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Analytical Approach to Study the Generalized Lane-Emden Model Arises in the Study of Stellar Configuration

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Abstract: In this paper, we consider the generalized Lane-Emden model which arises in the study of steller configuration. We came up with the nonlinear, multi-singular, initial value ordinary differential equations. Mathematical induction is used to verify the generalized non-iterative higher-order Lane-Emden type equation. We use various Homotopy Analysis Method (HAM) algorithms to find the convergent series solutions of the model. It is observed how the choice of initial value, increasing values of M in the polynomial nonlinearity y^M, and different choices of HAM algorithms impact the solution radius of convergence. Convergent series solutions obtained from HAM algorithms are compared with the traditional power series and Runge-Kutta-Fehlberg method (RKF45). The traditional series solution follows the actual solution in the domain where the actual solution is positive while HAM does not require domain restriction.

Keywords: Lane-Emden equation, homotopy analysis method, power series solution.

1 Introduction

In the second half of the 19^{th} century, astrophysicist J. Homer Lane derived and investigated an equation that models the equilibrium of stellar configurations [1]. The work was then extended [2] to the following equation which is now known as the Lane-Emden equation. The equation describes the polytropic models derived in [3] and given by

$$x^{-k}\frac{d}{dx}(x^k\frac{dy}{dx})+y^M=0. \tag{1}$$

y^M is the polynomial nonlinearity, and M is a constant whose value depends on the physical phenomena modeled by equation (1). The polynomial nonlinearity makes equation (1) to explore the thermal behavior of spherical clouds of gas under the mutual attraction of its molecules and subject to laws of thermodynamics. In addition, instead of using y^M, the nonlinearity e^y gives us the isothermal Lane-Emden equation.

Analytic and numeric studies of this model have been constructed by using different types of methods. Analytical solutions for Lane-Emden equation have been studied in [4,5]. There are numerous studies of this type of equation using the Adomian Decomposition Method(ADM) by [6,7,8,9,10]. A well known Variational Iteration Method(VIM) has been used in [10, 11,12,13] to study the Lane-Emden-Fowler equations. Homotopy Perturbation Method(HPM) which is another choice for finding a series solution to nonlinear problems has been used by [14] to investigate the Lane-Emden equation. [15] proposed that HPM is not a good choice for solving problems with strong nonlinearity. The same challenge was found in [16], where the same approach was used to find an analytical solution for their diffusive flux study for the biofilm modeling.

HAM was first developed in the doctoral dissertation by [17]. The HAM is based on homotopy theory from topology. The main idea of this method is to transform a given nonlinear differential equation into several linear

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equations and then to find a series solution of the original nonlinear problem. In [18,19,20,21] the HAM idea is applied to find the convergent series solution to Lane-Emden type equations. An iterative higher-order Lane-Emden type equation has the following form:

$$x^{-k}\frac{d^m}{dx^m}\left(x^k\frac{d^n y}{dx^n}\right)+y^M=0,$$

where the choices of integers m and n define the order of the model. The parameter k is called shape factor. We focus on the case of $M=3$ for the fifth-order Lane-Emden equation where analytical solutions can be found for $M=0,1,5$. [13] proposed the idea of extending the standard second order Lane-Emden equation to two different choices of third-order Lane-Emden-Fowler type equations. Later, [8,10] came up with the three fourth order Lane-Emden-Fowler type equations. They explored these higher-order models with the Adomian Decomposition Method and Variational Iteration Method. In the papers [8,13], the authors also proposed a generalized non-iterative form for any choice of higher-order model. Unlike the standard Lane-Emden equation, the higher-order Lane-Emden type equations are with multiple shape factor values k and with multiple singularities at $x=0$. [22] presented the generalized fractional order of Chebyshev orthogonal functions to construct an approximation to the solution of nonlinear Lane-Emden type equations of various orders. This paper verifies all the results numerically and matches their findings with [8,13].

We organize the paper in the following way: Section Two addresses the possibility of four different fifth order equations. Moreover, we verify the generalized non-iterative higher-order algorithm proposed in [10,13] for Lane-Emden type equations by mathematical induction. A comparison between the power series solution and numerical solution is presented ; it demonstrates that the increasing nonlinearity reduces the solution radius of convergence. Section Three handles the basic idea behind HAM. A detailed proof for one HAM construction is presented. It is shown that the auxiliary parameter $\hbar$ controls the radius of convergence and helps to obtain a convergent series solution of the problem. We find that the HAM algorithm gives a larger interval of convergence than that of the power series method. We also compare the significance of various HAM algorithms for the faster convergence of the obtained series solutions.

2 Model Equations and Power Series Analysis

2.1 Fifth-order Iterative Lane-Emden type Equations

To derive the fifth-order Lane-Emden type equation, we start with the generalized version of the standard form as

$$x^{-k}\frac{d^m}{dx^m}\left(x^k\frac{d^n y}{dx^n}\right)+y^M=0, \tag{2}$$

with shape factor $k>0$. To come up with the fifth-order linear operator, the following choices are made:

$$m+n=5, \text{ where } m,n\geq 1.$$

This means that we have the following possible choices for m and n

$$\begin{aligned} m=4, &\quad n=1\\ m=3, &\quad n=2\\ m=2, &\quad n=3\\ m=1, &\quad n=4.\end{aligned}$$

In this paper, we study the fifth-order model using the possible pair of $m=4$ and $n=1$, giving the fifth-order, multi-singular, nonlinear, initial value problem.

Initial condition in the fifth-order model equations is imposed in the same way for the standard second-order Emden-Fowler model. It is easy to observe that the obtained higher-order model has multiple singularities at $x=0$ with respect to their shape factors $k, k(k-1), k(k-1)(k-2), k(k-1)(k-2)(k-3)$. Moreover, the fourth and the fifth term in the model equations go to zero for the shape factor $k=2$, and in this case, the shape factors are 8 and 12 for the second and third term in the equation respectively. For $M=3$, we have the fifth-order of the first kind Lane-Emden model as

$$\begin{aligned}&\frac{d^5y}{dx^5}+\frac{8}{x}\frac{d^4y}{dx^4}+\frac{12}{x^2}\frac{d^3y}{dx^3}+y^3(x)=0\\ &y(0)=a, y^{(1)}(0)=y^{(2)}(0)=y^{(3)}(0)=y^{(4)}(0)=0.\end{aligned} \tag{3}$$

[13] considers the case when $m=2$ and $n=1$ to study one of three possible third-order Lane-Emden type model. They also considered the case when $m=3$ and $n=1$ to study one type of fourth order Lane-Emden type model and proposed the non-iterative generalized higher-order Lane-Emden type model, which has the form

$$\begin{aligned}&y^{(m+1)}+\sum_{r=1}^{m}\binom{m}{l}\frac{1}{x^r}\prod_{j=1}^{r}(k-j+1))y^{(m+1-r)}+f(y)=0\\ &y(0)=a, y^{(1)}=0, y^{(2)}...=y^{(m)}(0)=0.\end{aligned} \tag{4}$$

This generalized non-iterative model proposed in [10], can be used to study higher-order Lane-Emden type equations. In the following theorem, we prove this generalization through mathematical induction for $m\geq 1$.

2.2 Higher-order Generalization Verification

Theorem 21. *For $m \geq 1$ and $n = 1$, the generalized iterative higher-order derivative operator in the Emden-Fowler type equations is obtained as*

$$x^{-k}\frac{d^m}{dx^m}\left(x^k\frac{dy}{dx}\right) = y^{(m+1)} + \sum_{r=1}^{m}\binom{m}{r}\frac{1}{x^r}\prod_{j=1}^{r}(k-j+1))y^{(m+1-r)} \quad (5)$$

Proof. For $m = 1$, we have

$$L.H.S = x^{-k}\frac{d}{dx}\left(x^k\frac{dy}{dx}\right) = y^{(2)}(x) + \frac{k}{x}y^{(1)}(x)$$

and

$$R.H.S = y^{(2)} + \sum_{r=1}^{1}\binom{1}{r}\frac{1}{x^r}\prod_{j=1}^{r}(k-j+1))y^{(2-r)} = y^{(2)}(x) + \frac{k}{x}y^{(1)}(x).$$

Suppose that (5) is true for $m = l$. This means that we have

$$x^{-k}\frac{d^l}{dx^l}\left(x^k\frac{dy}{dx}\right) = y^{(l+1)} + \sum_{r=1}^{l}\binom{l}{r}\frac{1}{x^r}\prod_{j=1}^{r}(k-j+1))y^{(l+1-r)}. \quad (6)$$

To prove the claim for $m = l+1$, we apply the operator $x^{-k}\frac{d}{dx}\left(x^k[\cdot]\right)$ to both sides (6). We have

$$L.H.S = x^{-k}\frac{d^{l+1}}{dx^{l+1}}\left(x^k\frac{dy}{dx}\right).$$

$$\begin{aligned} R.H.S. &= y^{(l+1)} \\ &+ \left[\binom{l}{0}+\binom{l}{1}\right]\frac{k}{x}y^{(l)} + \left[\binom{l}{1}+\binom{l}{2}\right]\frac{k(k-1)}{x^2}y^{(l-1)} \\ &+ \ldots \\ &+ \left[\binom{l}{n}+\binom{l}{n+1}\right]\frac{k(k-1)(k-2)\ldots(k-n)}{x^{n+1}}y^{(l-n+1))} \\ &+ \ldots \\ &+ \left[\binom{l}{l}+\binom{l}{l+1}\right]\frac{k(k-1)\ldots(k-n+1))\ldots(k-l)}{x^{l+1}}y^{(1)}, \end{aligned}$$

or

$$\begin{aligned} R.H.S &= y^{(l+1)} + \binom{l+1}{1}\frac{k}{x}y^{(l)} + \binom{l+1}{2}\frac{k(k-1)}{x^2}y^{(l-1)} \\ &+ \ldots + \binom{l+1}{n}\frac{k(k-1)(k-2)\ldots(k-n)}{x^{n+1}}y^{(l-n+1))} \\ &+ \ldots + \binom{l+1}{l+1}\frac{k(k-1)\ldots(k-n+1))\ldots(k-l)}{x^{l+1}}y^{(1)} \\ &= y^{(l+2)} + \sum_{r=1}^{l+1}\binom{l+1}{r}\frac{1}{x^r}\prod_{j=1}^{r}(k-j+1))y^{(l+2-r)}. \end{aligned}$$

Therefore (5) is valid for all $m \in \mathbb{N}$.

2.3 Power Series Analysis

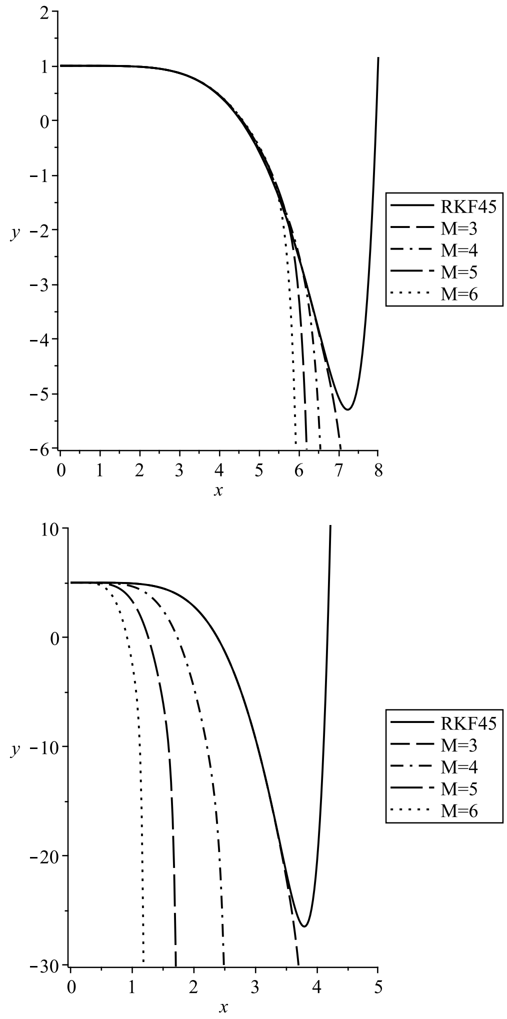
We look at the effects of increasing values of M in y^M on the interval of convergence for the solution of fifth-order Lane-Emden type model and analyze the whole situation with changing initial values in the model. Thus, we compare the 100th-order power series solution with Runge-Kutta-Fehlberg Method. In each of the figures 1(a-c), we plot the 100th-order series solution for increasing nonlinearities $M = 3,4,5$ and 6, with initial values $a = 1$, $a = 5$ and $a = 10$, respectively. When $a = 1$ in figure 1(a), the interval of convergence reduces from $(0,6.76)$ for $M = 3$ to $(0,5.44)$ for $M = 6$. Consider the situation when $a = 5$ in 1(b) where the interval of convergence shrinks from from $(0,3.53)$ for $M = 3$ to $(0,0.44)$ for $M = 6$. Finally, when $a = 10$ in 1(c), the interval of convergence in this case reduces from $(0,2.72)$ for $M = 3$ to $(0,0.20)$ for $M = 6$. In the cases where $a = 5$ and $a = 10$, we observe a substantial shrink in the interval of convergence rather than in the case where $a = 1$. Moreover, it is noticeable in all the figures 1(a-c) that increasing the initial values $a = 1,5,10$ in the problem reduces the interval of convergence.

3 HAM for the Fifth-order Model

3.1 Basic Idea of HAM

The idea of homotopy comes from topology. Homotopic equations for the nonlinear problems provide a mapping which starts from the initial guess to the actual solution of the problem. HAM does not even require a small parameter in the model. Moreover, it transforms a nonlinear problem into a recursive sequence of linear problems. Adding solutions to linear problems gives a convergent series solution of the linear/nonlinear problem.

Choice of the linear operator in the construction of the homotopy equation and the value of auxiliary parameter $\hbar$ in the homotopy equation play an important role in



(a) 100th-order series solution with $a=1$ (c) 100th-order series solution with $a=5$
(b) 100th-order series solution with $a=10$

Fig. 1: Numerical comparison for the fifth-order Lane-Emden type IVP with the 100th-order series solution when $a=1$ for increasing nonlinearities M in (a), when $a=5$ for increasing nonlinearities M in (b) and when $a=1$ for increasing nonlinearities M in (c).

finding a speedy convergent series solution. We name the auxiliary parameter $\hbar$ as a convergence controlling series solution because it doesn't only help to obtain a convergent solution but it also enhances the interval of convergence.

The starting point is the fifth-order Lane-Emden type equations:

$$x^{-2}\frac{d^4}{dx^4}\left(x^2\frac{dy}{dx}\right)+y^3(x)=0 \qquad (7)$$
$$y(0)=a, y^{(1)}(0)=y^{(2)}(0)=y^{(3)}(0)=y^{(4)}(0)=0,$$

equivalently

$$\frac{d^5y}{dx^5}+\frac{8}{x}\frac{d^4y}{dx^4}+\frac{12}{x^2}\frac{d^3y}{dx^3}+y^3(x)=0 \qquad (8)$$
$$y(0)=a, y^{(1)}(0)=y^{(2)}(0)=y^{(3)}(0)=y^{(4)}(0)=0.$$

We define for $p\in[0,1]$ the homotopy

$$\begin{aligned} H_1(x,p) &:= (1-p)\mathscr{L}\left[v(x;p)-y_0(x)\right] \\ &= \hbar p\left[\mathscr{L}v(x;p)+v^3(x;p)\right], \end{aligned} \qquad (9)$$

$$v(0;p)=a,$$
$$v^{(1)}(0;p)=v^{(2)}(0;p)=v^{(3)}(0;p)=v^{(4)}(0;p)=0,$$

where

$$[v]=x^{-2}\frac{\partial^4}{\partial x^4}\left(x^2\frac{\partial}{\partial x}\right)[v].$$

It is easy to verify that $p=1$ where it is equivalent to (8). Moreover, every function that satisfies the initial conditions is a valid solution of (9) for $p=0$. We expand $v(x;p)$ in a power series about $p=0$ to obtain

$$v(x;p)=v(x;0)+\sum_{m=0}^{+\infty}\left(\frac{v^{(m)}(x;p)|_{p=0}}{m!}\right)p^m. \qquad (10)$$

Formally, provided that there exists sufficient regularity in the problem and the series is convergent, we can write

$$v(x;1)=v(x;0)+\sum_{m=0}^{+\infty}\frac{v^{(m)}(x;p)|_{p=0}}{m!}. \qquad (11)$$

The terms under the sum of mth order deformation derivative are defined as

$$y_m(x):=\frac{v^{(m)}(x;p)|_{p=0}}{m!}. \qquad (12)$$

The solution of (8) is as follows:

$$y(x)=y_0+\sum_{m=1}^{+\infty}y_m(x), \qquad (13)$$

where $y_0(x)=v(x;0)$ and $y_m(x)$ is the mth-order approximation to the actual solution. It is expected that the mth-order gets closer to the actual solution. However, pieces of literature indicate that for some problems.

3.2 Recursive Linear Algorithm for $H_1(x;p)$

The deformation derivatives can be computed by analogy to the theory summarized by [18]. For our specific problem, we have the following algorithm.

Theorem 31. *For $y_0=a$, the deformation derivatives y_m associated with homotopy H_1 are obtained recursively as* *solutions of the initial value problem*

$$\mathscr{L}[y_m(x)] = (\hbar+\chi_m)\mathscr{L}[y_{m-1}(x)] + \hbar[\sum_{k=0}^{m-1} y_{m-1-k} \sum_{l=0}^{k} y_l y_{k-l}]$$
$$y_m(0) = a, y_m^{(1)}(0) = y_m^{(2)}(0) = y_m^{(3)}(0) = y_m^{(4)}(0) = 0, \tag{14}$$

where

$$\mathscr{L}[\cdot] = x^{-2}\frac{d^4}{dx^4}\left(x^2\frac{d}{dx}\right)[\cdot]$$

and

$$\chi_m = \begin{cases} 1 & m \le 1 \\ 0 & m > 1. \end{cases}$$

Proof. Substituting (10) into (9) and applying Leibniz' Rule, we obtain for the left hand side of (9)

$$\begin{aligned} L.H.S &= \sum_{l=0}^{m}\binom{m}{l}(1-p)^{(l)}\mathscr{L}[v(x;p)-y_0(x)]^{(m-l)} \\ &= m!\mathscr{L}\left[\frac{v^{(m)}(x;p)|_{p=0}}{m!} - \frac{v^{(m-1)}(x;p)|_{p=0}}{(m-1)!}\right] \\ &= m!\mathscr{L}[y_m(x) - \chi_m y_{m-1}(x)] \end{aligned}$$

and the right-hand side of (9)

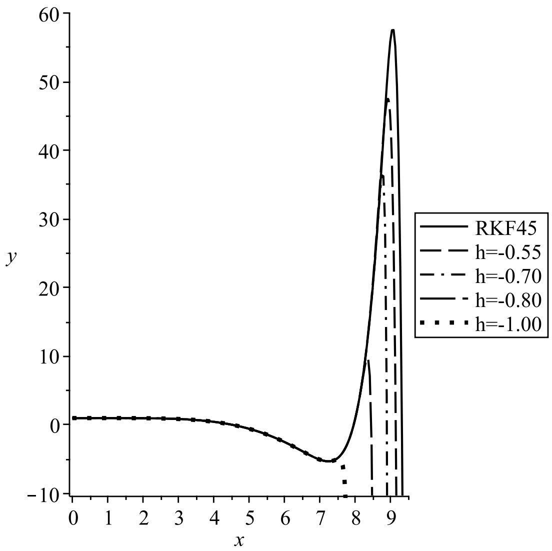
$$\begin{aligned} R.H.S &= \sum_{l=0}^{m}\binom{m}{l}(\hbar p)^{(l)}\left[\mathscr{L}v(x;p)+v^3(x;p)\right]^{(m-l)} \\ &= m!\hbar\left[\mathscr{L}\left(\frac{v^{(m-1)}(x;p)|_{p=0}}{(m-1)!}\right)\right. \\ &\quad + \sum_{k=0}^{m-1}\frac{v^{(m-1-k)}(x;p)|_{p=0}}{(m-1-k)!}\times \\ &\quad \left.\sum_{l=0}^{k}\frac{v^{(l)}(x;p)|_{p=0}}{l!}\frac{v^{(k-l)}(x;p)|_{p=0}}{(k-l)!}\right] \\ &= m!\hbar\left[\mathscr{L}y_{m-1}(x) + \sum_{k=0}^{m-1}y_{m-1-k}(x)\sum_{l=0}^{k}y_l(x)y_{k-l}(x)\right]. \end{aligned}$$

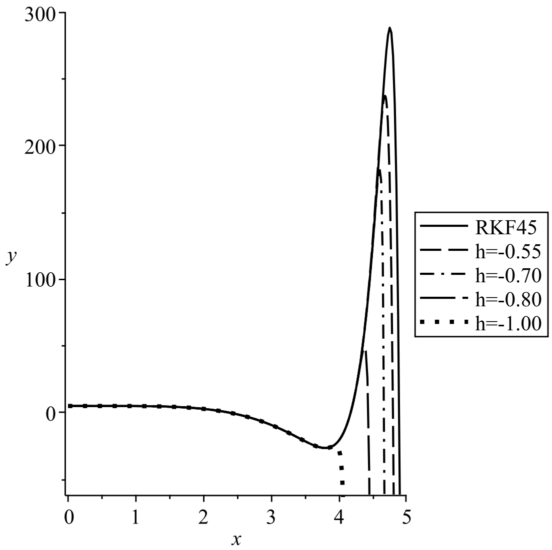
Thus, for the mth deformation derivative, we obtain with the initial conditions in equation (9), the following recursive linear initial value problem

$$\mathscr{L}[y_m(x)] = (\hbar+\chi_m)\mathscr{L}[y_{m-1}(x)] + \hbar[\sum_{k=0}^{m-1} y_{m-1-k} \sum_{l=0}^{k} y_l y_{k-l}]$$
$$y_m(0) = a, y_m^{(1)}(0) = y_m^{(2)}(0) = y_m^{(3)}(0) = y_m^{(4)}(0) = 0.$$

3.3 HAM vs. Numerical analysis for the fifth-order Lane-Emden Type Model

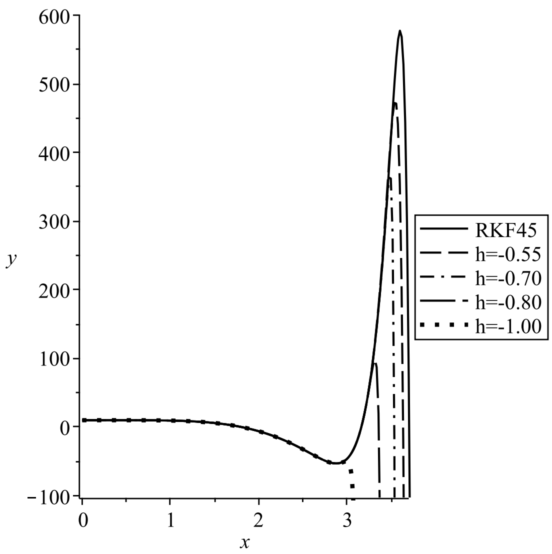
The HAM algorithm is based on the fact that the auxiliary parameter $\hbar$ which is used in constructing the homotopy

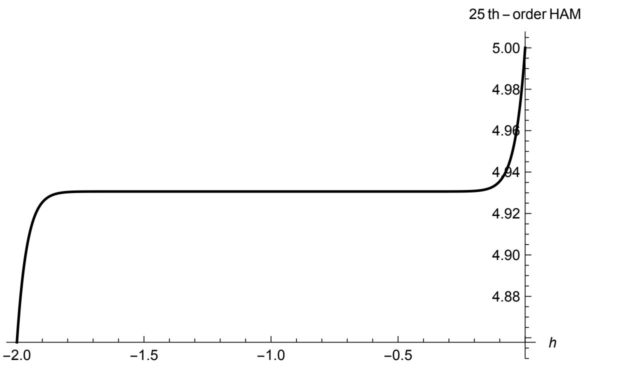




(a) 25th-order HAM series solution with $a = 1$

(b) 25th-order HAM series solution with $a = 5$





(c) 25th-order HAM series solution with $a = 10$ (d) h-curve for the 25th-order HAM series solution

Fig. 2: Numerical comparison for the fifth-order Lane-Emden type IVP with the 25th-order HAM series solution when $a = 1$ for various values of auxiliary parameter $\hbar$ in (a), when $a = 5$ for various values of auxiliary parameter $\hbar$ in (b) and when $a = 10$ for various values of auxiliary parameter $\hbar$ in (c).

equation doesn't only control the interval of convergence but it also help find the convergent series solution of the problem. We present the numerical comparison of model with various choices of auxiliary parameter values $\hbar = -0.55, -0.70, -0.80, -1.00$ for the 25th-order HAM solution based on the homotopy construction defined in (9). Figures 2(a-d) manifest that HAM controls the interval of convergence with various choices of the initial values $a = 1, 5, 10$. Furthermore, the increasing values of a in figures 2(a-d) reduce the interval of convergence in general. In figure 3(a) with $a = 1$, we have the interval of convergence for $\hbar = -0.55$ $(0, 8.78)$, $\hbar = -0.70$ $(0, 8.75)$, $\hbar = -0.80$ $(0, 8.30)$ and $\hbar = -1.00$ $(0, 7.39)$. In figure 3(b) with $a = 5$, we observe the maximum interval of convergence for $\hbar = -0.55$ is $(0, 3.53)$. Last, in figure 3(c) with $a = 10$, we observe the maximum interval of convergence for $\hbar = -0.55$ which is $(0, 4.68)$.

In figures 3(a-c), we present the numerical comparison of the fifth order Lane-Emden model with 25th-order HAM solution and 100th-order power series solution. It exhibits that the HAM solution converges faster and has a bigger interval of convergence with various choices of the initial values $a = 1, 5, 10$ than the traditional power series solution. Moreover, the increasing initial value reduces the interval of convergence in general. In figure 3(a) with $a = 1$, we compare the 25th-order HAM approximation and 100th-order power series solution with RKF45. We notice that the interval of convergence for the HAM solution is $(0, 8.90)$ and for power series is $(0, 7.29)$. It looks as if the HAM algorithm achieves a bigger interval of convergence with a smaller number of approximations than that is required in the case of power series solution. In figure 3(c) with $a = 5$, the intervals of convergence for HAM and power series is $(0, 0.81)$ and $(0, 4.66)$ respectively. Finally, in figure 3(c) with $a = 10$, the intervals of convergence for HAM and power series is $(0, 0.38)$ and $(0, 3.56)$ respectively. With increasing initial values, HAM interval of convergence becomes bigger than the traditional series method.

3.4 Possible HAM Algorithms for the Model Equations

We introduce three possible HAM algorithms in the following three theorems. We only prove the first theorem because the proofs of the second and the third are similar to that of the first. .
For the first possible algorithm, we define the homotopy

$$\begin{aligned} &H_2(x,p) := (1-p)\mathscr{L}[v(x;p) - y_0(x)] \\ &= \hbar p\left[\mathscr{L}v(x;p) + \tfrac{8}{x}\tfrac{\partial^4 v(x;p)}{\partial x^4} + \tfrac{12}{x^2}\tfrac{\partial^3 v(x;p)}{\partial x^3} + v^3(x;p)\right], \\ &v(0;p) = a, \\ &\quad v^{(1)}(0;p) = v^{(2)}(0;p) = v^{(3)}(0;p) = v^{(4)}(0;p) = 0, \end{aligned} \tag{15}$$

where $p \in [0,1]$ and

$$\mathscr{L}[v] = \frac{\partial^5}{\partial x^5}[v].$$

In this case, we have the following result:

Theorem 32. *For $y_0 = a$, the deformation derivative y_m associated with homotopy H_2 are obtained recursively as solutions of the initial value problem*

$$\begin{aligned} &\mathscr{L}[y_m(x)] = (\hbar + x_m)\mathscr{L}[y_{m-1}(x)] \\ &+ \quad \hbar[\tfrac{8}{x}y_{m-1}^{(4)} + \tfrac{12}{x^2}y_{m-1}^{(3)} + \sum_{k=0}^{m-1} y_{m-1-k}\sum_{l=0}^{k} y_l y_{k-l}] \\ &y_m(0) = a, y_m^{(1)}(0) = y_m^{(2)}(0) = y_m^{(3)}(0) = y_m^{(4)}(0) = 0, \end{aligned} \tag{16}$$

where

$$\mathscr{L}[\cdot] = \frac{d^5}{dx^5}[\cdot]$$

and

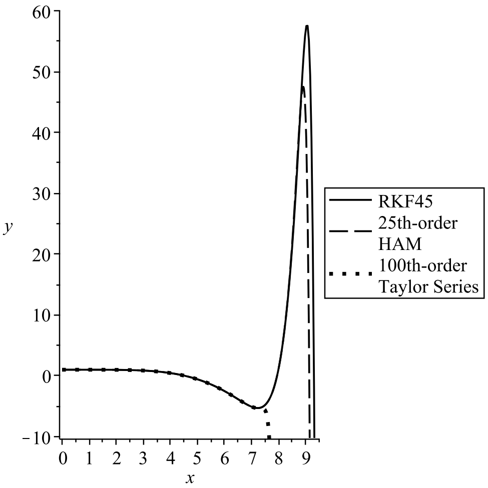
$$\chi_m = \begin{cases} 1 & m \le 1 \\ 0 & m > 1. \end{cases}$$

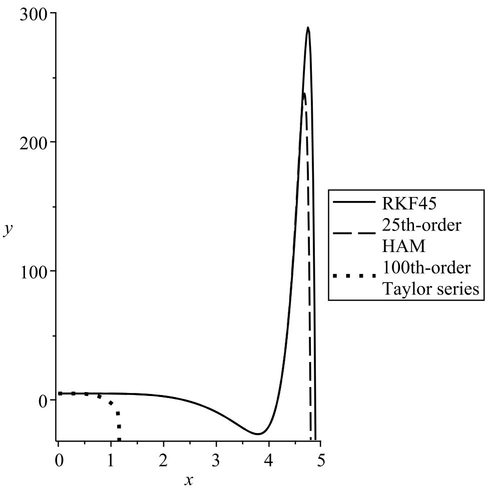
□

Proof. Substituting (10) into (15) and applying Leibniz' Rule, we obtain for the left hand side of (15)

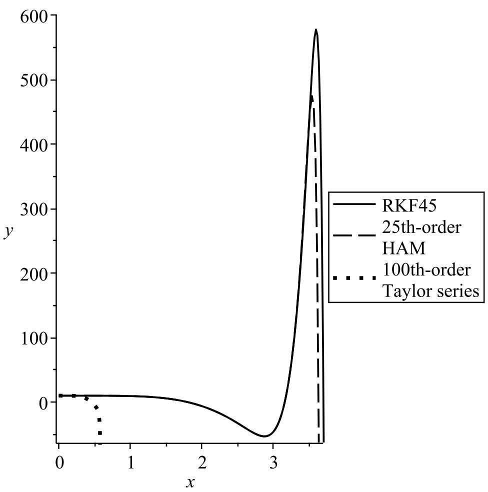
$$\begin{aligned} L.H.S &= \sum_{l=0}^{m}\binom{m}{l}(1-p)^{(l)}\mathscr{L}[v(x;p) - y_0(x)]^{(m-l)} \\ &= m!\,\mathscr{L}\left[\frac{v^{(m)}(x;p)|_{p=0}}{m!} - \frac{v^{(m-1)}(x;p)|_{p=0}}{(m-1)!}\right] \\ &= m!\,\mathscr{L}[y_m(x) - \chi_m y_{m-1}(x)] \end{aligned}$$

and the right-hand side of (15)





(a) 100th-order power series vs 25th order HAM for $a = 1$ (b) 100th-order power series vs 25th order HAM for $a = 5$



(c) 100th-order power series vs 25th order HAM for $a = 10$

Fig. 3: Numerical comparison between 25th-order HAM series solution and 100th-order power series solution when $a = 1$ in (a), when $a = 5$ in (b) and when $a = 10$ in (c).

R.H.S

$$
\begin{aligned}
&= \sum_{l=0}^{m} \binom{m}{l} (\hbar p)^{(l)} \left[\mathscr{L} v(x;p) + \frac{8}{x}\frac{\partial^4 v(x;p)}{\partial x^4} \right.\\
&\qquad \left. + \frac{12}{x^2}\frac{\partial^3 v(x;p)}{\partial x^3} + v^3(x;p) \right]^{(m-l)}
\end{aligned}
$$

$$
= m!\hbar \left[\begin{array}{c} \mathscr{L}\left(\frac{v^{(m-1)}(x;p)|_{p=0}}{(m-1)!}\right) + \frac{8}{x}\frac{\partial^4}{\partial x^4}\left(\frac{v^{(m-1)}(x;p)|_{p=0}}{(m-1)!}\right) \\ + \frac{12}{x^2}\frac{\partial^3}{\partial x^3}\left(\frac{v^{(m-1)}(x;p)|_{p=0}}{(m-1)!}\right) \\ + \sum_{k=0}^{m-1} \frac{v^{(m-1-k)}(x;p)|_{p=0}}{(m-1-k)!} \sum_{l=0}^{k} \frac{v^{(l)}(x;p)|_{p=0}}{l!}\frac{v^{(k-l)}(x;p)|_{p=0}}{(k-l)!} \end{array} \right]
$$

$$
\begin{aligned}
&= m!\hbar \left[\mathscr{L} y_{m-1}(x) + \frac{8}{x} y^{(4)}_{m-1} + \frac{12}{x^2} y^{(3)}_{m-1} \right.\\
&\qquad \left. + \sum_{k=0}^{m-1} y_{m-1-k}(x) \sum_{l=0}^{k} y_l(x) y_{k-l}(x) \right] \qquad (17)
\end{aligned}
$$

Thus, for the mth deformation derivative, with the initial conditions in equation (15), we obtain the recursive linear initial value problem.

$$
\begin{aligned}
&\mathscr{L}[y_m(x)] = (\hbar + x_m)\mathscr{L}[y_{m-1}(x)] + \hbar[\tfrac{8}{x} y^{(4)}_{m-1} + \tfrac{12}{x^2} y^{(3)}_{m-1}\\
&+ \sum_{k=0}^{m-1} y_{m-1-k} \sum_{l=0}^{k} y_l y_{k-l}]\\
&y_m(0) = a, y^{(1)}_m(0) = y^{(2)}_m(0) = y^{(3)}_m(0) = y^{(4)}_m(0) = 0
\end{aligned}
$$

□

For the next algorithm, we define the homotopy

$$
\begin{aligned}
H_3(x,p) &:= (1-p)\mathscr{L}[v(x;p) - y_0(x)]\\
&= \hbar p \left[\mathscr{L} v(x;p) + \frac{12}{x^2}\frac{\partial^3 v(x;p)}{\partial x^3} + v^3(x;p) \right], \qquad (18)\\
v(0;p) &= a, \quad v^{(1)}(0;p) = v^{(2)}(0;p)\\
&= v^{(3)}(0;p) = v^{(4)}(0;p) = 0,
\end{aligned}
$$

where $p \in [0,1]$ and

$$
\mathscr{L}[v] = \left(\frac{\partial^5}{\partial x^5} + \frac{8}{x}\frac{\partial^4}{\partial x^4} \right)[v].
$$

In this case, the algorithm is validated by the following theorem.

Theorem 33. *For $y_0 = a$, the deformation derivatives y_m associated with homotopy H_2 are obtained recursively as solutions of the initial value problem*

$$
\begin{aligned}
\mathscr{L}[y_m(x)] &= (\hbar + x_m)\mathscr{L}[y_{m-1}(x)]\\
&+ \hbar[\frac{12}{x^2} y^{(3)}_{m-1} + \sum_{k=0}^{m-1} y_{m-1-k} \sum_{l=0}^{k} y_l y_{k-l}] \qquad (19)\\
y_m(0) &= a, \ y^{(1)}_m(0) = y^{(2)}_m(0) = y^{(3)}_m(0) = y^{(4)}_m(0) = 0
\end{aligned}
$$

where

$$\mathscr{L}[\cdot] = \left(\frac{d^5}{dx^5} + \frac{8}{x}\frac{d^4}{dx^4}\right)[\cdot]$$

and

$$\chi_m = \begin{cases} 1 & m \le 1 \\ 0 & m > 1. \end{cases}$$

□

For the last algorithm, we define the homotopy

$$\begin{aligned} H_4(x,p) &:= (1-p)\mathscr{L}[v(x;p) - y_0(x)] \\ &= \hbar p\left[\frac{\partial^5 v(x;p)}{\partial x^5} + \mathscr{L}v(x;p) + v^3(x;p)\right] = 0, \\ & v(0;p) = a, \\ & v^{(1)}(0;p) = v^{(2)}(0;p) = v^{(3)}(0;p) \\ & = v^{(4)}(0;p) = 0, \quad (20) \end{aligned}$$

where $p \in [0,1]$ and

$$\mathscr{L}[v] = \left(\frac{8}{x}\frac{\partial^4}{\partial x^4} + \frac{12}{x^2}\frac{\partial^3}{\partial x^3}\right)[v].$$

Theorem 34. *For $y_0 = a$, the deformation derivatives y_m associated with homotopy H_2 are obtained recursively as solutions of the initial value problem*

$$\mathscr{L}[y_m(x)] = (\hbar + x_m)\mathscr{L}[y_{m-1}(x)] + \hbar[y^{(5)}_{m-1} + \sum_{k=0}^{m-1} y_{m-1-k} \sum_{l=0}^{k} y_l y_{k-l}] \quad (21)$$

$$y_m(0) = a, y_m^{(1)}(0) = y_m^{(2)}(0) = y_m^{(3)}(0) = y_m^{(4)}(0) = 0$$

where

$$\mathscr{L}[\cdot] = \left(\frac{8}{x}\frac{d^4}{dx^4} + \frac{12}{x^2}\frac{d^3}{dx^3}\right)[\cdot]$$

and

$$\chi_m = \begin{cases} 1 & m \le 1 \\ 0 & m > 1. \end{cases}$$

□

3.5 Convergence Interval Analysis for Various HAM Algorithms

The choice of linear operator plays a vital role in the construction of the homotopy equation. Figures 4(a-c) show comparison analysis on the choice of different HAM algorithms for solution convergence of interval for the considered problem. We observe the same pattern , i.e. the increasing initial values reduce the interval of convergence. Apart from that, each of the figures 4(a-c) shows that various HAM algorithm choices do not have much affect in achieving a more significant interval of convergence.

Table 1: c_m calculations are shown based on the sup-norm ratio of the successive solutions y_{m+1} & y_m which we obtained through HAM for the observed IVP

$c_m = \|\|y_{m+1}\|\|/\|\|y_m\|\|$	$a=1$; $[0,9.25]$	$a=5$; $[0,4.8]$	$a=10$; $[0,3.6]$
$c_1 = \|\|y_1\|\|/\|\|y_0\|\|$	0.955	0.746	0.977
$c_2 = \|\|y_2\|\|/\|\|y_1\|\|$	0.961	0.877	0.810
$c_3 = \|\|y_3\|\|/\|\|y_2\|\|$	0.978	0.888	0.962
$c_4 = \|\|y_4\|\|/\|\|y_3\|\|$	0.945	0.997	0.963
⋮	⋮	⋮	⋮
$c_{23} = \|\|y_{23}\|\|/\|\|y_{22}\|\|$	0.777	0.901	0.726
$c_{24} = \|\|y_{24}\|\|/\|\|y_{23}\|\|$	0.399	0.625	0.755
$c_{25} = \|\|y_{25}\|\|/\|\|y_{24}\|\|$	0.631	0.304	0.479

Now, we prove that the series (13) converges to the solution of the IVP. The proof is similar that proof of the Banach Fixed-Point Theorem. From our numerical simulations, we establish the fact that for two successive y_m and y_{m+1} we have the relation $\frac{||y_{m+1}||}{||y_m||} < 1$. Here $||.||$ denotes the usual sup norm. This means there is a constant c_m with $0 \le c_m < 1$, such that $||y_{m+1}|| \le c_m||y_m||$. We illustrate this fact in the next table, where several calculations are shown. The calculations are done for different values of the parameter a with its corresponding intervals.

Since the table shows that $c_m < 1$ for all m. We can take $c = \sup_{m\in\mathbb{N}} c_m$ which provides the necessary condition for the poof in [23] about the convergence of the HAM series solution of the IVP in (13) .

4 Conclusion

The present paper addressed , the fifth-order Lane-Emden model with multiple singularities. A generalized non-iterative algorithm for obtaining any higher-order Lane-Emden type equation was verified with mathematical induction. HAM was used to study the model equations semi-analytically. The study revealed that HAM is a better choice than the traditional power series approach. In addition, the choice of the auxiliary parameter $\hbar$ in the basic homotopic mapping of the higher ordered Lane-Emden model and the choice of a linear operator in this regard had a great influence not only on obtaining a convergent series solution but also on extending the radius of convergence for the solution. Moreover, the increasing values of a in the nonlinear term y^m shrink the interval of convergence for the series solution. The increasing initial values a in the model also reduce the interval of convergence in all the analysis we have made in the entire paper.

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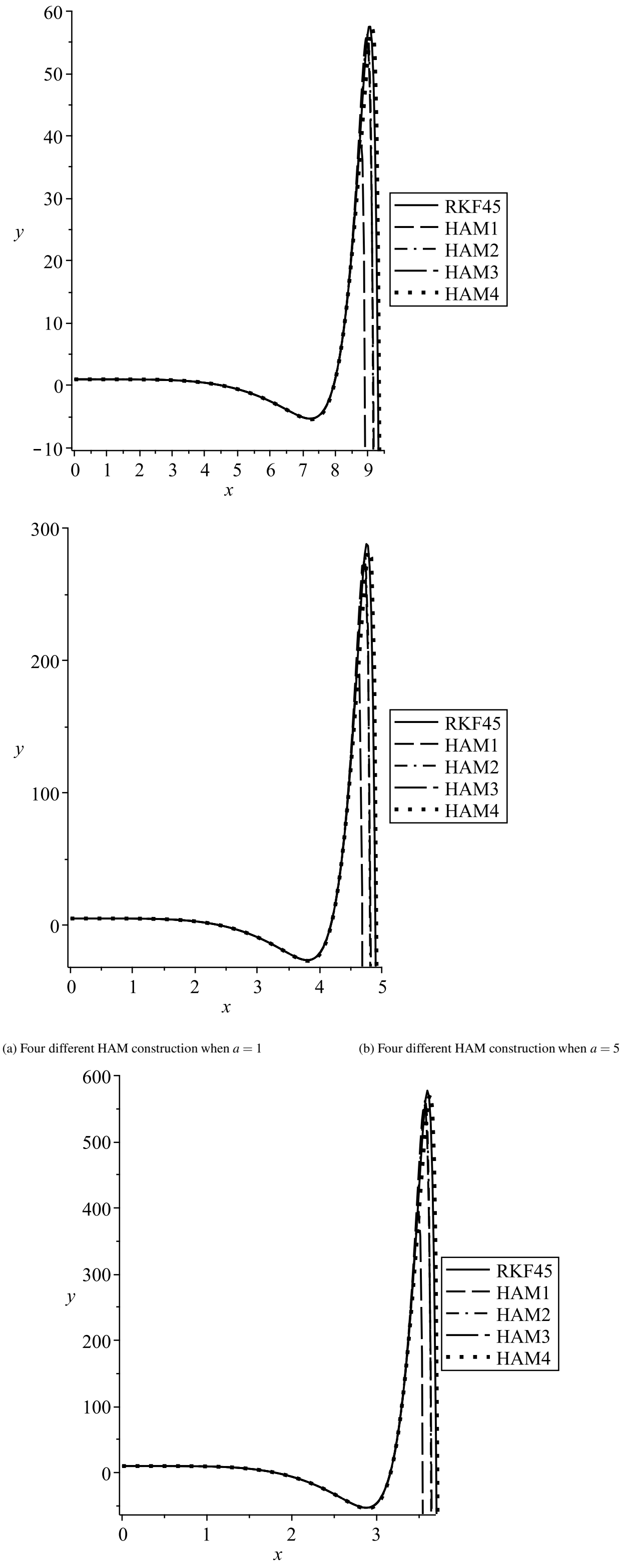

(a) Four different HAM construction when $a = 1$

(b) Four different HAM construction when $a = 5$

(c) Four different HAM construction when $a = 10$

Fig. 4: Numerical comparison between the four different homotopy constructions of 25th-order HAM series solution when $a = 1$ in (a), when $a = 5$ in (b) and when $a = 10$ in (c).

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