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Exponential Stability and Numerical Results of a Coupled System of Wave Equations with Indirect Control

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Abstract: The present paper addresses a coupled system of waves with indirect control. First, using Semigroup Theory we prove the existence of solution applying the Lumer-Phillips theorem. We show the exponential decay of the solution using Nakao's method. Computational experiments are conducted in the one-dimensional case in order to show that the dissipative energy properties are present. To obtain the time behavior of the energy of system, we resolve two numerical tests. Finally, in both tests, we conclude that the system, for one-dimensional case, is dissipative with exponential stability.

Keywords: Numerical properties, Nakao's method, Wave coupled system, Indirect control, Semigroups.

1 Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be a bounded open domain with smooth boundary Γ and $\alpha > 0$ a coupling parameter. Let the Lebesgue space and Sobolev spaces be $L^2(\Omega)$ and $H^1(\Omega)$ (see [1] for more details). We consider the following system of coupled wave equations

$$\begin{aligned} u_{tt} - \Delta u + u_t + \alpha v = 0, &\quad \text{in } \Omega \times \mathbb{R}, \\ v_{tt} - \Delta v + v_t + \alpha u = 0, &\quad \text{in } \Omega \times \mathbb{R}, \\ u(x,t) = v(x,t) = 0, &\quad \text{on } \Gamma \times \mathbb{R}, \\ (u,u_t)(x,0) = (u_0,u_1), &\quad \text{in } \Omega, \\ (v,v_t)(x,0) = (v_0,v_1), &\quad \text{in } \Omega. \end{aligned} \tag{1}$$

Here Δ is the Laplacian in the space variable x and $t > 0$ denotes time variable and $u = u(x,t)$. The problem (1) is well known as system with indirect stabilization or indirect control (see [2]) and it has been investigated by several authors in different frameworks see [3–11, 14–27] and the references therein. For the case of $v_t(x,t) = 0$, i.e. with one indirect control, the system (1) was taken into account in [4] under compatibility assumptions and proved the polynomial decay for the energy by interpolation method. Later, see [5], that considered Neumann and Robin boundary conditions, which describe different physical situations, such as hinged or clamped devices and under new compatibility assumptions, the same stabilization rate was obtained. The study of stability for coupled wave equations under various end conditions was given in [6]. For the uniform exponential stability of a linear system of compactly coupled wave equations and estimate to energy decay rate in case of nonlinear boundary feedbacks, see [7]. Nonlinear coupled system with memory condition at the boundary was considered in [8] where the energy decay with the same rate of the relaxation functions was proved. Recently, [9] addressed the exact controllability with Neumann boundary controls for a system of linear wave equations coupled in parallel by lower order terms on piecewise smooth domains of the plane. The result extends the same one obtained in [10] for linear case. In [11], the Method of Nakao was applied to prove the exponential decay of the solution and a numerical scheme by finite differences method was presented to numerical solution and the long-time decay energy. The present paper handles the exponential stability as well as the numerical consistence. It is organized, as follows: In Section Two, we prove the existence of solution applying the Lumer-Phillips theorem. In Section Three we prove the exponential stability by Nakao's method. In Section Four, we present the numerical scheme to testify consistence of the wave coupled system with indirect control in both components.

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2 Semigroup setup

Let $\mathscr{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ be a Hilbert with inner product

$$\langle U,V\rangle_{\mathscr{H}} = \int_\Omega [\nabla u_1 \cdot \nabla v_1 + u_2 v_2 + \nabla u_3 \cdot \nabla v_3 + u_4 v_4 + \alpha(u_1 v_3 + u_3 v_1)]dx$$

and associated norm

$$\|U\|^2_{\mathscr{H}} = \int_\Omega [|\nabla u_1|^2 + u_2^2 + |\nabla u_3|^2 + u_4^2 + \alpha(u_1 u_3 + u_3 u_1)]dx$$

where $U = (u_1,u_2,u_3,u_4)^T$ and $V = (v_1,v_2,v_3,v_4)^T$. Consider the elliptic operator $\mathscr{A}$ defined by

$$\mathscr{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \Delta & -I & -\alpha I & 0 \\ 0 & 0 & 0 & I \\ -\alpha I & 0 & \Delta & -I \end{pmatrix}$$

with domain

$$\mathscr{D}(\mathscr{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega).$$

Denoting $u_t = \varphi$, $v_t = \psi$, we formally get that $U = (u,\varphi,v,\psi)^T$ satisfies the Cauchy problem

$$\frac{dU}{dt} = \mathscr{A}U,$$
$$U(0) = U_0 = (u_0,\varphi_0,v_0,\psi_0)^T.$$

Using the inner product is easy to see that the operator $\mathscr{A}$ is dissipative, that is,

$$\mathscr{R}e\langle \mathscr{A}U,U\rangle_{\mathscr{H}} = -\int_\Omega |\varphi|^2 dx - \int_\Omega |\psi|^2 dx \le 0$$

Now, we will prove that $0 \in \rho(\mathscr{A})$ the resolvent set of $\mathscr{A}$. In order to do so, take $F = (f_1,f_2,f_3,f_4)^T \in \mathscr{H}$ and consider the equation $\mathscr{A}U = F$ which leads to

$$\varphi = f_1 \in H_0^1(\Omega), \quad (2)$$
$$\Delta u - \alpha v - \varphi = f_2 \in L^2(\Omega), \quad (3)$$
$$\psi = f_3 \in H_0^1(\Omega), \quad (4)$$
$$\Delta v - \alpha u - \psi = f_4 \in L^2(\Omega). \quad (5)$$

Remember that $v \in H_0^1(\Omega)$. From $v_t = \psi = f_3$, we get

$$v = \int_0^t f_3(s)\,ds \in L^2(\Omega).$$

Follows from (2), (3), (4) that

$$\Delta u = f_1 + f_2 + \alpha \int_0^t f_3(s)\,ds \in L^2(\Omega). \quad (6)$$

From standard Theory of Linear Elliptic Equations, it follows that (6) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. By the same argument we see that $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Hence, $U \in D(\mathscr{A})$ and as a consequence $0 \in \rho(\mathscr{A})$.

Theorem 21 *The operator $\mathscr{A}$ is the infinitesimal generator of C_0-semigroup of contractions $S(t) = e^{t\mathscr{A}}$ in $\mathscr{H}$.*

Proof. Since $D(\mathscr{A})$ is dense in $\mathscr{H}$, $\mathscr{A}$ is a dissipative operator and $0 \in \rho(\mathscr{A})$ by Lumer-Phillips'theorem the conclusion follows. (See [12], Theorem 1.2.4).

From Theory of Semigroups, we have

Theorem 22 *For $U_0 \in \mathscr{H}$ the system (1) has a unique weak solution*

$$U \in C(\mathbb{R}^+;\mathscr{H}).$$

Moreover, if $U_0 \in D(\mathscr{A})$, we have the following regularity

$$U \in C(\mathbb{R}^+;\, D(\mathscr{A})) \cap C^1(\mathbb{R}^+;\, \mathscr{H}).$$

Proof. (See [13], Theorem 7.4).

3 Exponential stability - Nakao's method

We start this section introducing the following result.

Lemma 31 *Let $E(t)$ be a bounded positive function, defined in $\mathbb{R}^+$ satisfying for some positive constant C_0*

$$\sup_{s\in[t,t+1]} E(s) \le C_0[E(t) - E(t+1)],\ \forall t \ge 0, \quad (7)$$

Then, there exist positive constants M and ω such that $E(t) \le M e^{-\omega t}$, $\forall t > 1$.

Proof. See ([14], Lemma 3, page 339).

We will use the previous lemma to get our principal result.

Theorem 32 *The solution $(u(x,t),v(x,t))$ of the system (1) satisfies*

$$E(t) \le M e^{-\omega t}, \forall t \ge 1, \quad (8)$$

where M and ω are positive constants and $E(t)$ is the full energy given by

$$E(t) = \frac{1}{2}|u_t|^2 + \frac{1}{2}|v_t^2| + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 + \alpha \int_\Omega uv\,dx.$$

Proof. Multiplying the first and second equations of the system (1) by u_t and v_t, respectively, and performing integration by parts in $(0,L)$, we deduce that the energy has an important property

$$\frac{d}{dt}E(t) = -(|u_t|^2 + |v_t|^2). \quad (9)$$

Integrating in $[t,t+1]$, we get

$$\int_t^{t+1} (|u_t|^2 + |v_t|^2)ds = E(t) - E(t+1) =: F^2(t). \quad (10)$$

Therefore, there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$|u_t(t_i)| + |v_t(t_i)| \le 4F(t),\ i = 1,2. \quad (11)$$

Multiplying $(1)_1$ by u and $(1)_2$ by v, performing integration by parts in $(0,L)$ and adding all, we obtain

$$\frac{d}{dt}\int_\Omega u_t u dx + \frac{d}{dt}\int_\Omega v_t v dx - |u_t|^2 - |v_t|^2 + |\nabla u|^2 + |\nabla v|^2 + 2\alpha\int_\Omega uv\,dx = 0$$

Integrating from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2}\left(|\nabla u|^2+|\nabla v|^2+2\alpha\int_\Omega uv\,dx\right)dt = \int_\Omega u_t(t_1)u(t_1)dx - \int_\Omega u_t(t_2)u(t_2)dx + \int_\Omega v_t(t_1)v(t_1)dx - \int_\Omega v_t(t_2)v(t_2)dx + \int_{t_1}^{t_2}(|u_t|^2+|v_t|^2)ds.$$

Applying Cauchy-Schwarz, we have

$$\int_{t_1}^{t_2}\left(|\nabla u|^2+|\nabla v|^2+2\alpha\int_\Omega uv,dx\right)dt \le |u_t(t_1)||u(t_1)| + |u_t(t_2)||u(t_2)| + |v_t(t_1)||v(t_1)| + |v_t(t_2)||v(t_2)| + \int_{t_1}^{t_2}(|u_t|^2+|v_t|^2)ds.$$

From (10) and (11)

$$\int_{t_1}^{t_2}\left(|\nabla u|^2+|\nabla v|^2+2\alpha\int_\Omega uv\,dx\right)dt \le 4F(t)(|u(t_1)| + |u(t_2)|) + 4F(t)(|v(t_1)|+|v(t_2)|) + F^2(t).$$

Applying Poincarè inequality, we obtain $C_p>0$ such that

$$\int_{t_1}^{t_2}\left(|\nabla u|^2+|\nabla v|^2+2\alpha\int_\Omega uv\,dx\right)dt \le 4F(t)C_p(|\nabla u(t_1)| + |\nabla u(t_2)|) + 4F(t)C_p(|\nabla v(t_1)|+|\nabla v(t_2)|) + F^2(t),$$

and then

$$\int_{t_1}^{t_2}\left(|\nabla u|^2+|\nabla v|^2+2\alpha\int_\Omega uv\,dx\right)dt \le 4C_pF(t)\sup_{s\in[t,t+1]}(|\nabla u(s)|+|\nabla v(s)|)+F^2(t).$$

Choosing $m=\max\{4C_p,1\}$, defining

$$G^2(t) := m\left[F(t)\sup_{s\in[t,t+1]}(|\nabla u(s)|+|\nabla v(s)|)+F^2(t)\right]$$

and adding

$$\int_{t_1}^{t_1}(|u_t|^2+|v_t|^2)ds$$

in both sides of the last inequality, we obtain

$$\int_{t_1}^{t_2}\left(|u_t(t)|^2+|v_t(t)|^2+|\nabla u|^2+|\nabla v|^2+2\alpha\int_\Omega uv\,dx\right)dt \le F^2(t)+G^2(t).$$

Therefore there exists $t^*\in[t_1,t_2]$ such that

$$|u_t(t^*)|^2+|v_t^2(t^*)|+|\nabla u(t^*)|^2+|\nabla v(t^*)|^2 + \alpha\int_\Omega u(t^*)v(t^*)dx \le 2[F^2(t)+G^2(t)]$$

from where follows that

$$E(t^*)\le 2(F^2(t)+G^2(t)). \tag{12}$$

Integrating (9) from t to t^*, we get

$$E(t)=E(t^*)+\int_t^{t^*}(|u_t(s)|^2+|v_t(s)|^2)\,ds$$

and then

$$E(t)\le E(t^*)+\int_t^{t+1}(|u_t(s)|^2+|v_t(s)|^2)\,ds.$$

Using (10) and (12), we have

$$\begin{aligned}\sup_{s\in[t,t+1]}E(s) &\le E(t^*)+\int_t^{t+1}(|u_t(s)|^2+|v_t(s)|^2)ds\\ &\le 2[F^2(t)+G^2(t)]+F^2(t)\\ &\le 3F^2(t)+ m\left[F(t)\sup_{s\in[t,t+1]}(|\nabla u(s)|+|\nabla v(s)|)+F^2(t)\right]\\ &\le (3+m)F^2(t)+ mF(t)\sup_{s\in[t,t+1]}(|\nabla u(s)|+|\nabla v(s)|)\\ &\le (3+m)F^2(t)+mF(t)\sup_{s\in[t,t+1]}(2\sqrt{2}\sqrt{E(s)})\\ &\le (3+m)F^2(t)+4m^2F^2(t)+\frac{1}{2}\sup_{s\in[t,t+1]}E(s),\end{aligned}$$

which immediately yields

$$\sup_{s\in[t,t+1]}E(s)\le C_0[E(t)-E(t+1)],\; C_0=2(3+m+4m^2).$$

The proof is complete.

4 Numerical Approaches

In this section, we analyze the analogue of (1) in 1- d, i.e.,

$$\begin{aligned} u_{tt}-u_{xx}+u_t+\alpha v=0, &\quad \text{in} \quad (0,1)\times(0,1),\\ v_{tt}-v_{xx}+v_t+\alpha u=0, &\quad \text{in} \quad (0,1)\times(0,1),\\ u(0,t)=u(1,t)=0, &\quad 0<t<1,\\ v(0,t)=v(1,t)=0, &\quad 0<t<1,\\ u(x,0)=u_0(x),\; u_t(x,0)=u_1(x), &\quad 0<x<1,\\ v(x,0)=v_0(x),\; v_t(x,0)=v_1(x), &\quad 0<x<1,\end{aligned} \tag{13}$$

where the full energy is given by

$$E(t) := \frac{1}{2}\int_0^1 u_t^2 dx + \frac{1}{2}\int_0^1 v_t^2 dx + \frac{1}{2}\int_0^1 u_x^2 dx + \frac{1}{2}\int_0^1 v_x^2 dx + \alpha\int_0^1 uv dx$$

and satisfies

$$\frac{d}{dt}E(t) = -\int_0^1 u_t^2 dx - \int_0^1 v_t^2 dx.$$

We note that assuming the decomposition given by $\phi := u+v$ we obtain the decoupled wave equations

$$\phi_{tt} - \phi_{xx} + \phi_t + \alpha\phi = 0, \quad \text{in} \quad (0,1)\times(0,1), \tag{14}$$
$$\phi(0,t) = \phi(1,t) = 0, \quad 0<t<1, \tag{15}$$
$$\phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), \quad 0<x<1, \tag{16}$$

and taking $\psi := u-v$ we obtain

$$\psi_{tt} - \psi_{xx} + \psi_t - \alpha\psi = 0, \quad \text{in} \quad (0,1)\times(0,1), \tag{17}$$
$$\psi(0,t) = \psi(1,t) = 0, \quad 0<t<1, \tag{18}$$
$$\psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \quad 0<x<1. \tag{19}$$

It is clear that $u := (\phi+\psi)/2$ and $v := (\phi-\psi)/2$ recover the equations (13). Moreover, the total energy of the system (14)-(16) is given by

$$F(t) := \frac{1}{2}\int_0^1 \phi_t^2 dx + \frac{1}{2}\int_0^1 \phi_x^2 dx + \frac{\alpha}{2}\int_0^1 \phi^2 dx, \tag{20}$$

and the total energy of the system (17)-(19) is given by

$$G(t) := \frac{1}{2}\int_0^1 \psi_t^2 dx + \frac{1}{2}\int_0^1 \psi_x^2 dx - \frac{\alpha}{2}\int_0^1 \psi^2 dx. \tag{21}$$

Proposition 41 *The solutions of the systems (14)-(16) and (17)-(19) are given by Fourier series developments*

$$\phi(x,t) = e^{-\frac{t}{2}}\sum_{k=1}^{\infty}\left[a_k^+ \sin\left(\sqrt{k^2\pi^2+\alpha t}\right) + b_k^+ \cos\left(\sqrt{k^2\pi^2+\alpha t}\right)\right]\sin(k\pi x),$$

$$\psi(x,t) = e^{-\frac{t}{2}}\sum_{k=1}^{\infty}\left[a_k^- \sin\left(\sqrt{k^2\pi^2-\alpha t}\right) + b_k^- \cos\left(\sqrt{k^2\pi^2-\alpha t}\right)\right]\sin(k\pi x),$$

where $a_k^\pm, b_k^\pm$ are the Fourier coefficients.

Proof. Assuming that

$$\phi(x,t) := \varphi(x)S^+(t) \ \text{ and } \ \psi(x,t) := \varphi(x)S^-(t) \quad \forall t\geq 0,\ x\in(0,1) \tag{22}$$

and substituting (22) into (14) (or 17), we obtain

$$\frac{S''^{\pm}(t)+S'^{\pm}(t)}{S^{\pm}(t)} = \left[\varphi'(x)\pm\alpha\varphi(x)\right]\frac{1}{\varphi(x)} = -\nu^{\pm}. \tag{23}$$

For non-trivial solutions we take $\nu^\pm > 0$. Then, we obtain the eigenvalue problem to the system (14)-(16) (or (17)-(19)) given by

$$-\varphi'(x) \pm \alpha\varphi(x) = \nu^\pm\varphi(x), \ \forall x\in(0,1) \tag{24}$$
$$\varphi(0) = \varphi(1) = 0. \tag{25}$$

Taking into account the homogeneous boundary conditions (25), we can assume that the eigenfunctions are given by $\varphi(x) = \sin(k\pi x)$. Hence, the eigenvalues are given by

$$\nu_k^\pm = k^2\pi^2 \pm \alpha, \ \forall k\in\mathbb{N}. \tag{26}$$

Returning to (23) we obtain the equation $S''(t)+\nu^\pm S(t) = 0$, for all $t\in[0,T]$. Solving it, we obtain

$$S_k^\pm(t) = a_k^\pm \sin\left(\sqrt{\nu_k^\pm t}\right)e^{-\frac{t}{2}} + b_k^\pm \cos\left(\sqrt{\nu_k^\pm t}\right)e^{-\frac{t}{2}}, \quad k\in\mathbb{N},\ \forall t>0, \tag{27}$$

where $a_k^\pm, b_k^\pm$ are the Fourier coefficients. Thus, the result is established.

Proposition 42 *The solutions of the systems (13) are given by Fourier series developments*

$$u(x,t) = \frac{1}{2}e^{-\frac{t}{2}}\sum_{k=1}^{\infty}\left[a_k^+ \sin\left(\sqrt{\nu_k^+ t}\right) + a_k^- \sin\left(\sqrt{\nu_k^+ t}\right) + b_k^+ \cos\left(\sqrt{\nu_k^- t}\right) + b_k^- \cos\left(\sqrt{\nu_k^- t}\right)\right]\sin(k\pi x),$$

$$v(x,t) = \frac{1}{2}e^{-\frac{t}{2}}\sum_{k=1}^{\infty}\left[a_k^+ \sin\left(\sqrt{\nu_k^+ t}\right) - a_k^- \sin\left(\sqrt{\nu_k^- t}\right) + b_k^+ \cos\left(\sqrt{\nu_k^+ t}\right) - b_k^- \cos\left(\sqrt{\nu_k^- t}\right)\right]\sin(k\pi x),$$

where $\nu_k^\pm = k^2\pi^2 \pm \alpha$ and $a_k^\pm, b_k^\pm$ are the Fourier coefficients.

Proof. The proof is immediate. We consider Proposition 41 and the fact that $u = (\phi+\psi)/2$ and $v = (\phi-\psi)/2$.

Now, investigate the numerical scheme in finite-difference applied to dissipative system (13). Given $J,N\in\mathbb{N}$, we set $\Delta x = \frac{1}{J+1}$, $\Delta t = \frac{1}{N+1}$ and introduce the nets

$$x_0 = 0 < x_1 = \Delta x < \ldots < x_J = J\Delta x < x_{J+1} = 1,$$
$$t_0 = 0 < t_1 = \Delta t < \ldots < t_N = N\Delta t < t_{N+1} = 1,$$

with $x_j = j\Delta x$ and $t_n = n\Delta t$, $j = 0,1,2,\ldots,J+1$, $n = 0,1,2,\ldots,N+1$. We consider the following finite difference discretization of (13) with $j = 1,2,\ldots,J$ and

$n = 1,2,...,N$

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + \alpha v_j^n = 0, \quad (28)$$

$$\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2} - \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2} + \frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} + \alpha u_j^n = 0, \quad (29)$$

$$u_0^n = u_{J+1}^n = v_0^n = v_{J+1}^n = 0, \quad (30)$$

$$(u_j^0, v_j^0) = (u_{0j}, v_{0j}), \quad (31)$$

$$\left(\frac{u_j^1 - u_j^0}{\Delta t}, \frac{v_j^1 - v_j^0}{\Delta t}\right) = (u_{1j}, v_{1j}). \quad (32)$$

Now, we define the energy of (28)-(32) by

$$E^n := \frac{\Delta x}{2}\sum_{j=0}^{J}\left|\frac{u_j^{n+1} - u_j^n}{\Delta t}\right|^2 + \frac{\Delta x}{2}\sum_{j=0}^{J}\left|\frac{v_j^{n+1} - v_j^n}{\Delta t}\right|^2 + \frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x}\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) + \frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta x}\frac{v_{j+1}^n - v_j^n}{\Delta x}\right) + \frac{\alpha\Delta x}{2}\sum_{j=0}^{J}\left(u_j^n v_j^{n+1} + u_j^{n+1} v_j^n\right).$$

Proposition 43 *Let* (u_j^n, v_j^n) *be a solution of the finite difference scheme (28)-(32). Then, for all* Δx, $\Delta t \in (0,1)$ *the discrete rate of change of energy of this numerical scheme at the* t_n *instant of time is given by*

$$\frac{E^n - E^{n-1}}{\Delta t} = -\Delta x\sum_{j=0}^{J}\left|\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}\right|^2 - \Delta x\sum_{j=0}^{J}\left|\frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t}\right|^2, \quad \forall\, n = 1,2,...,N.$$

Proof. Multiplying (28) by $\Delta x(u_j^{n+1} - u_j^{n-1})/2\Delta t$ and adding for $1 \le j \le J$, we have

$$\Delta x\sum_{j=1}^{J}\left(\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}\right) - \Delta x\sum_{j=1}^{J}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}\right) + \Delta x\sum_{j=1}^{J}\left|\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}\right|^2 + \alpha\Delta x\sum_{j=1}^{J} v_j^n\left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}\right) = 0.$$

For instance, considering the homogeneous Dirichlet boundary conditions, one has

$$\begin{aligned}\mathscr{I}_{1,n} &:= \Delta x\sum_{j=1}^{J}\left(\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}\right)\\ &= \frac{\Delta x}{2\Delta t\Delta t^2}\sum_{j=1}^{J}(u_j^{n+1} + u_j^{n-1})(u_j^{n+1} - u_j^{n-1}) - \frac{2\Delta x}{2\Delta t\Delta t^2}\sum_{j=1}^{J}u_j^n(u_j^{n+1} - u_j^{n-1})\\ &= \frac{\Delta x}{2\Delta t\Delta t^2}\sum_{j=1}^{J}(|u_j^{n+1}|^2 - |u_j^{n-1}|^2 - 2u_j^n u_j^{n+1} + 2u_j^n u_j^{n-1})\\ &= \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left|\frac{u_j^{n+1} - u_j^n}{\Delta t}\right|^2 - \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left|\frac{u_j^n - u_j^{n-1}}{\Delta t}\right|^2.\end{aligned}$$

$$\begin{aligned}\mathscr{I}_{2,n} &:= \Delta x\sum_{j=1}^{J}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}\right)\\ &= \frac{\Delta x}{2\Delta x^2\Delta t}\sum_{j=1}^{J}(u_{j+1}^n - u_j^n)(u_j^{n+1} - u_j^{n-1}) + \frac{\Delta x}{2\Delta x^2\Delta t}\sum_{j=1}^{J}(u_{j-1}^n - u_j^n)(u_j^{n+1} - u_j^{n-1})\\ &= \frac{\Delta x}{2\Delta x^2\Delta t}\sum_{j=0}^{J}(u_{j+1}^n u_j^{n+1} - u_{j+1}^n u_j^{n-1} - u_j^n u_j^{n+1} + u_j^n u_j^{n-1})\\ &+ \frac{\Delta x}{2\Delta x^2\Delta t}\sum_{j=0}^{J}(u_j^n u_{j+1}^{n+1} - u_j^n u_{j+1}^{n-1} - u_{j+1}^n u_{j+1}^{n+1} + u_{j+1}^n u_{j+1}^{n-1})\\ &= -\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x}\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) + \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}\frac{u_{j+1}^{n-1} - u_j^{n-1}}{\Delta x}\right).\end{aligned}$$

and

$$\begin{aligned}\mathscr{I}_{3,n} &:= \alpha\Delta x\sum_{j=1}^{J}v_j^n\left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}\right)\\ &= \frac{\alpha}{2}\frac{\Delta x}{\Delta t}\sum_{j=0}^{J}\left(v_j^n u_j^{n+1} - v_j^n u_j^{n-1}\right).\end{aligned}$$

Combining $\mathscr{I}_{1,n}$, $\mathscr{I}_{2,n}$, $\mathscr{I}_{3,n}$, we get

$$\frac{\Delta x}{2}\sum_{j=0}^{J}\left|\frac{u_j^{n+1}-u_j^n}{\Delta t}\right|^2-\frac{\Delta x}{2}\sum_{j=0}^{J}\left|\frac{u_j^{n}-u_j^{n-1}}{\Delta t}\right|^2$$
$$+\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{u_{j+1}^{n+1}-u_j^{n+1}}{\Delta x}\frac{u_{j+1}^{n}-u_j^{n}}{\Delta x}\right)-$$
$$\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{u_{j+1}^{n}-u_j^{n}}{\Delta x}\frac{u_{j+1}^{n-1}-u_j^{n-1}}{\Delta x}\right)$$
$$+\Delta x\Delta t\sum_{j=0}^{J}\left|\frac{u_j^{n+1}-u_j^{n-1}}{2}\right|^2+$$
$$\frac{\alpha\Delta x}{2}\sum_{j=0}^{J}\left(v_j^n u_j^{n+1}-v_j^n u_j^{n-1}\right)=0.$$

Analogously, multiplying the equation (29) by $\Delta x(v_j^{n+1}-v_j^{n-1})/2\Delta t$, we obtain

$$\frac{\Delta x}{2}\sum_{j=0}^{J}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2-\frac{\Delta x}{2}\sum_{j=0}^{J}\left|\frac{v_j^{n}-v_j^{n-1}}{\Delta t}\right|^2$$
$$+\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x}\frac{v_{j+1}^{n}-v_j^{n}}{\Delta x}\right)-$$
$$\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n}-v_j^{n}}{\Delta x}\frac{v_{j+1}^{n-1}-v_j^{n-1}}{\Delta x}\right)$$
$$+\Delta x\Delta t\sum_{j=0}^{J}\left|\frac{v_j^{n+1}-v_j^{n-1}}{2}\right|^2+$$
$$\frac{\alpha\Delta x}{2}\sum_{j=0}^{J}\left(u_j^n v_j^{n+1}-u_j^n v_j^{n-1}\right)=0.$$

Adding the two last equations, we can write

$$E^n-E^{n-1}=-\Delta x\Delta t\sum_{j=0}^{J}\left|\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}\right|^2-$$
$$\Delta x\Delta t\sum_{j=0}^{J}\left|\frac{v_j^{n+1}-v_j^{n-1}}{2\Delta t}\right|^2,\ \forall\, n=1,2,...,N. \tag{33}$$

Therefore, we obtain the desired result.

4.1 Stability Analysis

In order to obtain a numerical solution corresponding to Proposition 42, we will perform a stability analysis of the finite difference scheme (28)-(32). Anticipate the stability condition is given in the following proposition.

Proposition 44 (Stability conditions) *The system (28)-(32) is stable if and only if*

$$\Delta t\le\frac{\Delta x}{\sqrt{1+\frac{\alpha\Delta x^2}{4}}}. \tag{34}$$

Proof. The proof is an immediate consequence of the inequalities (53), (54).

In order to motivate our approach, let us recall the particular solutions of the Proposition 42 for the continuous problem (13). If we use the complex form (27) for the functions $S_k^{\pm}(t)$, these solutions take the form

$$u_k(x,t)=\frac{1}{2}e^{-\frac{t}{2}}\left(e^{\pm\sqrt{k^2\pi^2+\alpha t}}+e^{\pm\sqrt{k^2\pi^2-\alpha t}}\right)\sin(k\pi x), \tag{35}$$
$$v_k(x,t)=\frac{1}{2}e^{-\frac{t}{2}}\left(e^{\pm\sqrt{k^2\pi^2+\alpha t}}-e^{\pm\sqrt{k^2\pi^2-\alpha t}}\right)\sin(k\pi x). \tag{36}$$

Let $\varphi^k=(\varphi_{k,1},\varphi_{k,2},\dots,\varphi_{k,J},)$ be the vector with components $\varphi_{k,j}=\sin(k\pi x_j)$, $j=1,2,\dots,J$. For the finite difference scheme (28) – (32), we will consider possible solutions of the form

$$u_j^n=\frac{1}{2}\varphi_{k,j}a^n \quad\text{and}\quad v_j^n=\frac{1}{2}\varphi_{k,j}b^n, \tag{37}$$

where a,b is a complex number. The particular solutions given by (35) and (36) will always have the property that

$$|u_k(x,t)|\le 1 \quad\text{and}\quad |v_k(x,t)|\le 1.$$

It is, therefore, reasonable to demand that the particular solutions (37) of the difference scheme have a corresponding property. Thus, we shall require that

$$|a|\le 1 \quad\text{and}\quad |b|\le 1.$$

To be more precise in our assertion, we note that assuming the decomposition given by $\phi_j^n:=u_j^n+v_j^n$, we obtain the discrete wave equations for $j=1,2,...,J$ and $n=1,2,...,N$

$$\frac{\phi_j^{n+1}-2\phi_j^n+\phi_j^{n-1}}{\Delta t^2}-\frac{\phi_{j+1}^n-2\phi_j^n+\phi_{j-1}^n}{\Delta x^2}$$
$$+\frac{\phi_j^{n+1}-\phi_j^{n-1}}{2\Delta t}+\alpha\phi_j^n=0, \tag{38}$$
$$\phi_0^n=\phi_{J+1}^n=0, \tag{39}$$
$$\phi_j^0=\phi_{0j},\quad \frac{\phi_j^1-\phi_j^0}{\Delta t}=\phi_{1j} \tag{40}$$

and taking $\psi_j^n:=u_j^n-v_j^n$, we obtain

$$\frac{\psi_j^{n+1}-2\psi_j^n+\psi_j^{n-1}}{\Delta t^2}-\frac{\psi_{j+1}^n-2\psi_j^n+\psi_{j-1}^n}{\Delta x^2}$$
$$+\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}-\alpha\psi_j^n=0 \tag{41}$$
$$\psi_0^n=\psi_{J+1}^n=0, \tag{42}$$
$$\psi_j^0=\psi_{0j},\quad \frac{\psi_j^1-\psi_j^0}{\Delta t}=\psi_{1j}. \tag{43}$$

In order to create a stability criterion, we will perform a stability analysis of the finite difference scheme. For the finite difference scheme (38)-(40), we will consider possible solutions of the form

$$\phi_j^n=\varphi_{k,j}\gamma^n,\quad k,j=1,2,...,J,\quad n=1,2,...,N, \tag{44}$$

where γ is a complex number. We shall, therefore, require that

$$|\gamma|\le 1. \tag{45}$$

Assuming (44) and substituting into (38)-(40), we obtain

$$\varphi_j \frac{\gamma^{n+1}-2\gamma^n+\gamma^{n-1}}{\Delta t^2} - \gamma^n \frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{\Delta x^2} + \varphi_j \frac{\gamma^{n+1}-\gamma^{n-1}}{2\Delta t} + \alpha\varphi_j\gamma^n = 0.$$

So,

$$\frac{1}{\gamma^n}\left[\frac{\gamma^{n+1}-2\gamma^n+\gamma^{n-1}}{\Delta t^2} + \frac{\gamma^{n+1}-\gamma^{n-1}}{2\Delta t}\right] = \quad (46)$$

$$\frac{1}{\varphi_j}\left[\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{\Delta x^2} - \alpha\varphi_j\right] = -\nu. \quad (47)$$

For non-trivial solutions, we take $\nu > 0$. Then, we obtain the eigenvalue problem to the system (38) − (40) given by

$$-\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{\Delta x^2} + \alpha\varphi_j = \nu\varphi_j \quad (48)$$

$$\varphi_0 = \varphi_{J+1} = 0. \quad (49)$$

Based on, from (48) it follows that the corresponding eigenvalues

$$\nu_k = \frac{4}{\Delta x^2}\sin^2\left(\frac{k\pi\Delta x}{2}\right) + \alpha.$$

Therefore the left side of (46), we obtain

$$\gamma^2 - 2\left(\frac{2-s}{2+\Delta t}\right)\gamma + \left(\frac{2-\Delta t}{2+\Delta t}\right) = 0, \quad (50)$$

where $s = \nu_k\Delta t^2 = 4\frac{\Delta t^2}{\Delta x^2}\sin^2\left(\frac{k\pi\Delta x}{2}\right) + \alpha\Delta t^2$. Hence, if we let r be the mesh ratio $r = \Delta t/\Delta x$, $s \in (0, 4r^2 + \alpha\Delta t^2)$. To be more precise, let us note that the roots a of (50) will depend on s, i.e., $\gamma = \gamma(s)$. Since $s \in (0, 4r^2 + \alpha\Delta t^2)$, we will define the scheme to be stable as long as the roots $\gamma(s)$ satisfy (45) for all $s \in (0, 4r^2 + \alpha\Delta t^2)$.

Theorem 45 *Let $s \geq 0$ be given. The roots of (50) satisfy (45) if and only if $s \leq 4$.*

Proof. The roots of (50) are given by

$$\gamma_k^{\pm} = \frac{2 - s \pm \sqrt{s(s-4)+\Delta t^2}}{2+\Delta t}. \quad (51)$$

If $s = 0$, there are two roots given by $\gamma^{\pm} = \frac{2\pm\Delta t}{2+\Delta t}$, and if $s = 4$, $\gamma^{\pm} = \left\{-1, -\frac{2-\Delta t}{2+\Delta t}\right\}$. If $s \in (2-\sqrt{4-\Delta t^2}, 2+\sqrt{4-\Delta t^2})$ there are two complex roots γ^+ and γ^-. Written in polar coordinates, these are of the form

$$\gamma^+ = \rho e^{i\theta} \quad \text{and} \quad \gamma^- = \rho e^{-i\theta} \quad (52)$$

for $\rho > 0$ and $\theta \in (0, \pi)$. Furthermore, from (50) it follows that the product of the roots is $\frac{2-\Delta t}{2+\Delta t}$, i.e.,

$$\gamma^+\gamma^- = \rho^2 = \frac{2-\Delta t}{2+\Delta t} \leq 1.$$

Hence, the roots are of the form

$$\gamma^{\pm} = \left(1 - \frac{2\Delta t}{2+\Delta t}\right)^{\frac{1}{2}} e^{\pm i\theta},$$

and therefore the bound (45) holds. On the other hand, if $s > 4$, there exist two distinct real roots γ^+ and γ^-, with $\gamma^+\gamma^- = 1$. Hence, one of them must have absolute value greater than 1 in this case.

Corollary 46 (Stability conditions) *The roots of (50) will satisfy (45) for all $s \in (0, 4r^2 + \alpha\Delta t^2)$ with $r = \Delta t/\Delta x$ if and only if*

$$\Delta t \leq \frac{\Delta x}{\sqrt{1 + \frac{\alpha\Delta x^2}{4}}}. \quad (53)$$

If the mesh parameters satisfy this bound, the numerical solution behaves qualitatively as the exact solution. However, if the bound is violated, we observe oscillations in the numerical solution, but they are not present in the exact solution. A similar analysis shows that the stability condition of the problem (17) − (19) is given by

$$\Delta t \leq \frac{\Delta x}{\sqrt{1 - \frac{\alpha\Delta x^2}{4}}} \quad (54)$$

which is slightly less restrictive than the corresponding condition to (53).

4.2 Numerical Results

Two numerical tests, for one-dimensional case, will be developed to confirm the exponential stability of the system (1).

4.2.1 Vectorial form of discrete equations

The vector form of equations (28) - (32) is given by

$$U^{n+1} = A_{ab}U^n + cU^{n-1} + d\alpha V^n + bU_\Gamma^n \quad (55)$$

$$n = 0, 1, 2, \ldots$$

$$V^{n+1} = A_{ab}V^n + cV^{n-1} + d\alpha U^n + bV_\Gamma^n \quad (56)$$

where $A_{ab} = aI + bA$, I and $A = \text{Trid}(1,0,1)$ are the identity and tridiagonal matrices of order J, respectively. $U^n = (u_1, \ldots, u_J)^T_{J\times1}$, $U^0 = (u_1^0, \ldots, u_J^0)^T_{J\times1}$, $U^{-1} = (u_1^{-1}, \ldots, u_J^{-1})^T_{J\times1}$ is the ghost vector (below the initial condition line), $U_\Gamma^n = (u_0, 0, \ldots, 0, u_{J+1})^T_{J\times1}$, $a = \frac{4(\Delta x^2 - \Delta t^2)}{\Delta x^2(2+\Delta t)}$, $b = \frac{2\Delta t^2}{\Delta x^2(2+\Delta t)}$, $c = -\frac{2-\Delta t}{2+\Delta t}$ and $d = -\frac{2\Delta t^2}{2+\Delta t}$. We describe analogously for V.

4.2.2 Energy of system

In order to obtain the time behavior of the energy of system, we resolve two *numerical tests* with the the parameter $\alpha = 0.125$. It is important to note that to achieve the stability of the system, we need a maximum time ($T_{\max}$) and a stop criterion subject to a certain tolerance (Tol). To obtain the $T_{\max}$, we use the stability condition (53). The following describes the numerical tests.

Test 1
The initial conditions

$$u_0(x) = 1, \quad u_1 = -x, \quad v_0(x) = e^{-x}, \quad v_1(x) = 0.$$

Parameters

$$J = 99, \quad N = 7400, \quad [0, T_{\max}] = [0, 16], \quad Tol = 10^{-4}$$

Results

The results are shown in Figure 1. The dynamic energy is stopped when $n = 7.260$ iterations, i.e., were used 15.695 time units.

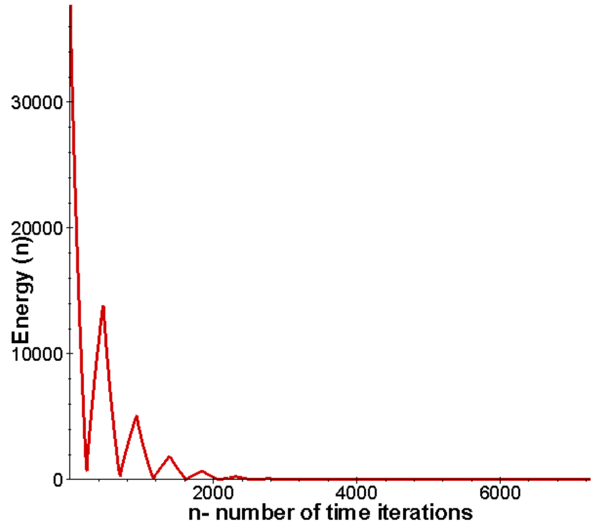

Fig. 1: Mechanical Energy Dissipation System (Test 1).

Test 2
The initial conditions

$$u_0(x) = \sin(\pi x), \quad u_1 = -\sin(\pi x), \quad v_0(x) = e^{-x}, \quad v_1(x) = 0.$$

Parameters

$$J = 99, \quad N = 6900, \quad [0, T_{\max}] = [0, 15], \quad Tol = 10^{-4}$$

Results

The results are manifested in Figure 2. The dynamic energy is stopped when $n = 6.773$ iterations, i.e., were used 14.720 time units.

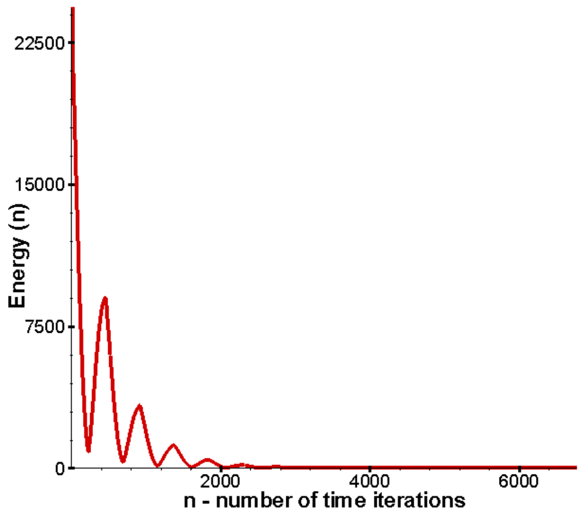

Fig. 2: Mechanical Energy Dissipation System (Test 2).

Note that, in both figures, the dynamic behavior of the system energy is stabilized. Stability occurs approximately after 3000 iterations. Finally, in both tests, we conclude that the system (1), for one-dimensional case, is dissipative with exponential stability.

5 Conclusion

The Nakao's method proved to be an efficient method for the demonstration of the exponential decay of the solution of coupled system of waves with indirect control. Numerical tests have shown the decay of the mechanical energy of the system.

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