft complete fuzzy soft normed linear space is called a fuzzy soft Banach space.

Theorem 2.1.[15] Every fuzzy soft normed linear space is a fuzzy soft metric space with the fuzzy soft metric $\tilde{d}(\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}) \tilde{=} \widetilde{\| \tilde{v}^1_{f_{1G(e_1)}} - \tilde{v}^2_{f_{2G(e_2)}} \|}$, for all $\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}} \tilde{\in} FSV(\tilde{U})$.

3 Main Results

This section introduces the concept of fuzzy soft inner product on fuzzy soft linear spaces as well as some properties and examples of fuzzy soft inner product spaces. In addition, fuzzy soft Cauchy-Schwartz inequality and more results are established. Moreover, fuzzy soft polarization identity, fuzzy soft parallelogram law and fuzzy soft continuity property are investigated. Furthermore, fuzzy soft orthogonality is defined. Finally, fuzzy soft Hilbert space is introduced.

Definition 3.1.(Fuzzy soft inner product space) Let $FSV(\tilde{U})$ be a fuzzy soft vector space. Then, the mapping $\widetilde{<\cdot,\cdot>} : FSV(\tilde{U}) \times FSV(\tilde{U}) \to \mathbb{C}(A)$ is said to be a (complex) fuzzy soft inner product on $FSV(\tilde{U})$ if $\widetilde{<\cdot,\cdot>}$ satisfies the following axioms:

(FSI1) $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}}>} \tilde{\geq} \tilde{0}$, for all $\tilde{v}_{f_{G(e)}} \tilde{\in} FSV(\tilde{U})$, and $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}}>} \tilde{=} \tilde{0} \Leftrightarrow \tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\theta}$.

(FSI2) $\widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}>} \tilde{=} \overline{\widetilde{<\tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^1_{f_{1G(e_1)}}>}}$, for all $\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}} \tilde{\in} FSV(\tilde{U})$, where bar denotes the complex conjugate of the fuzzy soft complex number.

(FSI3) $\widetilde{<\tilde{c}\cdot\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}>} = \tilde{c} \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}>}$, for all $\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}} \tilde{\in} FSV(\tilde{U})$ and for all fuzzy soft scalar $\tilde{c}$.

(FSI4) $\widetilde{<\tilde{v}^1_{f_{1G(e_1)}} + \tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^3_{f_{3G(e_3)}}>} \tilde{=} \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^3_{f_{3G(e_3)}}>} + \widetilde{<\tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^3_{f_{3G(e_3)}}>}$, for all $\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^3_{f_{3G(e_3)}} \tilde{\in} FSV(\tilde{U})$.

The fuzzy soft vector space $FSV(\tilde{U})$ with a fuzzy soft inner product $\widetilde{<\cdot,\cdot>}$ is said to be a complex fuzzy soft inner product space (shortly, fuzzy soft inner product space) and is denoted by $(\tilde{U}, \widetilde{<\cdot,\cdot>})$.

Remark 3.1. From the above Definition (3), we have the following:

1. Using the condition (FSI2), we get $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}}>} \tilde{=} \overline{\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}}>}}$. That is to say that $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}}>} \in \mathbb{R}(A)$.
2. Using the two conditions (FSI2) and (FSI3), we have $\widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{c}\cdot\tilde{v}^2_{f_{2G(e_2)}}>} \tilde{=} \overline{\tilde{c}} \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}>}$.
3. We have $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{\theta}>} \tilde{=} \tilde{0}$ and $\widetilde{<\tilde{\theta}, \tilde{v}_{f_{G(e)}}>} \tilde{=} \tilde{0}$ for all $\tilde{v}_{f_{G(e)}} \tilde{\in} FSV(\tilde{U})$.

Theorem 3.1. In fuzzy soft inner product spaces, we have the following:

1. $\widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}} + \tilde{v}^3_{f_{3G(e_3)}}>} \tilde{=} \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}>} + \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^3_{f_{3G(e_3)}}>}$,
for all $\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^3_{f_{3G(e_3)}} \tilde{\in} FSV(\tilde{U})$.
2. $\widetilde{<\tilde{v}^1_{f_{1G(e_1)}} + \tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^3_{f_{3G(e_3)}} + \tilde{v}^4_{f_{4G(e_4)}}>}$
$\tilde{=} \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^3_{f_{3G(e_3)}}>} + \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^4_{f_{4G(e_4)}}>}$
$+ \widetilde{<\tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^3_{f_{3G(e_3)}}>} + \widetilde{<\tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^4_{f_{4G(e_4)}}>}$,
for all $\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^3_{f_{3G(e_3)}}, \tilde{v}^4_{f_{4G(e_4)}} \tilde{\in} FSV(\tilde{U})$.
3. For fuzzy soft complex scalars $\tilde{\alpha}_i$, $i : 1, \ldots, n$ and $\tilde{\beta}_j$, $j : 1, \ldots, m$,

$$\widetilde{<\sum_{i=1}^{n} \tilde{\alpha}_i \tilde{v}^i_{f_{iG(e_i)}}, \sum_{j=1}^{m} \tilde{\beta}_j \tilde{u}^j_{f_{jG(e_j)}}>}$$
$$\tilde{=} \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{\alpha}_i \overline{\tilde{\beta}}_j \widetilde{<\tilde{v}^i_{f_{iG(e_i)}}, \tilde{u}^j_{f_{jG(e_j)}}>}.$$

4. If for fixed $\tilde{v}_{f_{G(e)}} \tilde{\in} FSV(\tilde{U})$, we have $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{u}_{g_{G(a)}}>} \tilde{=} \tilde{0}$ for all $\tilde{u}_{g_{G(a)}} \tilde{\in} FSV(\tilde{U})$.
Then, $\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\theta}$.

Proof.

1. Let $\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^3_{f_{3G(e_3)}} \tilde{\in} FSV(\tilde{U})$. Then, using the axioms (FSI2) and (FSI4) in Definition (3), we get:

$$\widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}} + \tilde{v}^3_{f_{3G(e_3)}}>}$$
$$\tilde{=} \overline{\widetilde{<\tilde{v}^2_{f_{2G(e_2)}} + \tilde{v}^3_{f_{3G(e_3)}}, \tilde{v}^1_{f_{1G(e_1)}}>}}$$
$$\tilde{=} \overline{\widetilde{<\tilde{v}^2_{f_{2G(e_2)}}, \tilde{v}^1_{f_{1G(e_1)}}>} + \widetilde{<\tilde{v}^3_{f_{3G(e_3)}}, \tilde{v}^1_{f_{1G(e_1)}}>}}$$
$$\tilde{=} \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^2_{f_{2G(e_2)}}>} + \widetilde{<\tilde{v}^1_{f_{1G(e_1)}}, \tilde{v}^3_{f_{3G(e_3)}}>}.$$

2. For $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \tilde{v}^4_{f_{4_{G(e_4)}}} \tilde{\in} FSV(\tilde{U})$, and using the previous item (1) as well as the axiom (FSI4) in Definition (3), we have:

$$\begin{aligned}&\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}} + \tilde{v}^4_{f_{4_{G(e_4)}}} >}\\ &\tilde{=} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}} + \tilde{v}^4_{f_{4_{G(e_4)}}} >}\\ &+ \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}} + \tilde{v}^4_{f_{4_{G(e_4)}}} >}\\ &\tilde{=} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}} >} + \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^4_{f_{4_{G(e_4)}}} >}\\ &+ \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}} >} + \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^4_{f_{4_{G(e_4)}}} >}.\end{aligned}$$

3. This is the general case and is obtained using the previous item (2), for arbitrary n and m.

4. Since for fixed $\tilde{v}_{f_{G(e)}} \tilde{\in} FSV(\tilde{U})$, $\widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{u}_{g_{G(a)}} >} \tilde{=} \tilde{0}$ for all $\tilde{u}_{g_{G(a)}} \tilde{\in} FSV(\tilde{U})$. If we take in particular $\tilde{v}_{f_{G(e)}} = \tilde{u}_{g_{G(a)}}$, then $\widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}} >} \tilde{=} \tilde{0}$. Hence $\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\theta}$.

Definition 3.2.(Real fuzzy soft inner product space) If the mapping $\widetilde{< \cdot, \cdot >}$ in the above Definition (3) is replaced by $\widetilde{< \cdot, \cdot >} : FSV(\tilde{U}) \times FSV(\tilde{U}) \to \mathbb{R}(A)$, then it is called a real fuzzy soft inner product space and its conditions (axioms (FSI1),(FSI3), (FSI4)) are the same, but the condition (FSI2) is replaced by

(FSI2(i)): $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{=} \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >}$, for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$.

Theorem 3.2.(Fuzzy soft Cauchy-Schwartz inequality) Let $(\tilde{U}, \widetilde{< \cdot, \cdot >})$ be a fuzzy soft inner product space, then

$$\begin{aligned}&|\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}|^2\\ &\tilde{\leq} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >},\end{aligned}$$

for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$.

Proof. If $\tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{=} \tilde{\theta}$, we have $\tilde{0} \tilde{=} |\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{\theta} >}|^2 \tilde{\leq} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} \widetilde{< \tilde{\theta}, \tilde{\theta} >} \tilde{=} \tilde{0}$.
Let $\tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\neq} \tilde{\theta}$ and let $\tilde{\lambda} \tilde{\in} \mathbb{C}(A)$. We have $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{\lambda} \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{\lambda} \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{\geq} \tilde{0}$. Then, using item (2) in Theorem (3), we have $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} - \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{\lambda} \tilde{v}^2_{f_{2_{G(e_2)}}} >} - \widetilde{< \tilde{\lambda} \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} + \widetilde{< \tilde{\lambda} \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{\lambda} \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{\geq} \tilde{0}$.
Therefore, using item (3) in Theorem (3), we get $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} - \overline{\tilde{\lambda}} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} - \tilde{\lambda} \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} + \tilde{\lambda} \overline{\tilde{\lambda}} \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{\geq} \tilde{0}$.
Hence, the following inequality is obtained:

$$\begin{aligned}&\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} - \overline{\tilde{\lambda}} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}\\ &- \tilde{\lambda} \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} + |\tilde{\lambda}|^2 \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{\geq} \tilde{0}.\end{aligned} \tag{1}$$

Then, Take

$$\tilde{\lambda} \tilde{=} \frac{\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}}{\widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}}, \ \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\neq} \tilde{\theta}. \tag{2}$$

Substituting from (2) in (1), we have

$$\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} - \frac{\overline{\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}}{\widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}} - \frac{\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \overline{\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}}}{\widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}} + \frac{|\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}|^2}{\widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}} \tilde{\geq} \tilde{0}.$$

Then, $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} - \tilde{2} \frac{|\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}|^2}{\widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}} + \frac{|\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}|^2}{\widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}} \tilde{\geq} \tilde{0}$.

Therefore, $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} - \frac{|\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}|^2}{\widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}} \tilde{\geq} \tilde{0}$.

Hence, $\frac{|\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}|^2}{\widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}} \tilde{\leq} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >}$.

That is to say that

$$\begin{aligned}&|\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}|^2\\ &\tilde{\leq} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}.\end{aligned} \tag{3}$$

Corollary 3.1. If $\tilde{v}^1_{f_{1_{G(e_1)}}}$ and $\tilde{v}^2_{f_{2_{G(e_2)}}}$ are fuzzy soft linearly dependent, then

$$|\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}|^2$$
$$\tilde{=}\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^1_{f_{1_{G(e_1)}}}>}\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>},$$

for all $\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}\tilde{\in}FSV(\tilde{U})$.

Proof. Since $\tilde{v}^1_{f_{1_{G(e_1)}}}$ and $\tilde{v}^2_{f_{2_{G(e_2)}}}$ are fuzzy soft linearly dependent, then $\tilde{v}^1_{f_{1_{G(e_1)}}}\tilde{=}\tilde{\alpha}\tilde{v}^2_{f_{2_{G(e_2)}}}$. Thus, using this, we have:

$$|\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}|^2$$
$$\tilde{=}|\widetilde{<\tilde{\alpha}\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}|^2$$
$$\tilde{=}|\tilde{\alpha}\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}|^2$$
$$\tilde{=}|\tilde{\alpha}|^2|\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}|^2$$
$$\tilde{=}|\tilde{\alpha}|^2\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}$$
$$\tilde{=}\widetilde{<\tilde{\alpha}\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{\alpha}\tilde{v}^2_{f_{2_{G(e_2)}}}>}\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}$$
$$\tilde{=}\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^1_{f_{1_{G(e_1)}}}>}\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}.$$

Theorem 3.3. A fuzzy soft inner product space $(\tilde{U},\widetilde{<\cdot,\cdot>})$ can be considered as a fuzzy soft normed space with the fuzzy soft norm $\widetilde{\|\tilde{v}_{f_{G(e)}}\|}\tilde{=}\sqrt{\widetilde{<\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}>}}$, for all $\tilde{v}_{f_{G(e)}}\tilde{\in}FSV(\tilde{U})$.

Proof. We have to prove the conditions of the fuzzy soft normed space stated in the Definition (2) as follows: $(FSN1)$ and $(FSN2)$ are trivial.

(FSN3)For all $\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}\tilde{\in}FSV(\tilde{U})$, we get:

$$\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}+\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2$$
$$\tilde{=}\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}+\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^1_{f_{1_{G(e_1)}}}+\tilde{v}^2_{f_{2_{G(e_2)}}}>}$$
$$\tilde{=}\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^1_{f_{1_{G(e_1)}}}>}+\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}$$
$$+\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^1_{f_{1_{G(e_1)}}}>}+\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}$$
$$\tilde{=}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2+\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}$$
$$+\overline{\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}}+\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2$$
$$\tilde{=}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2+\tilde{2}Re(\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>})+\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2$$
$$\tilde{\leq}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2+\tilde{2}|\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}|+\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2.$$

Then, using Theorem (3) (fuzzy soft Cauchy-Schwartz inequality), we obtain:

$$\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}+\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2\tilde{\leq}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2+\tilde{2}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|$$
$$+\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2$$
$$\tilde{=}(\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|+\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|)^2.$$

Then $\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}+\tilde{v}^2_{f_{2_{G(e_2)}}}}\|\tilde{\leq}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|+\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|$.

Remark 3.2. The fuzzy soft Cauchy-Schwartz inequality (3) can be rewritten using the above Theorem (3) as follows $|\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}|\tilde{\leq}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|$.

Proof. Using Theorem (3), we have:

$$\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^1_{f_{1_{G(e_1)}}}>}\tilde{=}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2, \quad (4)$$

and

$$\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}\tilde{=}\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2. \quad (5)$$

Then, substituting from (4) and (5) in (3), we obtain $|\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}}>}|^2\tilde{\leq}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2\|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2$. This completes the proof.

Remark 3.3. Since any fuzzy soft inner product space is a fuzzy soft normed space from Theorem (3) and any fuzzy soft normed space is fuzzy soft metric space from Theorem (2), then any fuzzy soft inner product space is a fuzzy soft metric space with

$$\tilde{d}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}) \tilde{=} \widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}\|} \tilde{=} \sqrt{\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>}},$$

for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$.

Example 3.1. The following spaces are examples of fuzzy soft inner product spaces:

1. $\mathbb{C}^n(A)$: the fuzzy soft complex Euclidean space (the space of all fuzzy soft $n-$dimensional complex numbers) is a complex fuzzy soft inner product space with the complex fuzzy soft inner product defined as follows $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{u}_{g_{G(a)}}>} \tilde{=} \sum_{i=1}^{n} \tilde{v}^i_{f_{i_{G(e_i)}}} \overline{\tilde{u}^i_{g_{i_{G(a_i)}}}}$, for all $\tilde{v}_{f_{G(e)}}, \tilde{u}_{g_{G(a)}} \tilde{\in} \mathbb{C}^n(A)$.
2. $\mathbb{R}^n(A)$: the fuzzy soft real Euclidean space (the space of all fuzzy soft $n-$dimensional real numbers) is a real fuzzy soft inner product space with the real fuzzy soft inner product defined as follows $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{u}_{g_{G(a)}}>} \tilde{=} \sum_{i=1}^{n} \tilde{v}^i_{f_{i_{G(e_i)}}} \overline{\tilde{u}^i_{g_{i_{G(a_i)}}}}$, for all $\tilde{v}_{f_{G(e)}}, \tilde{u}_{g_{G(a)}} \tilde{\in} \mathbb{R}^n(A)$.
3. $\ell_2(A)$: the space of all fuzzy soft square-summable sequences is a complex fuzzy soft inner product space with the complex fuzzy soft inner product defined as follows $\widetilde{<\tilde{v}_{f_{G(e)}}, \tilde{u}_{g_{G(a)}}>} \tilde{=} \sum_{i=1}^{\infty} \tilde{v}^i_{f_{i_{G(e_i)}}} \overline{\tilde{u}^i_{g_{i_{G(a_i)}}}}$, for all $\tilde{v}_{f_{G(e)}}, \tilde{u}_{g_{G(a)}} \tilde{\in} \ell_2(A)$.
4. $\mathscr{C}_{[\tilde{0},\tilde{1}]}(A)$: the space of all fuzzy soft complex-valued continuous functions on $[\tilde{0},\tilde{1}]$ is a complex fuzzy soft inner product space with the complex fuzzy soft inner product defined as follows $\widetilde{<\tilde{\eta}_{f_{G(e)}}, \tilde{\xi}_{g_{G(a)}}>} \tilde{=} \int_{i=\tilde{0}}^{\tilde{1}} \tilde{\eta}^i_{f_{i_{G(e_i)}}}(\tilde{x}) \overline{\tilde{\xi}^i_{g_{i_{G(a_i)}}}}(\tilde{x}) d\tilde{x}$, for all $\tilde{\eta}_{f_{G(e)}}, \tilde{\xi}_{g_{G(a)}} \in \mathscr{C}_{[\tilde{0},\tilde{1}]}(A)$.

Proof. For (1), (3) and (4), the proof is straightforward by applying the conditions of the complex fuzzy soft inner product space stated in Definition (3). For (2), the proof is straightforward by applying the conditions of the real fuzzy soft inner product space stated in Definition (3).

Theorem 3.4. Let $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ be a real fuzzy soft inner product space and let $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$. Then,

$$\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>} \tilde{=} \frac{\tilde{1}}{\tilde{4}}(\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 - \widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2), \quad (6)$$

for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$.

Proof. Using Theorem (3) and applying the properties stated in Theorem (3), we obtain:

$$\begin{aligned}
&\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 \\
&\tilde{=} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&\tilde{=} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}>} + \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&+ \widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}>} + \widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&\tilde{=} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}>} + \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&+ \overline{\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>}} + \widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>}.
\end{aligned}$$

Since $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ is a real fuzzy soft inner product space, then (FSI2(i)) in Definition (3) is satisfied, i.e. $\overline{\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>}} \tilde{=} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>}$ for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$. Using this condition, we have:

$$\begin{aligned}
&\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 \\
&\tilde{=} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}>} + \tilde{2}\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&+ \widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>},
\end{aligned} \quad (7)$$

and similarly, we get:

$$\begin{aligned}
&\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 \\
&\tilde{=} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&\tilde{=} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}>} + \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, -\tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&+ \widetilde{<-\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}>} + \widetilde{<-\tilde{v}^2_{f_{2_{G(e_2)}}}, -\tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&\tilde{=} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}>} - \tilde{2}\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>} \\
&+ \widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>}.
\end{aligned} \quad (8)$$

Subtracting (8) from (7), we have:

$$\begin{aligned}
&\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 - \widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 \\
&\tilde{=} \tilde{4}\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, -\tilde{v}^2_{f_{2_{G(e_2)}}}>}.
\end{aligned} \quad (9)$$

Dividing both sides of Equation (9) by $\tilde{4}$ gives us $\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>} \tilde{=} \frac{\tilde{1}}{\tilde{4}}(\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 - \widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2)$, which is the required formula (6).

Theorem 3.5.(Fuzzy soft polarization identity) Let $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ be a complex fuzzy soft inner product space. Then, we can write the fuzzy soft polarization identity in the following formula, for all $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$:

$$
\begin{aligned}
&< \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} \frac{\tilde{1}}{\tilde{4}}(\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 - \|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&+ \tilde{i}\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 - \tilde{i}\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2).
\end{aligned} \tag{10}
$$

Proof. Using the conditions of Definition (3) (fuzzy soft inner product space), applying the properties which are stated in Theorem (3) and with the help of Theorem (3), we obtain:

$$
\begin{aligned}
&\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&+ < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} >,
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
&\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > - < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&- < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} >.
\end{aligned} \tag{12}
$$

In addition, we have:

$$
\begin{aligned}
&\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&+ < \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + \bar{\tilde{i}} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&+ \tilde{i} < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > - \tilde{i} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&+ \tilde{i} < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} >,
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
&\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} -\tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&+ < -\tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < -\tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} -\tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > - \bar{\tilde{i}} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&- \tilde{i} < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&\tilde{=} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + \tilde{i} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} > \\
&- \tilde{i} < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^1_{f_{1_{G(e_1)}}} > + < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} >,
\end{aligned} \tag{14}
$$

where $\bar{\tilde{i}} = -\tilde{i}$.
Subtracting (12) from (11), we get:

$$
\begin{aligned}
&\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 - \|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&\tilde{=} \tilde{2} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} -\tilde{v}^2_{f_{2_{G(e_2)}}} > + \tilde{2} < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} -\tilde{v}^1_{f_{1_{G(e_1)}}} >.
\end{aligned} \tag{15}
$$

Also, subtracting (14) from (13), we have:

$$
\begin{aligned}
&\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 - \|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&\tilde{=} -\tilde{2}\tilde{i} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} -\tilde{v}^2_{f_{2_{G(e_2)}}} > + \tilde{2}\tilde{i} < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} -\tilde{v}^1_{f_{1_{G(e_1)}}} >.
\end{aligned} \tag{16}
$$

Then, multiplying both sides of Equation (16) by $\tilde{i}$, we get:

$$
\begin{aligned}
&\tilde{i}\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 - \tilde{i}\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&\tilde{=} \tilde{2} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} -\tilde{v}^2_{f_{2_{G(e_2)}}} > - \tilde{2} < \tilde{v}^2_{f_{2_{G(e_2)}}} \widetilde{,} -\tilde{v}^1_{f_{1_{G(e_1)}}} >.
\end{aligned} \tag{17}
$$

Therefore, adding (15) and (17), we obtain:

$$
\begin{aligned}
&\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 - \|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&+ \tilde{i}\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{+} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 - \tilde{i}\|\tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{-} \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}\|^2 \\
&\tilde{=} \tilde{4} < \tilde{v}^1_{f_{1_{G(e_1)}}} \widetilde{,} \tilde{v}^2_{f_{2_{G(e_2)}}} >.
\end{aligned} \tag{18}
$$

Hence, dividing both sides of Equation (18) by $\tilde{4}$ we have:

$$
\begin{aligned}
&< \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&\tilde{=}\frac{\tilde{1}}{\tilde{4}}(\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 - \|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 \\
&+ \tilde{i}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 - \tilde{i}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{i}\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2),
\end{aligned}
$$

which is the required formula (10).

Theorem 3.6.(Fuzzy soft parallelogram law) Let $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ be a fuzzy soft inner product space and let $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$. Then, the fuzzy soft parallelogram law holds as follows:

$$
\begin{aligned}
&\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 + \|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 \\
&\tilde{=}\tilde{2}(\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2 + \|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2).
\end{aligned} \tag{19}
$$

Proof. Using Theorem (3) and applying the properties stated in Theorem (3), we obtain:

$$
\begin{aligned}
&\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 \\
&\tilde{=}< \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&\tilde{=}< \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}} > + < \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&+ < \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}} > + < \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&\tilde{=}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2 + \|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 + < \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&+ < \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}} >,
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
&\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 \\
&\tilde{=}< \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&\tilde{=}< \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}} > - < \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&- < \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}} > + < \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&\tilde{=}\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2 + \|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 - < \widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}} > \\
&- < \widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}} >.
\end{aligned} \tag{21}
$$

Then, adding (20) and (21), we get:

$$
\begin{aligned}
&\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 + \|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 \\
&\tilde{=}\tilde{2}(\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\|^2 + \|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2),
\end{aligned}
$$

which is the required formula (19).

Example 3.2. As a special case of the above-mentioned Theorem (3) (Fuzzy soft parallelogram law), if we take $\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}}}\| \tilde{=} \|\widetilde{\tilde{v}^2_{f_{2_{G(e_2)}}}}\| \tilde{=} \tilde{1}$, then we have $\|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 + \|\widetilde{\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}}\|^2 \tilde{=} \tilde{4}$.

Theorem 3.7.(Fuzzy soft continuity property) Let $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ be a fuzzy soft inner product space. Suppose that $\lim_{n\to\infty} \tilde{v}^n_{f_{n_{G(en)}}} \tilde{=} \tilde{v}^0_{f_{0_{G(e_0)}}}$ and $\lim_{n\to\infty} \tilde{u}^n_{g_{n_{G(an)}}} \tilde{=} \tilde{u}^0_{g_{0_{G(a_0)}}}$. Then $\lim_{n\to\infty} < \widetilde{\tilde{v}^n_{f_{n_{G(en)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} > \tilde{=} < \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^0_{g_{0_{G(a_0)}}}} >$. That is to say that the fuzzy soft inner product is a fuzzy soft continuous function from $FSV(\tilde{U}) \times FSV(\tilde{U})$ to $\mathbb{C}(A)$.

Proof. We have $\forall \tilde{\varepsilon} \tilde{>} \tilde{0}$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$,

$$
\begin{aligned}
&|< \widetilde{\tilde{v}^n_{f_{n_{G(en)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} > - < \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^0_{g_{0_{G(a_0)}}}} >| \\
&\tilde{=}|< \widetilde{\tilde{v}^n_{f_{n_{G(en)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} > - < \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} > \\
&+ < \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} > - < \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^0_{g_{0_{G(a_0)}}}} >| \\
&\tilde{\leq}|< \widetilde{\tilde{v}^n_{f_{n_{G(en)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} > - < \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} >| \\
&+ |< \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} > - < \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^0_{g_{0_{G(a_0)}}}} >| \\
&\tilde{=}|< \widetilde{\tilde{v}^n_{f_{n_{G(en)}}} - \tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} >| \\
&+ |< \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^n_{g_{n_{G(an)}}} - \tilde{u}^0_{g_{0_{G(a_0)}}}} >|.
\end{aligned}
$$

Then, using Theorem (3) (fuzzy soft Cauchy-Schwartz inequality), we obtain:

$$
\begin{aligned}
&|< \widetilde{\tilde{v}^n_{f_{n_{G(en)}}}, \tilde{u}^n_{g_{n_{G(an)}}}} > - < \widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^0_{g_{0_{G(a_0)}}}} >| \\
&\tilde{\leq}\|\widetilde{\tilde{v}^n_{f_{n_{G(en)}}} - \tilde{v}^0_{f_{0_{G(e_0)}}}}\| \|\widetilde{\tilde{u}^n_{g_{n_{G(an)}}}}\| \\
&+ \|\widetilde{\tilde{v}^0_{f_{0_{G(e_0)}}}}\| \|\widetilde{\tilde{u}^n_{g_{n_{G(an)}}} - \tilde{u}^0_{g_{0_{G(a_0)}}}}\|.
\end{aligned} \tag{22}
$$

We have $\lim_{n\to\infty} \|\widetilde{\tilde{u}^n_{g_{n_{G(an)}}}}\| \tilde{=} \|\widetilde{\tilde{u}^0_{g_{0_{G(a_0)}}}}\|$ from the given that $\lim_{n\to\infty} \tilde{u}^n_{g_{n_{G(an)}}} \tilde{=} \tilde{u}^0_{g_{0_{G(a_0)}}}$ and since every fuzzy soft convergent sequence is fuzzy soft bounded, then $\|\widetilde{\tilde{u}^n_{g_{n_{G(an)}}}}\|$ is fuzzy soft bounded, i.e.

$$
\|\widetilde{\tilde{u}^n_{g_{n_{G(an)}}}}\| \tilde{\leq} \tilde{k}_1. \tag{23}
$$

Since $\lim\limits_{n\to\infty} \tilde{v}^n_{f_{n_{G(e_n)}}} \tilde{=} \tilde{v}^0_{f_{0_{G(e_0)}}}$, then $\forall \tilde{\varepsilon}_1 \tilde{>} \tilde{0}$, $\exists n_1 \in \mathbb{N}$ such that

$$\widetilde{\|\tilde{v}^n_{f_{n_{G(e_n)}}} - \tilde{v}^0_{f_{0_{G(e_0)}}}\|} \tilde{<} \tilde{\varepsilon}_1, \ \forall n \geq n_1, \qquad (24)$$

and since $\lim\limits_{n\to\infty} \tilde{u}^n_{g_{n_{G(a_n)}}} \tilde{=} \tilde{u}^0_{g_{0_{G(a_0)}}}$, then $\forall \tilde{\varepsilon}_2 \tilde{>} \tilde{0}$, $\exists n_2 \in \mathbb{N}$ such that

$$\widetilde{\|\tilde{u}^n_{g_{n_{G(a_n)}}} - \tilde{u}^0_{g_{0_{G(a_0)}}}\|} \tilde{<} \tilde{\varepsilon}_2, \ \forall n \geq n_2. \qquad (25)$$

Take $n_0 = \max(n_1, n_2)$, and

$$\widetilde{\|\tilde{v}^0_{f_{0_{G(e_0)}}}\|} \tilde{=} \tilde{k}_2. \qquad (26)$$

Therefore, substituting from (23),(24),(25) and (26) in (22), we get:

$$|\widetilde{< \tilde{v}^n_{f_{n_{G(e_n)}}}, \tilde{u}^n_{g_{n_{G(a_n)}}} >} - \widetilde{< \tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^0_{g_{0_{G(a_0)}}} >}| \tilde{\leq} \tilde{\varepsilon}_1 \tilde{k}_1 + \tilde{k}_2 \tilde{\varepsilon}_2 \tilde{=} \tilde{\varepsilon}.$$

That is to say that, $\forall \tilde{\varepsilon} \tilde{>} \tilde{0}$, $\exists n_0 = \max(n_1, n_2) \in \mathbb{N}$ such that $|\widetilde{< \tilde{v}^n_{f_{n_{G(e_n)}}}, \tilde{u}^n_{g_{n_{G(a_n)}}} >} - \widetilde{< \tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^0_{g_{0_{G(a_0)}}} >}| \tilde{\leq} \tilde{\varepsilon}$, $\forall n \geq n_0$. Hence,
$\lim\limits_{n\to\infty} \widetilde{< \tilde{v}^n_{f_{n_{G(e_n)}}}, \tilde{u}^n_{g_{n_{G(a_n)}}} >} \tilde{=} \widetilde{< \tilde{v}^0_{f_{0_{G(e_0)}}}, \tilde{u}^0_{g_{0_{G(a_0)}}} >}$.

Definition 3.3.(Fuzzy soft orthogonality) Let $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ be a fuzzy soft inner product space and $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$. Then, $\tilde{v}^1_{f_{1_{G(e_1)}}}$ is said to be fuzzy soft orthogonal to $\tilde{v}^2_{f_{2_{G(e_2)}}}$, written $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\perp} \tilde{v}^2_{f_{2_{G(e_2)}}}$, if $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{=} \tilde{0}$.

Remark 3.4.

1. It is clear that if $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\perp} \tilde{v}^2_{f_{2_{G(e_2)}}}$, then $\tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\perp} \tilde{v}^1_{f_{1_{G(e_1)}}}$.
2. We have $\tilde{v}_{f_{G(e)}} \tilde{\perp} \tilde{\theta}$, for all $\tilde{v}_{f_{G(e)}} \tilde{\in} FSV(\tilde{U})$, since $\widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{\theta} >} \tilde{=} \tilde{0}$.
3. Since $\widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}} >} \tilde{=} \tilde{0}$ if and only if $\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\theta}$, then $\tilde{\theta}$ is the only fuzzy soft element (point) fuzzy soft orthogonal to itself.

Theorem 3.8. Let $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ be a fuzzy soft inner product space and let $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$. If $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\perp} \tilde{v}^2_{f_{2_{G(e_2)}}}$, then

$$\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 \tilde{=} \widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}}\|}^2 + \widetilde{\|\tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2. \qquad (27)$$

Proof. Since $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\perp} \tilde{v}^2_{f_{2_{G(e_2)}}}$, then $\tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\perp} \tilde{v}^1_{f_{1_{G(e_1)}}}$. Therefore $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{=} \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} \tilde{=} \tilde{0}$. Using Theorem (3) and applying the properties stated in Theorem (3), we get:

$$\begin{aligned}
&\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 \\
&\tilde{=} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} + \tilde{v}^2_{f_{2_{G(e_2)}}} >} \\
&\tilde{=} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} + \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \\
&+ \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} + \widetilde{< \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \\
&\tilde{=} \widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}}\|}^2 + \widetilde{\|\tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2.
\end{aligned}$$

Hence the Equation (27) holds.

Corollary 3.2. If $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\perp} \tilde{v}^2_{f_{2_{G(e_2)}}}$ and $\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}}\|} \tilde{=} \widetilde{\|\tilde{v}^2_{f_{2_{G(e_2)}}}\|} \tilde{=} \tilde{1}$, then $\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}\|} \tilde{=} \sqrt{\tilde{2}}$.

Proof. $\widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}} - \tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 \tilde{=} \widetilde{\|\tilde{v}^1_{f_{1_{G(e_1)}}}\|}^2 + \widetilde{\|\tilde{v}^2_{f_{2_{G(e_2)}}}\|}^2 \tilde{=} \tilde{1} + \tilde{1} \tilde{=} \tilde{2}$.

Definition 3.4.(Fuzzy soft orthogonal complement) Let $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ be a fuzzy soft inner product space. Let $\tilde{\omega}$ be a non-empty fuzzy soft subset of $FSV(\tilde{U})$. The fuzzy soft set of all fuzzy soft elements (fuzzy soft points) $\tilde{v}^1_{f_{1_{G(e_1)}}}$ of $FSV(\tilde{U})$ which are fuzzy soft orthogonal to $\tilde{\omega}$ is denoted by $\tilde{\omega}^{\tilde{\perp}}$, and is called the fuzzy soft orthogonal complement of $\tilde{\omega}$. That is to say that

$$\tilde{\omega}^{\tilde{\perp}} \tilde{=} \{\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} FSV(\tilde{U}) : \tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\perp} \tilde{v}^2_{f_{2_{G(e_2)}}}, \forall \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} \tilde{\omega}\}.$$

Theorem 3.9. For fuzzy soft orthogonal complement, we have the following properties:

1. $\{\tilde{\theta}\}^{\tilde{\perp}} \tilde{=} \tilde{U}$ and $\tilde{U}^{\tilde{\perp}} \tilde{=} \{\tilde{\theta}\}$.
2. $\tilde{\omega} \tilde{\cap} \tilde{\omega}^{\tilde{\perp}} \tilde{=} \{\tilde{\theta}\}$.
3. $\tilde{\omega}^{\tilde{\perp}\tilde{\perp}} \tilde{=} \tilde{\omega}$.
4. If $\tilde{\omega}_1 \tilde{\subseteq} \tilde{\omega}_2$, then $\tilde{\omega}_2^{\tilde{\perp}} \tilde{\subseteq} \tilde{\omega}_1^{\tilde{\perp}}$.

Proof.

1. Clear from the above Definition (3).
2. Let $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega} \tilde{\cap} \tilde{\omega}^{\tilde{\perp}}$, then $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega}$ and $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega}^{\tilde{\perp}}$. Therefore, from the above Definition (3), we have $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{=} \tilde{0}, \forall \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} \tilde{\omega}$. Since $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega}$, then $\widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}} >} \tilde{=} \tilde{0}$. Hence $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{=} \tilde{\theta}$. This completes the proof.
3. $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega}^{\tilde{\perp}\tilde{\perp}} \Leftrightarrow \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{=} \tilde{0}, \forall \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} \tilde{\omega}^{\tilde{\perp}} \Leftrightarrow \tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega}$.

4.Let $\tilde{\omega}_1 \tilde{\subset} \tilde{\omega}_2$. Then, we get:

$$\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega}_2^{\tilde{\perp}} \Leftrightarrow \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{=} \tilde{0}, \forall \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} \tilde{\omega}_2$$
$$\Leftrightarrow \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{=} \tilde{0}, \forall \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} \tilde{\omega}_1$$
$$\Leftrightarrow \tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega}_1^{\tilde{\perp}}.$$

Definition 3.5. Let $(\tilde{U}, \widetilde{< \cdot, \cdot >})$ be a fuzzy soft inner product space. Let $\tilde{\omega}_1$ and $\tilde{\omega}_2$ be two non-empty fuzzy soft subsets of $FSV(\tilde{U})$. We say that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are fuzzy soft orthogonal, written $\tilde{\omega}_1 \tilde{\perp} \tilde{\omega}_2$, if every $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\in} \tilde{\omega}_1$ and every $\tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} \tilde{\omega}_2$ are fuzzy soft orthogonal.
It should be noted that if $\tilde{\omega}_1 \tilde{\perp} \tilde{\omega}_2$, then $\tilde{\omega}_2 \tilde{\perp} \tilde{\omega}_1$.
Definition 3.6.(Fuzzy soft Hilbert space) Let $(\tilde{U}, \widetilde{< \cdot, \cdot >})$ be a fuzzy soft inner product space. Then, this space, which is fuzzy soft complete in the induced fuzzy soft norm stated in Theorem (3), is called a fuzzy soft Hilbert space, denoted by $(\tilde{H}, \widetilde{< \cdot, \cdot >})$ (shortly $\tilde{H}$). It is clear that every fuzzy soft Hilbert space is a fuzzy soft Banach space.

4 Conclusion and Further Research

Introducing the fuzzy version or the soft version of topics like metric spaces, normed spaces, Banach spaces, operators, dual spaces, functionals, inner product spaces, Hilbert spaces, operators on Hilbert spaces and many other topics has been investigated by several mathematicians. On the other hand, combining fuzzy and soft sets together is not only more general than using only one of them but also gives us more extended and accurate results. Few researchers have explored some of those general extensions concepts such as fuzzy soft normed spaces and fuzzy soft metric spaces. Continuing their work, we have defined fuzzy soft inner product on fuzzy soft linear spaces using the concept of fuzzy soft point. Moreover, some related properties of fuzzy soft inner product spaces have been introduced. Furthermore, some examples of fuzzy soft inner product spaces have been investigated. To make the picture complete, we have established fuzzy soft Cauchy-Schwartz inequality and many more related results. Moreover, fuzzy soft polarization identity, fuzzy soft parallelogram law and fuzzy soft continuity property have been investigated. In addition, we have introduced fuzzy soft orthogonality in fuzzy soft inner product spaces. Finally, fuzzy soft Hilbert space has been defined. Our work can be considered a generalization for the well-known (crisp) inner product spaces. This type of investigations fills some gaps in the literature. The authors can introduce new results using similar techniques in this paper. Some properties in fuzzy soft Hilbert spaces will be addressed in further investigations depending on the definition of fuzzy soft Hilbert space given in this paper. Finally, further topics need to be covered by applying fuzzy soft notion on them.

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Conflict of interest

The authors declare that they have no conflict of interest.

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