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Some Novel Laplace-Transform Based Integrals via Hypergeometric Techniques

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Abstract: In this paper, we investigate and evaluate a number of novel Laplace-transform based integrals under suitable convergence conditions. In our derivations, we make use of the hypergeometric approach involving algebraic properties of the Pochhammer symbol and classical summation theorems of the hypergeometric series ${}_2F_1(1)$, ${}_2F_1(-1)$ and ${}_4F_3(-1)$. We also obtain the Laplace transforms of arbitrary powers of some finite series, which contain the hyperbolic sine and cosine functions with different arguments, in terms of the hypergeometric and Beta functions. Moreover, we derive the Laplace transforms of the even and odd positive integer powers of the trigonometric sine and cosine functions with different arguments, as well as their combinations in products involving two, three or four functions at a time. Finally, several interesting special cases of the main results are considered.

Keywords: Pochhammer symbol; Gamma and Beta functions; Generalized hypergeometric functions; Gauss, Kummer and Clausen hypergeometric functions; Summation and multiplication theorems; Laplace transforms; Hyperbolic functions; Trigonometric Functions.

1 Introduction, Definitions and Preliminaries

Throughout this paper, we use the following standard notations:

$$\mathbb{N} := \{1,2,3,\cdots\}, \quad \mathbb{N}_0 := \{0,1,2,3,\cdots\} = \mathbb{N}\cup\{0\}$$

and

$$\mathbb{Z}^- := \{-1,-2,-3,\cdots\} = \mathbb{Z}_0^- \setminus \{0\} \quad (\mathbb{Z}_0^- = \{0,-1,-2,\cdots\}).$$

Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of positive and negative real numbers, respectively, and $\mathbb{C}$ denotes the set of complex numbers.
In terms of the familiar (Euler's) Gamma function $\Gamma(z)$, we denote by $(\lambda)_\nu$ $(\lambda,\nu \in \mathbb{C})$ the *general* Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}), \end{cases} \tag{1}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient in (1) exists. It is easily observed from the definition (1) that

$$(\lambda)_{\nu+\sigma} = (\lambda)_\nu\,(\lambda+\nu)_\sigma \qquad (\lambda,\nu,\sigma\in\mathbb{C}).$$

For two complex numbers α and β, the classical Beta function $B(\alpha,\beta)$ is defined by (see, for example, [22, p.

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26, Eq. (48)])

$$B(\alpha,\beta):=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & \left(\min\{\Re(\alpha),\Re(\beta)\}>0\right)\\[2mm] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).\end{cases} \tag{2}$$

A natural extension of the celebrated Gauss hypergeometric function ${}_2F_1$ is the generalized hypergeometric function ${}_pF_q$ with p numerator parameters:

$$\alpha_j\in\mathbb{C}\qquad (j=1,\cdots,p)$$

and q denominator parameters:

$$\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\qquad (j=1,\cdots,q),$$

which is defined by (see, for example, [21, p. 23, Eq. (23) *et seq.*])

$${}_pF_q\left(\begin{matrix}\alpha_1,\cdots,\alpha_p;\\ \\ \beta_1,\cdots,\beta_q;\end{matrix}\; z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \tag{3}$$

where $p,q\in\mathbb{N}_0$ and the hypergeometric series in (3)

(i) converges absolutely for $|z|<\infty$ if $p\leqq q$,
(ii) converges absolutely for $|z|<1$ if $p=q+1$, and
(iii) diverges for all z $(z\neq 0)$ if $p>q+1$.
Furthermore, if we set

$$\omega:=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j,$$

then it is known that the generalized hypergeometric ${}_pF_q$ series in (3) (with $p=q+1$) is

I. absolutely convergent for $|z|=1$ if $\Re(\omega)>0$,
II. conditionally convergent for $|z|=1$ $(z\neq 1)$ if $-1<\Re(\omega)\leqq 0$, and
III. divergent for $|z|=1$ if $\Re(\omega)\leqq -1$.

Other than the above-mentioned Gauss hypergeometric function ${}_2F_1$, some important special cases of the generalized hypergeometric function ${}_pF_q$ include (for example) the binomial series ${}_1F_0$, Kummer's confluent hypergeometric function ${}_1F_1$, Clausen's hypergeometric function ${}_3F_2$, and so on.

We now recall a number of known results that will be needed in our investigation (see, for details, [6, 8, 20–22]).

Gauss's Summation Theorem:

$${}_2F_1\left(\begin{matrix}a,b;\\ \\ c;\end{matrix}\; 1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{4}$$

$$\left(\Re(c-a-b)>0;\ c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Kummer's First Summation Theorem:

$${}_2F_1\left(\begin{matrix}a,b;\\ \\ 1+a-b;\end{matrix}\; -1\right)=\frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1+\frac{a}{2}-b\right)\Gamma(1+a)} \tag{5}$$

$$\left(\Re(b)<1;\ 1+a-b\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Summation Theorem for the Hypergeometric Series ${}_4F_3(-1)$:

$${}_4F_3\left(\begin{matrix}a,1+\frac{a}{2},b,c;\\ \\ \frac{a}{2},1+a-b,1+a-c;\end{matrix}\; -1\right)=\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \tag{6}$$

$$\left(\Re(a-2b-2c)>-2;\ \frac{a}{2},1+a-b,1+a-c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Gauss-Legendre Multiplication Theorem for the Gamma Function:

$$\Gamma(mz)=(2\pi)^{\frac{1}{2}(1-m)}\,m^{mz-\frac{1}{2}}\prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right)$$
$$\left(z\neq 0,-\frac{1}{m},-\frac{2}{m},\cdots;\ m\in\mathbb{N}\right), \tag{7}$$

which, in the special case when $m=2$, is known as Legendre's duplication formula for the Gamma Function. Moreover, for every positive integer $m\in\mathbb{N}$, we find from (7) that (see [22, p. 22, Eq. (26)])

$$(\lambda)_{mn}=m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n\qquad \left(m\in\mathbb{N};\ n\in\mathbb{N}_0\right). \tag{8}$$

Reflection Formula for the Gamma Function:

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}\qquad \left(z\neq 0,\pm 1,\pm 2,\cdots\right). \tag{9}$$

Next, if a given function $f(t)$ is well defined for all real values of $t>0$, then the Laplace transform of $f(t)$ with the parameter $s\in\mathbb{C}$, denoted by $g(s)$, is given by the following infinite integral:

$$\mathscr{L}[f(t);s]=\int_0^{\infty}e^{-st}\,f(t)\,dt=g(s)\left(\Re(p)>0\right) \tag{10}$$

and the inverse Laplace transform of $g(s)$ is given by

$$\mathscr{L}^{-1}[g(s);t]=\frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty}e^{st}\,g(s)\,ds=f(t)\qquad \left(\mathrm{i}=\sqrt{-1}\right), \tag{11}$$

where κ is a real constant which exceeds the real parts of all of the singularities of the function $g(s)$ in the complex s-plane. For the tables of the Laplace and the inverse

Laplace transforms, one may refer to [9] and [19]. We choose to recall here the following results:

$$\mathscr{L}[\sin(\lambda t):s] = \frac{\lambda}{\lambda^2+s^2} \quad (\Re(s) > |\Im(\lambda)|) \tag{12}$$

and

$$\mathscr{L}[\cos(\lambda t):s] = \frac{s}{\lambda^2+s^2} \quad (\Re(s) > |\Im(\lambda)|). \tag{13}$$

We also record here the following finite series representations of the positive integer powers of the hyperbolic functions:

$$\sum_{k=0}^{m-1}\binom{2m}{k}\cosh[(2m-2k)\beta x] + \frac{1}{2}\binom{2m}{m} = \frac{1}{2}\left(e^{\beta x}+e^{-\beta x}\right)^{2m} = 2^{2m-1}\cosh^{2m}(\beta x) \tag{14}$$

and

$$\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\sinh[(2n-2k)\gamma x] + \frac{(-1)^n}{2}\binom{2n}{n} = \frac{1}{2}\left(e^{\gamma x}-e^{-\gamma x}\right)^{2n} = 2^{2n-1}\sinh^{2n}(\gamma x), \tag{15}$$

where m and n are positive integers;

$$\sum_{\ell=0}^{p}(-1)^\ell\binom{2p+1}{\ell}\sinh[(2p+1-2\ell)\lambda x] = \frac{1}{2}\left(e^{\lambda x}-e^{-\lambda x}\right)^{2p+1} = 2^{2p}\sinh^{2p+1}(\lambda x) \tag{16}$$

and

$$\sum_{\ell=0}^{q}\binom{2q+1}{\ell}\cosh\{(2q+1-2\ell)\mu x\} = \frac{1}{2}\{e^{\mu x}+e^{-\mu x}\}^{2q+1} = 2^{2q}\cosh^{2q+1}(\mu x), \tag{17}$$

where p and q are non-negative integers.

Proof. We take the right-hand side of Eq. (14) and apply the binomial expansion. We thus obtain

$$\begin{aligned}\frac{1}{2}\left(e^{\beta x}+e^{-\beta x}\right)^{2m} &= \frac{1}{2}\sum_{k=0}^{2m}\binom{2m}{k}\left(e^{\beta x}\right)^{2m-k}\left(e^{-\beta x}\right)^k \\ &= \frac{1}{2}\sum_{k=0}^{2m} T_k \\ &= \frac{1}{2}\Big[(T_0+T_{2m})+(T_1+T_{2m-1})+(T_2+T_{2m-2})+\cdots \\ &\quad +(T_{m-3}+T_{m+3})+(T_{m-2}+T_{m+2})+(T_{m-1}+T_{m+1})+T_m\Big].\end{aligned} \tag{18}$$

Now, putting the values of

$$T_0, T_1, T_2, \cdots, T_{m-2}, T_{m-1}, T_m, T_{m+1}, T_{m+2}, \cdots, T_{2m-2}, T_{2m-1}, T_{2m}$$

in Eq. (18) and applying the binomial expansion once again, we get the left-hand side of Eq. (14). Similarly, we can prove Eq. (15).

Next, considering the right-hand side of Eq. (16), we get

$$\begin{aligned}\frac{1}{2}\left(e^{\lambda x}-e^{-\lambda x}\right)^{2p+1} &= \frac{1}{2}\sum_{\ell=0}^{2p+1}\binom{2p+1}{\ell}\left(e^{\lambda x}\right)^\ell\left(-e^{-\lambda x}\right)^{2p+1-\ell} \\ &= \frac{1}{2}\sum_{\ell=0}^{2p+1} U_\ell \\ &= \frac{1}{2}\Big[(U_0+U_{2p+1})+(U_1+U_{2p})+(U_2+U_{2p-1})\cdots \\ &\quad +(U_{p-2}+U_{p+3})+(U_{p-1}+U_{p+2})+(U_p+U_{p+1})\Big].\end{aligned} \tag{19}$$

Now, putting the values of

$$U_0, U_1, U_2, \cdots, U_{p-1}, U_p, U_{p+1}, U_{p+2}, \cdots, U_{2p-1}, U_{2p}, U_{2p+1}$$

in Eq. (19) and applying again the binomial expansion, after simplifications we get the left-hand side of Eq. (16). Similarly, it is easy to prove Eq. (17).

In a series of papers, Moll *et al.* [3–5, 14–18] evaluated many definite integrals in the tables of Gradshteyn and Ryzhik (see [10] and [11]) by changing the independent variables. Our approach in this paper is to obtain the analytical evaluations of some novel integrals by using the hypergeometric approach, which is significantly different from that of Moll *et al.* [3–5, 14–18]. This paper is organized as follows. In Section 2, we obtain the Laplace transforms of an arbitrary power of some finite series containing the hyperbolic sine and cosine functions. The analytical evaluations of some novel integrals are presented in Section 3. In Section 4, the Laplace transforms of positive integer powers of the trigonometric sine and cosine functions as well as their combinations involving products are derived. Various interesting special cases of some of our novel integrals are addressed in Section 5. Finally, in Section 6, we present our concluding remarks and observations.

2 Laplace Transforms of Finite Series Containing the Hyperbolic Sine and Cosine Functions

The following known results (20) to (23) are presented in the tables ([9, p. 163, Entry (5) and Entry (6)]; see also [10, pp. 384–385, Entry (3.541)(1) and Entry (3.542)(1)] and [8, p. 11, Eq. (25)]):

$$\int_0^\infty e^{-sx}[\cosh(\gamma x)-1]^\nu dx$$
$$= \frac{2\nu}{2^\nu(s+\nu\gamma)} B\left(\frac{s}{\gamma}-\nu, 2\nu\right) \tag{20}$$
$$= \frac{1}{2^\nu(\gamma)} B\left(\frac{s}{\gamma}-\nu, 2\nu+1\right), \tag{21}$$

where $\Re(\gamma)>0$, $\Re(\nu)>-\frac{1}{2}$, $\Re(s)>\Re(\gamma\nu)$ and

$$\frac{s}{\gamma}-\nu+1\in\mathbb{C}\setminus\mathbb{Z}_0^-;$$

$$\int_0^\infty e^{-sx}[\sinh(\lambda x)]^\nu dx$$
$$= \frac{\nu}{2^\nu(s+\lambda\nu)} B\left(\frac{s}{2\lambda}-\frac{\nu}{2}, \nu\right) \tag{22}$$
$$= \frac{1}{2^{1+\nu}(\lambda)} B\left(\frac{s}{2\lambda}-\frac{\nu}{2}, 1+\nu\right), \tag{23}$$

where $\Re(\lambda)>0$, $\Re(\nu)>-1$, $\Re(s)>\Re(\lambda\nu)$ and

$$\frac{s}{2\lambda}-\frac{\nu}{2}+1\in\mathbb{C}\setminus\mathbb{Z}_0^-.$$

In this section, our work is motivated by the above known results (20) to (23). We derive each of the following Laplace transforms of arbitrary powers of some finite series, in terms of hypergeometric functions.

I. Laplace Transform of an Arbitrary Power of the First Finite Series Containing the Hyperbolic Cosine Function:

$$\int_0^\infty e^{-sx}\left[\sum_{j=0}^{m-1}\binom{2m}{j}\cosh\left((2m-2j)\beta x\right)+\frac{1}{2}\binom{2m}{m}\right]^\nu dx$$
$$= \frac{1}{2^\nu(s-2m\nu\beta)}\,{}_2F_1\left(\begin{matrix}-2m\nu, \frac{s}{2\beta}-m\nu; \\ \frac{s}{2\beta}-m\nu+1;\end{matrix}\; -1\right), \tag{24}$$

where $\Re(m\nu)>-1$, $\Re(s-2m\beta\nu)>0$, $\Re(\beta)>0$,

$$\frac{s}{2\beta}-m\nu+1\in\mathbb{C}\setminus\mathbb{Z}_0^-$$

and m is a positive integer.

II. Laplace Transform of an Arbitrary Power of the Second Finite Series Containing the Hyperbolic Cosine Function:

$$\int_0^\infty e^{-sx}\left[\sum_{j=0}^{n-1}(-1)^j\binom{2n}{j}\cosh\left((2n-2j)\gamma x\right)+\frac{(-1)^n}{2}\binom{2n}{n}\right]^\nu dx$$
$$= \frac{1}{2^\nu(s-2n\nu\gamma)}\,{}_2F_1\left(\begin{matrix}-2n\nu, \frac{s}{2\gamma}-n\nu; \\ \frac{s}{2\gamma}-n\nu+1;\end{matrix}\; 1\right)$$
$$= \frac{2^{1-\nu}(n\nu)}{s+2n\nu\gamma} B\left(\frac{s}{2\gamma}-n\nu, 2n\nu\right), \tag{25}$$

where $\Re(2n\nu)>-1$, $\Re(s-2n\gamma\nu)>0$, $\Re(\gamma)>0$,

$$\frac{s}{2\gamma}-n\nu+1\in\mathbb{C}\setminus\mathbb{Z}_0^-$$

and n is a positive integer.

III. Laplace Transform of an Arbitrary Power of the Third Finite Series Containing the Hyperbolic Sine Function:

$$\int_0^\infty e^{-sx}\left[\sum_{k=0}^{p}(-1)^k\binom{2p+1}{k}\sinh\left((2p+1-2k)\lambda x\right)\right]^\nu dx$$
$$= \frac{1}{2^\nu(s-2p\nu\lambda-\lambda\nu)}\,{}_2F_1\left(\begin{matrix}-2p\nu-\nu, \frac{s}{2\lambda}-p\nu-\frac{\nu}{2}; \\ \frac{s}{2\lambda}-p\nu-\frac{\nu}{2}+1;\end{matrix}\; 1\right)$$
$$= \frac{(2p\nu+\nu)}{2^\nu(s+2p\nu\lambda+\nu\lambda)} B\left(\frac{s}{2\lambda}-p\nu-\frac{\nu}{2}, 2p\nu+\nu\right), \tag{26}$$

where $\Re(2p\nu+\nu)>-1$, $\Re(s-2p\lambda\nu-\lambda\nu)>0$, $\Re(\lambda)>0$,

$$\frac{s}{2\lambda}-p\nu-\frac{\nu}{2}+1\in\mathbb{C}\setminus\mathbb{Z}_0^-$$

and p is a non-negative integer.

IV. Laplace Transform of an Arbitrary Power of the Fourth Finite Series Containing the Hyperbolic Sine Function:

$$\int_0^\infty e^{-sx}\left[\sum_{\ell=0}^{q}\binom{2q+1}{\ell}\sinh\left((2q+1-2\ell)\mu x\right)\right]^\nu dx$$
$$= \frac{1}{2^\nu(s-2q\nu\mu-\mu\nu)}\,{}_2F_1\left(\begin{matrix}-2q\nu-\nu, \frac{s}{2\mu}-q\nu-\frac{\nu}{2}; \\ \frac{s}{2\mu}-q\nu-\frac{\nu}{2}+1;\end{matrix}\; -1\right), \tag{27}$$

where $\Re(2q\nu+\nu)>-2$, $\Re(s-2q\mu\nu-\mu\nu)>0$, $\Re(\mu)>0$,

$$\frac{s}{2\mu}-q\nu-\frac{\nu}{2}+1\in\mathbb{C}\setminus\mathbb{Z}_0^-$$

and q is a non-negative integer.

Proof. We take the νth power on both sides of the equations (14) to (17). We then multiply both sides of the resulting equations by e^{-sx} and integrate with respect to x over the interval $(0,\infty)$. Finally, by using Laplace-transform formulas (12) and (13), we get the results stated in (24) to (27).

3 A Set of Novel Integrals in Terms of the Beta Functions

As we observed in Section 1, Moll *et al.* [3–5, 14–18] evaluated many definite integrals in the tables of Gradshteyn and Ryzhik (see [10] and [11]) by changing the independent variables. Our approach in this section is to obtain the analytical evaluations of the following novel integrals by using the hypergeometric approach, which is significantly different from that of Moll *et al.* [3–5, 14–18].

V. The fifth novel integral is given by

$$\int_0^\infty \frac{\cosh(2\alpha t)}{[\cosh(st)]^{2\beta}}dt = \frac{4^{\beta-1}}{s}\, B\left(\beta+\frac{\alpha}{s}, \beta-\frac{\alpha}{s}\right), \tag{28}$$

where $\Re(\beta) < 1$, $\Re(s) > 0$,

$$\Re\left(\beta \pm \frac{\alpha}{s}\right) > 0 \quad \text{and} \quad \beta, \beta \pm \frac{\alpha}{s}+1 \in \mathbb{C}\setminus\mathbb{Z}_0^-.$$

VI. The sixth novel integral is given by

$$\int_0^\infty \frac{[\sinh(x)]^\alpha}{[\cosh(x)]^\beta}dx = \frac{1}{2}\, B\left(\frac{1+\alpha}{2}, \frac{\beta-\alpha}{2}\right), \tag{29}$$

where $\Re(\alpha) > -1$, $-2 < \Re(\alpha-\beta) < 0$ and

$$\frac{\beta \pm \alpha + 2}{2} \in \mathbb{C}\setminus\mathbb{Z}_0^-.$$

VII. The seventh novel integral is given by

$$\int_0^\infty \frac{\cos(ax)}{[\cosh(\beta x)]^\nu}dx$$
$$= \frac{2^{\nu-2}}{\beta\Gamma(\nu)}\,\Gamma\left(\frac{\nu}{2}+\frac{\mathrm{i}a}{2\beta}\right)\Gamma\left(\frac{\nu}{2}-\frac{\mathrm{i}a}{2\beta}\right) \tag{30}$$
$$= B\left(\frac{\nu}{2}+\frac{\mathrm{i}a}{2\beta}, \frac{\nu}{2}-\frac{\mathrm{i}a}{2\beta}\right), \tag{31}$$

where $\Re(\beta) > 0$, $\Re(\nu) < 2$, $\Re(\nu\beta \pm \mathrm{i}a) > 0$ and

$$\frac{\nu}{2}, \frac{\nu}{2} \pm \frac{\mathrm{i}a}{2\beta}+1 \in \mathbb{C}\setminus\mathbb{Z}_0^-.$$

Proof. Let us denote the left-hand side of Eq. (28) by $\mho(\alpha,\beta,p)$. Then, by using some well-known identities that involve the hyperbolic functions and the familiar binomial function, we obtain

$$\mho(\alpha,\beta,p)$$
$$= 2^{2\beta-1}\int_0^\infty \left[e^{-2\beta pt}(e^{2\alpha t}+e^{-2\alpha t})\,{}_1F_0\left(\begin{matrix}2\beta; \\ -;\end{matrix}\; -e^{-2pt}\right)\right]dt, \tag{32}$$

when $\Re(p) > 0$ and $|-e^{-2pt}| < 1$ for all $t > 0$.

Now, we replace the function ${}_1F_0$ in (32) by its hypergeometric series given by (3) for $p-1 = q = 0$ and change the order of integration and summation in the resulting equation under the following condition:

$$\Re\left(\beta \pm \frac{\alpha}{p}\right) > 0.$$

Then, upon appropriately using known formulas for the Laplace transforms, followed by usages of some algebraic properties of the Pochhammer symbol and the classical Beta function, the classical summation theorem (6), and some simplifications, we arrive at the right-hand side of the result asserted by Eq. (28) under the stated conditions.

Similar are our demonstrations of the results asserted by Eq. (29) and Eq. (30). The involved details are being omitted here as an exercise for the interested reader.

4 Laplace Transforms of Positive Integer Powers of the Trigonometric Sine and Cosine Functions and Their Combinations

The following results (33) to (40) are motivated by the work given in the tables (see [11, pp. 150–156, Entry (3), Entry (7), Entry (47), Entry(51)]; see also [10, Section (3.611-4.146)]):

$$\int_0^\infty e^{-sx}\cos^{2m}(\beta x)dx$$
$$= \frac{1}{2^{2m-1}}\sum_{j=0}^{m-1}\left[\binom{2m}{j}\frac{s}{[(2m-2j)\beta]^2+s^2}\right] + \frac{1}{2^{2m}}\binom{2m}{m}\frac{1}{s} \tag{33}$$
$$= \frac{1}{2^{2m}(s-2\mathrm{i}m\beta)}\;{}_2F_1\left(\begin{matrix}-2m, -\frac{\mathrm{i}s+2m\beta}{2\beta}; \\ -\frac{\mathrm{i}s+2m\beta}{2\beta}+1;\end{matrix}\; -1\right), \tag{34}$$

where $\Re(s) > 2m|\Im(\beta)|$, $\mathrm{i} = \sqrt{-1}$ and m is a positive integer,

$$\int_0^\infty e^{-sx}\sin^{2n}(\gamma x)dx$$
$$=\frac{(-1)^n}{2^{2n-1}}\sum_{j=0}^{n-1}\left[(-1)^j\binom{2n}{j}\frac{s}{[(2n-2j)\gamma]^2+s^2}\right]$$
$$+\frac{1}{2^{2n}}\binom{2n}{n}\frac{1}{s} \tag{35}$$
$$=\frac{(2n)!}{2^{2n}(s+2\mathrm{i}n\gamma)\left(1+\frac{\mathrm{i}s}{2\gamma}\right)_n\left(-\frac{s}{2\gamma}\right)_n}, \tag{36}$$

where $\Re(s) > 2n|\Im(\gamma)|$ and n is a positive integer,

$$\int_0^\infty e^{-sx}\sin^{2p+1}(\lambda x)dx$$
$$=\frac{(-1)^p}{2^{2p}}\sum_{k=0}^{p}\left[(-1)^k\binom{2p+1}{k}\frac{(2p+1-2k)\lambda}{[(2p+1-2k)\lambda]^2+s^2}\right] \tag{37}$$
$$=\frac{(2p+1)!\,\lambda}{2^{2p}(s-\mathrm{i}\lambda)[s+\mathrm{i}\lambda(2p+1)]\left(\frac{3\lambda+\mathrm{i}s}{2\lambda}\right)_p\left(\frac{\lambda-\mathrm{i}s}{2\lambda}\right)_p}, \tag{38}$$

where $\Re(s) > (2p+1)|\Im(\lambda)|$ and p is a non-negative integer, and

$$\int_0^\infty e^{-sx}\cos^{2q+1}(\mu x)dx$$
$$=\frac{1}{2^{2q}}\sum_{\ell=0}^{q}\left[\binom{2q+1}{\ell}\frac{s}{[(2q+1-2\ell)\mu]^2+s^2}\right] \tag{39}$$
$$=\frac{1}{2^{2q+1}[s-\mathrm{i}\,(2q+1)\mu]}$$
$$\cdot {}_2F_1\left(\begin{matrix}-2q-1,-\frac{\mathrm{i}s+(2q+1)\mu}{2\mu};\\ -\frac{\mathrm{i}s+(2q+1)\mu}{2\mu}+1;\end{matrix}\ -1\right), \tag{40}$$

where $\Re(s) > (2q+1)|\Im(\mu)|$ and q is a non-negative integer.

Proof. We first multiply both sides of the following equations (41) to (44) by e^{-sx} and then integrate each member of the resulting equation with respect to x over the interval $(0,\infty)$. Finally, by the appropriate use of known and readily-accessible Laplace-transform formulas, we get the results stated in (33) to (40).

$$\sum_{j=0}^{m-1}\left[\binom{2m}{j}\cos\left((2m-2j)\beta x\right)\right]+\frac{1}{2}\binom{2m}{m}$$
$$=2^{2m-1}\cos^{2m}(\beta x), \tag{41}$$

$$\sum_{j=0}^{n-1}\left[(-1)^j\binom{2n}{j}\cos\left((2n-2j)\gamma x\right)\right]+\frac{(-1)^n}{2}\binom{2n}{n}$$
$$=(-1)^n\,2^{2n-1}\sin^{2n}(\gamma x), \tag{42}$$

$$\sum_{k=0}^{p}\left[(-1)^k\binom{2p+1}{k}\sin\left((2p+1-2k)\lambda x\right)\right]$$
$$=(-1)^p\,2^{2p}\sin^{2p+1}(\lambda x) \tag{43}$$

and

$$\sum_{\ell=0}^{q}\left[\binom{2q+1}{\ell}\cos\left((2q+1-2\ell)\mu x\right)\right]$$
$$=2^{2q}\cos^{2q+1}(\mu x). \tag{44}$$

Using some elementary relations in the equations (14) to (17) with suitable adjustments of the involred parameters, we can obtain the above results (41) to (44). Moreover, the Laplace transforms of the products of two functions, three functions and four functions, which have different arguments and different powers taken together, can be derived with the help of the aforementioned four equations (41) to (44). We choose to list these integrals as interesting consequences of the above-mentioned results (41) to (44) as follows.

- **Ten Integrals Associated with the Laplace Transforms of the Products of Two Functions:**

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)dx, \tag{45}$$
$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)dx, \tag{46}$$
$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n+1}(\gamma x)dx, \tag{47}$$
$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n}(\gamma x)dx, \tag{48}$$
$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\sin^{2n}(\gamma x)dx, \tag{49}$$
$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\sin^{2n+1}(\gamma x)dx, \tag{50}$$
$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\sin^{2n+1}(\gamma x)dx, \tag{51}$$
$$\int_0^\infty e^{-sx}\cos^{2m}(\beta x)\cos^{2n}(\gamma x)dx, \tag{52}$$
$$\int_0^\infty e^{-sx}\cos^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)dx \tag{53}$$

and

$$\int_0^\infty e^{-sx}\cos^{2m}(\beta x)\cos^{2n+1}(\gamma x)dx. \tag{54}$$

- **Twenty Integrals Associated with the Laplace Transforms of the Products of Three Functions:**

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p}(\lambda x)dx, \tag{55}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p+1}(\lambda x)dx, \tag{56}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p}(\lambda x)dx, \tag{57}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p+1}(\lambda x)dx, \tag{58}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p}(\lambda x)dx, \tag{59}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p+1}(\lambda x)dx, \tag{60}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p}(\lambda x)dx, \tag{61}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p+1}(\lambda x)dx, \tag{62}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2p+1}(\gamma x)\sin^{2n}(\lambda x)dx, \tag{63}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p+1}(\lambda x)dx, \tag{64}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n}(\gamma x)\sin^{2p+1}(\lambda x)dx, \tag{65}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n}(\gamma x)\cos^{2p}(\lambda x)dx, \tag{66}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\sin^{2n}(\gamma x)\sin^{2p}(\lambda x)dx, \tag{67}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\sin^{2n}(\gamma x)\sin^{2p+1}(\lambda x)dx, \tag{68}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\sin^{2n+1}(\gamma x)\sin^{2p}(\lambda x)dx, \tag{69}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\sin^{2n+1}(\gamma x)\sin^{2p+1}(\lambda x)dx, \tag{70}$$

$$\int_0^\infty e^{-sx}\cos^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p}(\lambda x)dx, \tag{71}$$

$$\int_0^\infty e^{-sx}\cos^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p+1}(\lambda x)dx, \tag{72}$$

$$\int_0^\infty e^{-sx}\cos^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p}(\lambda x)dx \tag{73}$$

and

$$\int_0^\infty e^{-sx}\cos^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p+1}(\lambda x)dx. \tag{74}$$

- **Thirty-Six Integrals Associated with the Laplace Transforms of the Products of Four functions:**

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q}(\mu x)dx, \tag{75}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{76}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p}(\lambda x)\cos^{2q}(\mu x)dx, \tag{77}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{78}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p+1}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{79}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p+1}(\lambda x)\cos^{2q}(\mu x)dx, \tag{80}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\sin^{2p+1}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{81}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p}(\lambda x)\cos^{2q}(\mu x)dx, \tag{82}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{83}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p+1}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{84}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q}(\mu x)dx, \tag{85}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{86}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{87}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p+1}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{88}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p+1}(\lambda x)\cos^{2q}(\mu x)dx, \tag{89}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p+1}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{90}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p}(\lambda x)\cos^{2q}(\mu x)dx, \tag{91}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{92}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p+1}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{93}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q}(\mu x)dx, \tag{94}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{95}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n+1}(\gamma x)\sin^{2p}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{96}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p+1}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{97}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n}(\gamma x)\sin^{2p+1}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{98}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n}(\gamma x)\sin^{2p+1}(\lambda x)\cos^{2q}(\mu x)dx, \tag{99}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\cos^{2n}(\gamma x)\cos^{2p}(\lambda x)\cos^{2q}(\mu x)dx, \tag{100}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\sin^{2n}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q}(\mu x)dx, \tag{101}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\sin^{2n}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{102}$$

$$\int_0^\infty e^{-sx}\sin^{2m}(\beta x)\sin^{2n}(\gamma x)\sin^{2p+1}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{103}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\sin^{2n+1}(\gamma x)\sin^{2p}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{104}$$

$$\int_0^\infty e^{-sx}\sin^{2m+1}(\beta x)\sin^{2n+1}(\gamma x)\sin^{2p+1}(\lambda x)\sin^{2q+1}(\mu x)dx, \tag{105}$$

$$\int_0^\infty e^{-sx}\cos^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p}(\lambda x)\cos^{2q}(\mu x)dx, \tag{106}$$

$$\int_0^\infty e^{-sx}\cos^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{107}$$

$$\int_0^\infty e^{-sx}\cos^{2m}(\beta x)\cos^{2n}(\gamma x)\cos^{2p+1}(\lambda x)\cos^{2q+1}(\mu x)dx, \tag{108}$$

$$\int_0^\infty e^{-sx}\cos^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p}(\lambda x)\cos^{2q+1}(\mu x)dx \tag{109}$$

and

$$\int_0^\infty e^{-sx}\cos^{2m+1}(\beta x)\cos^{2n+1}(\gamma x)\cos^{2p+1}(\lambda x)\cos^{2q+1}(\mu x)dx. \tag{110}$$

The above-mentioned combinations of the Laplace transforms in the integrals (45) to (110) are not recorded in the tables (see, for example, [9, pp. 150–160, Section 4.7]; see also [11] and [19]). However, these Laplace-transform integrals can be evaluated by substituting known finite-series representations of the corresponding even and odd positive integer powers of the trigonometric sine and cosine functions with the help of the equations (41) and (44). In the resulting product of finite summations, we can appropriately use various known identities and then apply such Laplace-transform formulas as (for example) (12) and (13). We choose to omit the details involved in such evaluations.

5 Special Cases of the Integrals Evaluated in Section 2 and Section 3

The following integrals from (111) to (120) are not found in the literature of integral transforms:

- If we set $m=1$ and replace β by $\beta/2$ in Eq. (24), we get

$$\int_0^\infty e^{-sx}[\cosh(\beta x)+1]^\nu dx = \frac{1}{2^\nu(s-\beta\nu)}\,{}_2F_1\left(\begin{matrix}-2\nu,\ \frac{s}{\beta}-\nu;\\ \frac{s}{\beta}-\nu+1;\end{matrix}\ -1\right), \tag{111}$$

where

$$\Re(\beta)>0,\ \Re(\nu)>-1,\ \Re(s)>\Re(\beta\nu)$$

and

$$\frac{s}{\beta}-\nu+1\in\mathbb{C}\setminus\mathbb{Z}_0^-.$$

- If we set $m=2$ and replace β by $\beta/4$ in Eq. (24), we have

$$\int_0^\infty e^{-sx}\left[\cosh(\beta x)+4\cosh\left(\frac{\beta x}{2}\right)+3\right]^\nu dx = \frac{1}{2^\nu(s-\nu\beta)}\,{}_2F_1\left(\begin{matrix}-4\nu,\frac{2s}{\beta}-2\nu;\\ \frac{2s}{\beta}-2\nu+1;\end{matrix}\ -1\right), \tag{112}$$

where $\Re(\beta)>0$, $\Re(\nu)>-\frac{1}{2}$, $\Re(s)>\Re(\beta\nu)$ and

$$\frac{2s}{\beta}-2\nu+1\in\mathbb{C}\setminus\mathbb{Z}_0^-.$$

- If we set $m=3$ and replace β by $\beta/6$ in Eq. (24), we obtain

$$\int_0^\infty e^{-sx}\left[\cosh(\beta x)+6\cosh\left(\frac{2\beta x}{3}\right)+15\cosh\left(\frac{\beta x}{3}\right)+10\right]^\nu dx = \frac{1}{2^\nu(s-\nu\beta)}\,{}_2F_1\left(\begin{matrix}-6\nu,\frac{3s}{\beta}-3\nu;\\ \frac{3s}{\beta}-3\nu+1;\end{matrix}\ -1\right), \tag{113}$$

where $\Re(\beta)>0$, $\Re(\nu)>-\frac{1}{3}$, $\Re(s)>\Re(\beta\nu)$ and

$$\frac{3s}{\beta}-3\nu+1\in\mathbb{C}\setminus\mathbb{Z}_0^-.$$

- If we set $n=1$ and replace γ by $\gamma/2$ Eq. (25), we get Eq. (20).

- If we set $n=2$ and replace γ by $\gamma/4$ in Eq. (25), we obtain

$$\int_0^\infty e^{-sx}\left[\cosh(\gamma x)-4\cosh\left(\frac{\gamma x}{2}\right)+3\right]^\nu dx = \frac{4\nu}{2^\nu(s+\gamma\nu)}\,B\left(\frac{2s}{\gamma}-2\nu,\ 4\nu\right), \tag{114}$$

where $\Re(\gamma)>0$, $\Re(\nu)>-\frac{1}{4}$, $\Re(s)>\Re(\gamma\nu)$ and

$$\frac{2s}{\gamma}-2\nu+1\in\mathbb{C}\setminus\mathbb{Z}_0^-.$$

- If we set $n=3$ and replace γ by $\gamma/6$ in Eq. (25), we have

$$\int_0^\infty e^{-sx}\left[\cosh(\gamma x)-6\cosh\left(\frac{2\gamma x}{3}\right)+15\cosh\left(\frac{\gamma x}{3}\right)-10\right]^\nu dx = \frac{6\nu}{2^\nu(s+\gamma\nu)}\,B\left(\frac{3s}{\gamma}-3\nu,6\nu\right), \tag{115}$$

where $\Re(\gamma)>0$, $\Re(\nu)>-\frac{1}{6}$, $\Re(s)>\Re(\gamma\nu)$ and

$$\frac{3s}{\gamma}-3\nu+1\in\mathbb{C}\setminus\mathbb{Z}_0^-.$$

- If we set $p=0$ in Eq. (26), we get Eq. (22).

- If we set $p=1$ and replace λ by $\lambda/3$ in Eq. (26), we have

$$\int_0^\infty e^{-sx}\left[\sinh(\lambda x)-3\sinh\left(\frac{\lambda x}{3}\right)\right]^\nu dx, = \frac{3\nu}{2^\nu(s+\lambda\nu)}\,B\left(\frac{3s}{2\lambda}-\frac{3\nu}{2},\ 3\nu\right), \tag{116}$$

where $\Re(\lambda) > 0$, $\Re(\nu) > -\frac{1}{3}$, $\Re(s) > \Re(\lambda\nu)$ and

$$\frac{3s}{2\lambda} - \frac{3\nu}{2} + 1 \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

- If we set $p = 2$ and replace λ by $\lambda/5$ in Eq. (26), we obtain

$$\int_0^\infty e^{-sx}\left[\sinh(\lambda x) - 5\sinh\left(\frac{3\lambda x}{5}\right) + 10\sinh\left(\frac{\lambda x}{5}\right)\right]^\nu dx = \frac{5\nu}{2^\nu(s+\lambda\nu)}\, B\left(\frac{5s}{2\lambda} - \frac{5\nu}{2}, 5\nu\right), \tag{117}$$

where $\Re(\lambda) > 0$, $\Re(\nu) > -\frac{1}{5}$, $\Re(s) > \Re(\lambda\nu)$ and

$$\frac{5s}{2\lambda} - \frac{5\nu}{2} + 1 \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

- If we set $q = 0$ in Eq. (27), we get

$$\int_0^\infty e^{-sx}[\cosh(\mu x)]^\nu dx = \frac{1}{2^\nu(s-\mu\nu)}\, {}_2F_1\left(\begin{matrix} -\nu, \frac{s}{2\mu} - \frac{\nu}{2}; \\ \\ \frac{s}{2\mu} - \frac{\nu}{2} + 1; \end{matrix} \; -1\right), \tag{118}$$

where $\Re(\mu) > 0$, $\Re(\nu) > -2$, $\Re(s) > \Re(\mu\nu)$ and

$$\frac{s}{2\mu} - \frac{\nu}{2} + 1 \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

- If we set $q = 1$ and replace μ by $\mu/3$ in Eq. (27), we have

$$\int_0^\infty e^{-sx}\left[\cosh(\mu x) + 3\cosh\left(\frac{\mu x}{3}\right)\right]^\nu dx = \frac{1}{2^\nu(s-\mu\nu)} \cdot {}_2F_1\left(\begin{matrix} -3\nu, \frac{3s}{2\mu} - \frac{3\nu}{2}; \\ \\ \frac{3s}{2\mu} - \frac{3\nu}{2} + 1; \end{matrix} \; -1\right), \tag{119}$$

where $\Re(\mu) > 0$, $\Re(\nu) > -\frac{2}{3}$, $\Re(s) > \Re(\mu\nu)$ and

$$\frac{3s}{2\mu} - \frac{3\nu}{2} + 1 \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

- If we set $q = 2$ and replace μ by $\mu/5$ in Eq. (27), we obtain

$$\int_0^\infty e^{-sx}\left[\cosh(\mu x) + 5\cosh\left(\frac{3\mu x}{5}\right) + 10\cosh\left(\frac{\mu x}{5}\right)\right]^\nu dx = \frac{1}{2^\nu(s-\mu\nu)}\, {}_2F_1\left(\begin{matrix} -5\nu, \frac{5s}{2\mu} - \frac{5\nu}{2}; \\ \\ \frac{5s}{2\mu} - \frac{5\nu}{2} + 1; \end{matrix} \; -1\right), \tag{120}$$

where $\Re(\mu) > 0$, $\Re(\nu) > -\frac{2}{5}$, $\Re(s) > \Re(\mu\nu)$ and

$$\frac{5s}{2\mu} - \frac{5\nu}{2} + 1 \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

The following integrals from (121) to (131), containing the integrands as a quotient of hyperbolic and trigonometric functions, are evaluated by applying Cauchy's Residue Theorem and are available in any standard text book on the analysis functions of a complex variable:

- If we put $p = 1$, $\alpha = 0$ and $\beta = \mu$ in Eq. (28), we get

$$\int_0^\infty \frac{1}{\{\cosh(t)\}^{2\mu}} dt = 4^{\mu-1} B(\mu,\mu) \qquad (\Re(\mu) > 0). \tag{121}$$

- If we set $2\alpha = a$, $p = b$ and $2\beta = 1$ in Eq. (28), we obtain

$$\int_0^\infty \frac{\cosh(at)}{\cosh(bt)} dt = \frac{\pi}{2b}\sec\left(\frac{a\pi}{2b}\right) \qquad (b > |a|). \tag{122}$$

- If we set $2\alpha = a$, $p = \pi$ and $2\beta = 1$ in Eq. (28), we get

$$\int_0^\infty \frac{\cosh(at)}{\cosh(\pi t)} dt = \frac{1}{2}\sec\left(\frac{a}{2}\right) \qquad (-\pi < a < \pi). \tag{123}$$

- If we set $2\alpha = a$, $p = 1$ and $2\beta = 1$ in Eq. (28), we have

$$\int_0^\infty \frac{\cosh(at)}{\cosh(t)} dt = \frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right) \qquad (|a| < 1). \tag{124}$$

- If we set $\alpha = 0$ and $2\beta = 1$ in Eq. (28), we obtain

$$\int_0^\infty \frac{dt}{\cosh(at)} = \frac{\pi}{2a} \qquad (a > 0). \tag{125}$$

- If we set $\beta = 1$ in Eq. (28), we have

$$\int_0^\infty \frac{\cosh(2\alpha t)}{\cosh^2(pt)} dt = \frac{\pi}{p}\operatorname{cosec}\left(\frac{\pi\alpha}{p}\right). \tag{126}$$

- If we set $\alpha = 0$ and $\beta = \mu$ in Eq. (29), we get

$$\int_0^\infty \frac{1}{\{\cosh(t)\}^\mu} dt = \frac{1}{2} B\left(\frac{1}{2}, \frac{\mu}{2}\right) \qquad (\Re(\mu) > 0). \tag{127}$$

- If we set $\nu = 2n$ in Eq. (30), we obtain

$$\int_0^\infty \frac{\cos(ax)}{[\cosh(\beta x)]^{2n}} dx = \frac{2^{2n-2}}{\beta\Gamma(2n)}\Gamma\left(n + \frac{ia}{2\beta}\right)\Gamma\left(n - \frac{ia}{2\beta}\right), \tag{128}$$

where $\Re(\beta) > 0$ and n is a positive integer.

- If we set $a=0$, $\beta=1$ and $\nu=2$ in Eq. (30), we get

$$\int_0^\infty \frac{1}{\cosh^2(x)}dx=1. \tag{129}$$

- If we set $\nu=2$ in Eq. (30), we have

$$\int_0^\infty \frac{\cos(ax)}{\cosh^2(\beta x)}dx=\frac{\pi a}{2\beta^2\sinh\left(\frac{\pi a}{2\beta}\right)}, \tag{130}$$

where $\Re(\beta)>0$ and $a>0$.

- If we set $\beta=1$ in Eq. (130) and differentiate both sides of the resulting equation with respect to a by applying the Leibniz rule for differentiation under the sign of integration, we obtain

$$\int_0^\infty \frac{x\sin(ax)}{\cosh^2(x)}dx=\frac{2\pi\sinh\left(\frac{a\pi}{2}\right)-a\pi^2\cosh\left(\frac{a\pi}{2}\right)}{4\sinh^2\left(\frac{\pi a}{2}\right)}, \tag{131}$$

where $a>0$.

6 Conclusion

Here, in this paper, we have evaluated several definite integrals containing the quotients of hyperbolic sine and consine functions in terms of the Beta functions. We have also obtained the Laplace transforms of an arbitrary power of some finite series containing the hyperbolic sine and cosine functions and the Laplace transforms of positive integer powers of the trigonometric sine and cosine functions in terms of the Gauss hypergeometric function and the Beta function. Various analogous integrals of the hyperbolic and trigonometric functions, which may be different from those presented here, can also be evaluated by means of the hypergeometric approach which we have used in this paper.

For motivating further researches along the lines described in this paper, we choose to include a number of recent works [23–26] which address hypergeometric summation theorems, evaluation of definite integrals, and other results involving (for example) the Laplace transform.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

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