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Action of Ornstein-Uhlenbeck Semigroup on (w_1,w_2)-Tempered Ultradistributions

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Abstract: Using a previously obtained structure theorem for (w_1,w_2)-tempered ultradistributions by the classical Riesz representation theorem, we investigate the action of the Ornstein-Uhlenbeck semigroup on (w_1,w_2)-tempered ultradistributions. As a result, we observe that these tempered ultradistributions can be represented as boundary values to the heat equation $u_t - Au = 0$, for $t > 0$.

Keywords: Ornstein-Uhlenbeck Semigroup, Short-Time Fourier Transform, Structure Theorem, Tempered Ultradistributions.

1 Introduction

Distributions are a special class of generalized functions devised by L. Schwartz in order to provide a satisfactory framework for the Fourier transform (see [1]). Recently, the theory of distributions has been used in microlocal analysis, signal processing, image processing and wavelets.

The Schwartz space $\mathfrak{S}$, as defined by L. Schwartz (see [2]), consists of all $C^\infty(\mathbb{R}^n)$ functions φ such that the functions and their derivatives decay rapidly at infinity; $\left\|x^\alpha \partial^\beta \varphi\right\|_\infty < \infty$ for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$. The dual space of $\mathfrak{S}$ is the space $\mathfrak{S}'$ of tempered distributions. In 1963, the theory of ultradistributions was introduced by A. Beurling as a generalization of Schwartz distributions. This generalization aimed to find an appropriate context for his work on pseudo-analytic extensions (see [3]).

In 1967, G. Björck introduced the Beurling-Björck space $\mathfrak{S}_w$ of test functions for tempered ultradistributions which expanded the space $\mathfrak{S}'$ of tempered distributions, and extended the work of Hörmander on existence, nonexistence, and regularity of solutions of differential equations with constant coefficient in addition to studying the convolution (see [4]). The Beurling-Björck space $\mathfrak{S}_w$, as defined by G. Björck, consists of all $C^\infty(\mathbb{R}^n)$ functions φ such that $\left\|e^{kw(x)}\partial^\beta \varphi\right\|_\infty < \infty$ and $\left\|e^{kw(x)}\partial^\beta \widehat{\varphi}\right\|_\infty < \infty$ for all $\beta \in \mathbb{N}^n$, where w is a subadditive weight function satisfying the classical Beurling conditions. The topological dual $\mathfrak{S}'_w$ of $\mathfrak{S}_w$ is a space of generalized functions, called $w-$tempered ultradistributions. When $w(x) = \log(1+|x|)$, the Beurling- Björck space $\mathfrak{S}_w$ becomes the Schwartz space $\mathfrak{S}$ (see [5] and [6]).

In [7], the authors introduced the space $\mathfrak{S}_{w_1,w_2}$ of all $C^\infty(\mathbb{R}^n)$ functions φ such that $\left\|e^{kw_1(x)}\partial^\beta \varphi\right\|_\infty < \infty$ and $\left\|e^{kw_2(x)}\partial^\beta \widehat{\varphi}\right\|_\infty < \infty$ for all $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$, where w_1 and w_2 are two weights satisfying the classical Beurling conditions. The topological dual $\mathfrak{S}'_{w_1,w_2}$ of $\mathfrak{S}_{w_1,w_2}$ is a space of generalized functions, called (w_1,w_2)-tempered ultradistributions. Moreover, they proved a structure theorem for functionals $T \in \mathfrak{S}'_{w_1,w_2}$ using the classical Riesz representation theorem.

In this paper, we use the structure theorem for (w_1,w_2)-tempered ultradistributions by the classical Riesz representation theorem obtained in [7] to investigate the action of the Ornstein-Uhlenbeck semigroup on (w_1,w_2)-tempered ultradistributions. As a result, we observe that these tempered ultradistributions can be represented as boundary valus to the heat equation $u_t - Au = 0$, for $t > 0$. We prove that given $\varphi \in \mathfrak{S}_{w_1,w_2}$, there is a solution $\varphi_t(x)$ of the heat equation, for which $\varphi_t(x)$ converges to φ in $\mathfrak{S}_{w_1,w_2}$, in the strong dual topology. Our work is inspired by a substantial body of work on the generalized functions of Gelfand-Shilov spaces pioneered by Hamed Obiedat and Lloyd Edgar [8].

The symbols C^∞, C_0^∞, L^p, etc., denote the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. $|\cdot|$

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indicates the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p-norm in the space L^p, where $1 \le p \le \infty$. In general, we work on the Euclidean space $\mathbb{R}^n$ till we find a more appropriate one. Partial derivatives will be denoted by ∂^α, where α is a multi-index $(\alpha_1, ..., \alpha_n)$ in $\mathbb{N}_0^n$. We will use the standard abbreviations $|\alpha| = \alpha_1 + ... + \alpha_n$, $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$. The Fourier transform of a function f is denoted by $\mathscr{F}(f)$ or $\widehat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x)\, dx$. With $\mathscr{C}_0$ we denote the Banach space of continuous functions vanishing at infinity with supremum norm. A Fréchet spaces are a locally convex topological vector spaces that are completely metrizable.

2 Preliminary definitions and results

In [9], J. Chung et al. proved symmetric characterizations for Gelfand-Shilov spaces via the Fourier transform in terms of the growth of the function and its Fourier transform which imposes no conditions on the derivative.

Theorem 1. *Given $w_1, w_2 \in \mathscr{M}_c$, the space $\mathfrak{S}_{w_1,w_2}$ can be described as a set as well as topologically by*

$$\mathfrak{S}_{w_1,w_2} = \left\{ \begin{array}{c} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all} \\ k = 0, 1, 2, ..., \; p_k(\varphi) < \infty, q_k(\varphi) < \infty \end{array} \right\},$$

where $p_k(\varphi) = \left\| e^{k w_1(x)} \varphi \right\|_\infty$, $q_k(\varphi) = \left\| e^{k w_2(\xi)} \widehat{\varphi} \right\|_\infty$.

The space $\mathfrak{S}_{w_1,w_2}$, equipped with the family of semi-norms

$$\mathscr{N} = \{p_k, q_k : k \in \mathbb{N}_0\},$$

is a Fréchet space.

Now, we present the restrictive definition of the space $\mathscr{M}_c$ of admissible functions w (see [10], page 14).

Definition 1. *([10]) With $\mathscr{M}_c$ we indicate the space of functions $w : \mathbb{R}^n \to \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where*

1. *$\Omega : [0, \infty) \to [0, \infty)$ is increasing, continuous and concave,*
2. *$\Omega(0) = 0$,*
3. *$\int_{\mathbb{R}} \frac{\Omega(t)}{(1+t^2)} dt < \infty$,*
4. *$\Omega(t) \ge a + b \ln(1+t)$ for some $a \in \mathbb{R}$ and some $b > 0$.*

Standard classes of functions w in $\mathscr{M}_c$ are given by

$$w(x) = |x|^d \text{ for } 0 < d < 1, \text{ and } w(x) = p \ln(1 + |x|) \text{ for } p > 0.$$

Remark. If $N > \frac{n}{b}$ is an integer, then

$$C_N = \int_{\mathbb{R}^n} e^{-N w(x)} dx < \infty, \text{ for all } w \in \mathscr{M}_c,$$

where b is the constant in Condition 4 of Definition 1

Remark. If $\tau \in \mathbb{R}^n$, there exist $N \in \mathbb{N}$ and a constant $C > 0$ such that $|\tau| \le C e^{N w(\tau)}$. In fact, since

$$|\tau| \le 1 + |\tau| = e^{\ln(1+|\tau|)}$$

and applying Condition 4 of Definition 1, there exist $a \in \mathbb{R}$ and $b > 0$ such that

$$\ln(1 + |\tau|) \le \frac{w(\tau) - a}{b}.$$

Hence,

$$\begin{aligned} |\tau| &\le 1 + |\tau| = e^{\ln(1+|\tau|)} \\ &\le e^{\frac{w(\tau)-a}{b}} = e^{-\frac{a}{b}} e^{\frac{w(\tau)}{b}} \\ &\le C e^{N w(\tau)} \end{aligned}$$

where $C = e^{-\frac{a}{b}} > 0$ and $N > \frac{n}{b}$ is an integer.

The following remark benefits the proof of the main theorem.

Remark. Using the concavity property of $w(x)$ and that $w(0) = 0$ in Definition 1 we have $w(e^{-t}x) \ge e^{-t} w(x)$ for $t \ge 0$. Indeed,

$$\begin{aligned} w(e^{-t}x) &= w(e^t e^{-t} e^{-t} x) \\ &= w(e^{-t}(e^t e^{-t} x) + (1 - e^{-t})(0)) \\ &\ge e^{-t} w(e^t e^{-t} x) + (1 - e^{-t}) w(0) \\ &= e^{-t} w(x) + (1 - e^{-t})(0) \\ &= e^{-t} w(x) \end{aligned}$$

3 Characterization of the dual space $\mathfrak{S}'_{w_1,w_2}$

Theorem 2. *([11]) Given a functional L in the topological dual of the space $\mathscr{C}_0$, there exists a unique regular complex Borel measure μ so that*

$$L(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu.$$

Moreover, the norm of the functional L is equal to the total variation $|\mu|$ of the measure μ. Conversely, any such measure μ defines a continuous linear functional on $\mathscr{C}_0$.

In [7], the authors employ Theorem 1 to prove the following structure theorem for functionals in $\mathfrak{S}'_{w_1,w_2}$.

Theorem 3. *([7]) Given $L \in \mathfrak{S}_{w_1,w_2} \to \mathbb{C}$, then the following statements are equivalent :*

(i) $L \in \mathfrak{S}'_{w_1,w_2}$
(ii) There exist two regular complex Borel measures μ_1 and μ_2 of finite total variation and $k \in \mathbb{N}_0$ such that

$$L = e^{k w_1(x)} d\mu_1 + e^{k w_2(\xi)} d\mu_2,$$

in the sense of $\mathfrak{S}'_{w_1,w_2}$.

The following corollary indicates an application of the structure theorem of $\mathfrak{S}'_{w_1,w_2}$ stated in Theorem 3.

Corollary 1.([7]) *If $T \in \mathfrak{S}'_{w_1,w_2}$ and $\varphi \in \mathfrak{S}_{w_1,w_2}$, then the functional $T * \varphi$ defined by*

$$\langle T * \varphi, \phi \rangle = \langle T_y, (\varphi_z, \phi(x+y)\rangle$$

coincides with the functional given by integration against the function $\psi(x) = \langle T_y, \varphi(x-y)\rangle$.

4 Ornstein-Uhlenbeck Semigroup action on $\mathfrak{S}'_{w_1,w_2}$

The second-order differential operator defined by

$$A = -\frac{1}{2}\Delta - x \cdot \nabla,$$

where Δ denotes the Laplacian operator in $\mathbb{R}^n$ and ∇ is the gradient, is called Ornstein-Uhlenbeck operator in $\mathbb{R}^n$. The semi-group generated by Ornstein-Uhlenbeck operator A is Ornstein-Uhlenbeck semi-group acting on the Hilbert space $L^2(\gamma)$ where γ is the normalized Gaussian measure. The Ornstein-Uhlenbeck semi-group $(P_t)_{t\geq 0} = (e^{At})_{t\geq 0}$ is given by

$$P_t\varphi(x) = \int_{\mathbb{R}^n} M_t(x,y)\varphi(y)d\gamma(y) = \langle M_t(x,y), \varphi(y)\rangle, \quad (1)$$

where $M_t(x,y)$ and $t > 0$ is the Mehler kernel and P_0 is the identity. The closed expression of the Mehler kernel $M_t(x,y)$ given by

$$M_t(x,y) = \frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} \quad (2)$$

allows us to establish a connection between Mehler's kernel and the heat kernel

$$k_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad (3)$$

see [12]. After applying a dilation to the variable x, the Ornstein-Uhlenbeck semigroup is a reparametrization of the heat semigroup. Thus, it is not a convolution semigroup. Indeed, if $\delta_a f(x) = f(ax)$ is the dilation operator by a, and $\{\mathscr{F}_t\}_{t\geq 0}$ is the operator semigroup, $P_t\varphi(x)$ has the following representation

$$\begin{aligned} P_t\varphi(x) &= (k_{(1-e^{-2t})/4} * f)(e^{-t}x) \\ &= \delta_{e^{-t}}\left[k_{(1-e^{-2t})/4} * f\right](x) \\ &= \delta_{e^{-t}}\mathscr{F}_{(1-e^{-2t})/4} f(x), \end{aligned}$$

where

$$\mathscr{F}_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y)\,dy, \; t > 0, \quad (4)$$

is the heat semigroup.

Observe that $M_t(x,\cdot)$ and $M_t(\cdot,y)$ are both in $\mathfrak{S}_{w_1,w_2}$ for all $\alpha,\beta > 1$ because both have exponential decay which implies that the operator P_t is well defined. Then for $T \in \mathfrak{S}'_{w_1,w_2}$ and $P_t = e^{At}$ where $A = -\frac{1}{2}\Delta - x\cdot\nabla$, we can write the action of P_t on $\mathfrak{S}'_{w_1,w_2}$ as

$$\langle T * P_t, \varphi\rangle = \langle T_y, < M_t(x,y), \varphi(x)\rangle\rangle, \; \varphi \in \mathfrak{S}_{w_1,w_2}. \quad (5)$$

To prove that $T * P_t \to T$ as $t \to 0^+$ in strong dual topology, it is enough to prove the following result.

Theorem 4. *Let B be a bounded subset of $\mathfrak{S}_{w_1,w_2}$ and $\varphi \in \mathfrak{S}_{w_1,w_2}$. Then $\varphi_t(x) = \langle M_t(x,\cdot), \varphi(x)\rangle \to \varphi$ in $\mathfrak{S}_{w_1,w_2}$ as $t \to 0^+$ uniformly on B.*

Proof. Recall that $\int_{\mathbb{R}^n} M_t(x,y)dx = e^{nt}$. We can write

$$\begin{aligned} I &= e^{kw_1(y)}\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - e^{kw_1(y)}\varphi(y) \\ &= e^{kw_1(y)}(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - \varphi(y)) \\ &= e^{kw_1(y)}(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - e^{-nt}\varphi(y)\int_{\mathbb{R}^n} M_t(x,y)dx) \\ &= e^{kw_1(y)}(\int_{\mathbb{R}^n} M_t(x,y)(\varphi(x) - e^{-nt}\varphi(y))dx) \\ &= e^{kw_1(y)}(\int_{\mathbb{R}^n} M_t(x,y)(\varphi(x) - \varphi(y) + \varphi(y) - e^{-nt}\varphi(y))dx. \end{aligned}$$

Taking the absolute value for both sides and applying the triangle inequality, we get

$$\begin{aligned} |I| &= \left| e^{kw_1(y)}\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - e^{kw_1(y)}\varphi(y)\right| \\ &\leq \int_{\mathbb{R}^n} e^{kw_1(y)} M_t(x,y)\left|\varphi(x) - \varphi(y) + \varphi(y) - e^{-nt}\varphi(y)\right| dx \\ &\leq \int_{\mathbb{R}^n} e^{kw_1(y)} M_t(x,y)\left|\varphi(x) - \varphi(y)\right| dx \\ &\quad + (1 - e^{-nt})\int_{\mathbb{R}^n} e^{kw_1(y)} M_t(x,y)\left|\varphi(y)\right| dx \\ &= I_1 + I_2. \end{aligned}$$

We estimate I_2 as follows:

$$\begin{aligned} I_2 &= (1 - e^{-nt})\int_{\mathbb{R}^n} e^{kw_1(y)} M_t(x,y)\left|\varphi(y)\right| dx \\ &\leq e^{nt}(1 - e^{-nt})\left|\left|e^{kw_1}\varphi\right|\right|_\infty. \end{aligned}$$

Using explicit formula for $M_t(x,y)$ and making the change of variable $u = \frac{y - e^{-t}x}{\sqrt{1-e^{-2t}}}$, we estimate I_1 as follows:

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} e^{kw_1(y)} M_t(x,y)\,|\varphi(x)-\varphi(y)|\,dx \\
&= \frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}} \int_{\mathbb{R}^n} e^{kw_1(y)} e^{-|u|^2} \Big|\varphi(\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}}) \\
&\quad -\varphi(y)\Big| \frac{(1-e^{-2t})^{n/2}}{e^{-nt}} du \\
&= \frac{e^{nt}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{kw_1(y)} e^{-|u|^2} \Big|\varphi(\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}}) - \varphi(\frac{y}{e^{-t}}) \\
&\quad +\varphi(\frac{y}{e^{-t}}) - \varphi(y)\Big| du \\
&\leq \frac{e^{nt}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{kw_1(y)} e^{-|u|^2} \Big|\varphi(\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}}) - \varphi(\frac{y}{e^{-t}})\Big| du \\
&\quad + \frac{e^{nt}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{kw_1(y)} e^{-|u|^2} \Big|\varphi(\frac{y}{e^{-t}}) - \varphi(y)\Big| du.
\end{aligned}
$$

Using Mean Value Theorem, there is a point u' on the line segment $\mathscr{L}_1$ from $\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}}$ to $\frac{y}{e^{-t}}$ and a point u'' on the line segment $\mathscr{L}_2$ from $\frac{y}{e^{-t}}$ to y such that

$$
\Big|\varphi(\frac{y-u\sqrt{1-e^{-2t}}}{e^{-t}}) - \varphi(\frac{y}{e^{-t}})\Big| = \frac{|u|\sqrt{1-e^{-2t}}}{e^{-t}} |\nabla\varphi(u')|
$$

and

$$
\Big|\varphi(\frac{y}{e^{-t}}) - \varphi(y)\Big| = \frac{|y|(1-e^{-t})}{e^{-t}} |\nabla\varphi(u'')|
$$

respectively. Thus, the estimate for I_1 above now becomes

$$
\begin{aligned}
I_1 &\leq \frac{e^{nt}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{kw_1(y)} e^{-|u|^2} \frac{|u|\sqrt{1-e^{-2t}}}{e^{-t}} |\nabla\varphi(u')|\,du \\
&\quad + \frac{e^{nt}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{kw_1(y)} e^{-|u|^2} \frac{|y|(1-e^{-t})}{e^{-t}} |\nabla\varphi(u'')|\,du.
\end{aligned}
$$

Using $|y| \leq |u''|$ and applying Remark 2 for $|u''|$, then

$$
|y| \leq Ce^{Nw_1(y)} \leq Ce^{Nw_1(u'')} \tag{6}
$$

for some integer N and constant $C>0$. Therefore,

$$
\begin{aligned}
I_1 &\leq \frac{e^{nt}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{kw_1(u')} e^{-|u|^2} \frac{|u|\sqrt{1-e^{-2t}}}{e^{-t}} |\nabla\varphi(u')|\,du \\
&\quad + \frac{e^{nt}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{kw_1(u'')} e^{-|u|^2} \frac{Ce^{Nw_1(u'')}(1-e^{-t})}{e^{-t}} |\nabla\varphi(u'')|\,du \\
&\leq \pi^{-n/2} e^{(n+1)t}\sqrt{1-e^{-2t}} \left\|e^{kw_1}\nabla\varphi\right\|_\infty \left\|ue^{-|u|^2}\right\|_1 \\
&\quad + C\pi^{-n/2}e^t(1-e^{-t}) \left\|e^{(N+k)w_1}\nabla\varphi\right\|_\infty \left\|e^{-|u|^2}\right\|_1.
\end{aligned}
$$

The estimates obtained for I_1 and I_2 imply that $I_1 \to 0$ and $I_2 \to 0$ as $t \to 0^+$ uniformly on B. Hence,

$$
\left\|e^{kw_1}(\int_{\mathbb{R}^n} M_t(x,\cdot)\varphi(x)dx - \varphi(\cdot))\right\|_\infty \to 0 \text{ as } t \to 0^+ \tag{7}
$$

uniformly on B as well. Now we prove that

$$
\left\|e^{kw_2}(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - \varphi(y))(\zeta)\right\|_\infty
$$

approaches 0 as $t \to 0^+$ uniformly on B. To do this, we write

$$
\begin{aligned}
I' &= \left|e^{kw_2}(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx - \varphi(y))(\zeta)\right| \\
&= \left|e^{kw_2(\zeta)}(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx)(\zeta) - e^{kw_2(\zeta)}(\varphi(y))(\zeta)\right| \\
&= \Big|e^{kw_2(\zeta)} \frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}} (\int_{\mathbb{R}^n} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} \varphi(x)dx)(\zeta) \\
&\quad - e^{kw_2(\zeta)}\widehat{\varphi}(\zeta)\Big| \\
&= \Big|e^{kw_2(\zeta)}(\frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}} e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} \widehat{\varphi}(e^{-t}\zeta) \\
&\quad - \widehat{\varphi}(\zeta))\Big| \\
&= \Big|e^{kw_2(\zeta)}(\frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}} e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} \widehat{\varphi}(e^{-t}\zeta) \\
&\quad - \widehat{\varphi}(e^{-t}\zeta) + \widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta))\Big| \\
&\leq \pi^{-n/2} e^{kw_2(\zeta)} (1-e^{-2t})^{-n/2} \Big|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} \widehat{\varphi}(e^{-t}\zeta) \\
&\quad - \widehat{\varphi}(e^{-t}\zeta)\Big| + e^{kw_2(\zeta)} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
&\leq \pi^{-n/2} e^{ke^t w_2(e^{-t}\zeta)} (1-e^{-2t})^{-n/2} \Big|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1\Big| \\
&\quad \left|\widehat{\varphi}(e^{-t}\zeta)\right| + e^{kw_2(\zeta)} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
&\leq \pi^{-n/2} e^{k[e^t]w_2(e^{-t}\zeta)} (1-e^{-2t})^{-n/2} \Big|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1\Big| \\
&\quad \left|\widehat{\varphi}(e^{-t}\zeta)\right| + e^{kw_2(\zeta)} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
&\leq \pi^{-n/2} e^{k([e^t]+1)w_2(e^{-t}\zeta)} (1-e^{-2t})^{-n/2} \Big|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1\Big| \\
&\quad \left|\widehat{\varphi}(e^{-t}\zeta)\right| + e^{kw_2(\zeta)} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
&\leq \pi^{-n/2} e^{2kw_2(e^{-t}\zeta)} (1-e^{-2t})^{-n/2} \Big|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1\Big| \\
&\quad \left|\widehat{\varphi}(e^{-t}\zeta)\right| + e^{kw_2(\zeta)} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
&\leq \pi^{-n/2} (1-e^{-2t})^{-n/2} \Big|e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} - 1\Big| \left\|e^{2kw_2}\widehat{\varphi}\right\|_\infty \\
&\quad + e^{kw_2(\zeta)} \left|\widehat{\varphi}(e^{-t}\zeta) - \widehat{\varphi}(\zeta)\right| \\
&= A_1 + A_2.
\end{aligned}
$$

where we used Remark 2 in the second inequality above. Since $e^{-\frac{(1-e^{-2t})|\zeta|^2}{4}} \to 1$ as $t \to 0^+$ uniformly on compact subsets of $\mathbb{R}^n$, the first term A_1 converges to 0 uniformly on B. Applying the Mean Value Theorem for the second term A_2, there exists a point τ on the line segment from $e^{-t}\zeta$ to ζ such that

$$\left|\widehat{\varphi}(e^{-t}\zeta)-\widehat{\varphi}(\zeta)\right| = (1-e^{-t})\left|\zeta\right|\left|\nabla\widehat{\varphi}(\tau)\right| \qquad (8)$$

Using Remark 2, we estimate $|\widehat{\varphi}(e^{-t}\zeta)-\widehat{\varphi}(\zeta)|$ as follows:

$$\begin{aligned}\left|\widehat{\varphi}(e^{-t}\zeta)-\widehat{\varphi}(\zeta)\right| &= (1-e^{-t})|\zeta||\nabla\widehat{\varphi}(\tau)| \\ &\le (1-e^{-t})|\tau||\nabla\widehat{\varphi}(\tau)| \\ &\le C(1-e^{-t})e^{-t}e^{Nw_2(\tau)}|\nabla\widehat{\varphi}(\tau)| \\ &\le C\left|\left|e^{Nw_2}\nabla\widehat{\varphi}\right|\right|_\infty (1-e^{-t})e^{-t},\end{aligned}$$

which implies that A_2 converges to 0 as $t \to 0^+$. Hence,

$$\left|\left|e^{kw_2}\left(\int_{\mathbb{R}^n} M_t(x,y)\varphi(x)dx-\varphi(y)\right)(\zeta)\right|\right|_\infty$$

converges to 0 uniformly on B as $t \to 0^+$. This completes the proof of Theorem 4.

5 Conclusion

The Laplacian operator and the heat semigroup serve as prototypes for elliptic differential operators and semigroups of operators, respectively. The Ornstein-Uhlenbeck operator and the Ornstein-Uhlenbeck semigroup play the role of the Laplacian and of the heat semigroup if the Lebesgue measure is replaced by the standard Gaussian measure γ in an infinite-dimensional setting, as in equation (1). For our application, Theorem 4 implies that the functionals in the dual space $\mathfrak{S}'_{w_1,w_2}$ can be realized as boundary values to the differential equation $\frac{\partial}{\partial t}u - Au = 0,\ t > 0$. This approach extends the result obtained in [13] where it proved that in the sense of the strong dual topology of the Beurling-Björck space $\mathfrak{S}'_w$, the w-tempered distributions can be realized as boundary values to solutions of the generalized heat equation, just as with the Gauss-Weierstrass semigroup. It is the same result obtained in [8] for the functionals in the dual space of Gelfand-Shilov spaces.

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References

[1] L. Schwartz, Une caractérization de l'space S de Schwartz, *C. R. Acad. Sci. Paris Sér.* I **316**, 23–25 (1993).
[2] L. Schwartz, *Théorie des distributions*, 3rd ed., Hermann, Paris, 200–240, (1966).
[3] A. Beurling, On quasi-analiticity and general distributions, Notes by P.L. Duren, lectures delivered at the AMS Summer Institute on Functional Analysis, Stanford University, August, (1961).
[4] G. Björck, Linear partial differential operators and generalized distributions, *Ark. Mat.* **6**, 351-407 (1965-1967).
[5] J. Alvarez and H. Obiedat, Characterizations of the Schwartz space S and the Beurling–Björck space Sw, *Cubo* **6**, 167–183 (2004).
[6] S.Y. Chung, D. Kim and S.k. Kim, Structure of the extended Fourier hyperfunctions. *Japan. J. Math.* **19**, 217-226 (1994).
[7] H. Obiedat, W. Shatanawi and M. Yasein, Structure theorem for functionals in the space $\mathfrak{S}'_{w_1,w_2}$, *int. j. of pure and applied math*, **2008**, 9 pages (2008).
[8] H. M. Obiedat and L. S. Moyo, Ornstein-Uhlenbeck Semigroup on the dual space of Gelfand-Shilov spaces of Beurling type, *Bol. Soc. Paran. Mat*, in press (2018).
[9] S.Y. Chung, D. Kim, S. Lee, Characterizations for Beurling-Björck space and Schwartz space. *Proc. Amer. Math. Soc.* **125**, 3229-3234 (1997).
[10] H.J. Schmeisser and H. Triebel, *Topics in Fourier Analysis and Function Spaces*, John Willey & Sons, New York, 14–16, (1987).
[11] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill Inc., New York, 157–160, (1991).
[12] W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, Switzerland, 40–43, (2019).
[13] J. Alvarez, M. S. Eydenberg, and H. Obiedat, The action of operator semigroups on the topological dual of the Beurling—Björck space, *Journal of Mathematical Analysis and Applications*, **339**, 405–418 (2008).

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