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Generalized Schwarz Algorithm For A Class Of Variational Inequalities

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Abstract: We consider a decomposition to m-subdomains of the obstacle problem, which is modeled by a variational inequality of first species, using the auxiliary sequences, and we have proved a alternating relation between the solutions on each subdomain. We also proved a geommetrical convergence between the nth iteration and the solution of the initial problem, we obtained a result on the error estimate that contains a logarithmic factor with an extra power of $|\log(h)|$.

Keywords: L^∞ error estimate. Finite element method. Variational inequalities. Schwarz algorithm.

1 Introduction

The Schwarz alternating method of decomposing the domain has lately been found to be effective means for solving elliptic partial differential equations on a multi processing computing system. Pierre-Louis Lions represented the starting point of an intense research activity to develop this tool of calculation, see [1]-[3]. In this paper, we are interested in the Schwarz alternating method which is used to solve a class of elliptic variational inequality in the context of overlapping nonmatching grids, precisely in the error analysis in the maximum norm of obstacle type problems. The maximum error analysis of overlapping nonmatching grids for the obstacle problem which Ω is the union of two subdomains has been investigated in [4]. The same error analysis of a nonmatching grids for linear and nonlinear elliptic partial differential equations as well as elliptic quasi-variational inequalities has been addressed in [5]-[9]. In this paper we consider a domain Ω which is the union of m overlapping sub-domains where each sub-domain has its own triangulation.To prove the main result, we introduce the m continuous and discrete Schwarz sequences as well as prove a main result concerning the error estimate of solution in L^∞-norm, taking into account the combination of geometrical convergence and uniform convergence of finite element approximation.
This paper consists of two parts: In the first, we formulate the problem of continuous and discrete elliptic variational inequality we show the monotonicity and stability of discrete solution, and define the Schwarz algorithm for m subdomains with overlapping nonmatching grids. In the second part, we establish m auxiliary Schwarz sequences, and prove the main result of this work.

2 The generalized Schwarz alternating method

2.1 Elliptic obstacle problem

Let Ω be a convex domain in $\mathbb{R}^2$ with suffciently smooth boundary $\partial\Omega$.
We consider the bilinear form

$$a(u,v) = \int_\Omega (\nabla u.\nabla v)dx, \tag{1}$$

the linear form

$$(f,v) = \int_\Omega f(x).v(x)dx, \tag{2}$$

the right hande-side

$$f \in L^\infty(\Omega), \tag{3}$$

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the obstacle

$$\Psi \in W^{2,\infty}(\Omega) \quad \text{such that} \quad \Psi \geq 0 \quad on \quad \partial\Omega, \tag{4}$$

and the nonempty convex set

$$K_g = \{v \in H^1(\Omega) : v = g \quad on \partial\Omega, v \leq \Psi in \quad \Omega\}, \tag{5}$$

where g is a regular function defined on $\partial\Omega$.

We consider the obstacle problem: find $u \in K_g$ such that

$$a(u, v-u) \geq (f, v-u), \quad \forall v \in K_g, \tag{6}$$

Let V_h be the space of finite elements consisting of continuous piecewise linear functions. The discrete counterpart of (6) consists of finding $u_h \in K_{gh}$ such that

$$a(u_h, v-u_h) \geq (f, v-u_h), \quad \forall v \in K_{gh}, \tag{7}$$

where

$$K_{gh} = \{\ v \in v_h : v = \pi_h g \quad \text{on}\partial\Omega, v \leq r_h \Psi \quad \text{in}\Omega\}, \tag{8}$$

π_h is an interpolation operator on $\partial\Omega$, and r_h is the usual finite element restriction operator on Ω. The lemma below establishes a monotonicity property of the solution of (6) with respect to the obstacle and the boundary condition.

Lemma 2.1*Let $(\Psi, g); (\tilde{\Psi}, \tilde{g})$ be a pair of data, and $u = \sigma(\Psi, g); \tilde{u} = \tilde{\sigma}(\tilde{\Psi}, \tilde{g})$ the corresponding solutions of (6). If $\Psi \geq \tilde{\Psi}$ and $g \geq \tilde{g}$, then $\sigma(\Psi, g) \geq \sigma(\tilde{\Psi}, \tilde{g})$.*

*Proof.*let $v = min(0, u-\tilde{u})$. In the region where v is negative $(v < 0)$,we have

$$u < \tilde{u} \leq \psi \leq \tilde{\psi} \tag{9}$$

which means that the obstacle is inactive for u.
Thus, for v, we have

$$a(u, v) = (f, v) \tag{10}$$

$$\tilde{u} + v \leq \tilde{\psi} \tag{11}$$

so

$$a(\tilde{u}, v) = (f, v) \tag{12}$$

Subtracting (10) and (12) from each other, we obtain

$$a(\tilde{u}-u, v) \geq 0 \tag{13}$$

but,

$$a(v, v) = a(u-\tilde{u}, v) = -a(\tilde{u}-u, v) \leq 0 \tag{14}$$

so

$$v = 0 \tag{15}$$

Then,

$$u \geq \tilde{u} \tag{16}$$

which completes the proof.
The proof for the discrete case is similar.

Proposition 2.2*Under the notations and conditions of the preceding lemma, we have*

$$\|u-\tilde{u}\|_{L^\infty(\Omega)} \leq \|\Psi - \tilde{\Psi}\|_{L^\infty(\Omega)} + \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}, \tag{17}$$

*Proof.*Setting

$$\phi \leq \|\psi - \tilde{\psi}\|_{L^\infty(\Omega)} + \|g-\tilde{g}\|_{L^\infty(\partial\Omega)} \tag{18}$$

we have

$$\psi \leq \tilde{\psi} + \psi - \tilde{\psi} \leq \tilde{\psi} + |\psi - \tilde{\psi}| \leq \tilde{\psi} + \|\psi - \tilde{\psi}\|_{L^\infty(\Omega)}$$

$$\tag{19}$$

$$\leq \psi + \|\psi - \tilde{\psi}\|_{L^\infty(\Omega)} + \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}$$

Hence,

$$\psi \leq \tilde{\psi} + \phi \tag{20}$$

On the other hand, we have

$$g \leq \tilde{g} + g - \tilde{g} \leq \tilde{g} + |g-\tilde{g}| \leq \tilde{g} + \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}$$

$$\tag{21}$$

$$\leq g + \|g-\tilde{g}\|_{L^\infty(\partial\Omega)} + \|\psi - \tilde{\psi}\|_{L^\infty(\Omega)}$$

so

$$g \leq \tilde{g} + \phi \tag{22}$$

Now, making use of **Lemma2.1**, we obtain

$$\sigma(\psi, g) \leq \sigma(\psi + \phi, g + \phi) = \sigma(\tilde{\psi}, \tilde{g}) + \phi \tag{23}$$

or

$$\sigma(\psi, g) - \sigma(\tilde{\psi}, \tilde{g}) \leq \phi \tag{24}$$

Similarly, interchanging the roles of the couples (ψ, g) and $(\tilde{\psi}, \tilde{g})$, we obtain

$$\sigma(\tilde{\psi}, \tilde{g}) - \sigma(\psi, g) \leq \phi \tag{25}$$

The proof for the discrete case is similar.

Remark 2.3*if $\psi = \tilde{\psi}$, then (17) becomes*

$$\|u-\tilde{u}\|_{L^\infty(\Omega)} \leq \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}, \tag{26}$$

Theorem 2.4*[10] Under conditions (3) and (4), there exists a constant C independent of h such that*

$$\|u-u_h\|_{L^\infty(\Omega)} \leq ch^2 \log|h|^2, \tag{27}$$

2.2 The continious Schwarz sequences

Consider the model obstacle problem: find $u \in K_0(g=0)$ such that

$$a(u,v) \geq f(v-u) \quad \forall v \in K_0, \tag{28}$$

We decompose Ω into m overlapping subdomains such that

$$\Omega = \bigcup_{i=1}^{m} \Omega_i, \quad \Omega_i \cap \Omega_j \neq \emptyset, \quad i = \overline{1,m}, \quad j = \overline{1,m}, \quad i \neq j \tag{29}$$

and u satisfies the local regularity condition

$$u/\Omega_i \in W^{2,p}(\Omega_i); \quad 2 < p < \infty, \tag{30}$$

We denote by $\partial\Omega_i$ the boundary of Ω_i, and $\Gamma_{ij} = \partial\Omega_i \cap \Omega_j, i \neq j$. The intersection of $\bar{\Gamma_i}$ and $\bar{\Gamma_j}$; $i \neq j$ is assumed to be empty. Choosing $u_0 = \Psi$, we define the alternating Schwarz sequences (u_i^{n+1}) on Ω_i such that $u_i^{n+1} \in K$ solves

$$a_i(u_i^{n+1}, v - u_i^{n+1}) \geq (f_i, v - u_i^{n+1}) \quad \text{in} \quad \Omega_i$$

$$u_i^{n+1} = u_j^{n+1_{ij}} \quad \text{on} \quad \Gamma_{ij} \tag{31}$$

where $i = \overline{1,m}$, $j = \overline{1,m}$, $i \neq j$ and

$$1_{ij} = \begin{cases} 1 & if \quad i > j \\ 0 & if \quad i < j \end{cases}$$

2.3 Geometrical convergence

Theorem 2.5*The sequences (u_1^{n+1}), (u_2^{n+1}),...,(u_m^{n+1}); $n \geq 0$ produced by the Schwarz alternating method converge geometrically to the solution u of the obstacle problem (28). More precisely, there exist m constants k_1, k_2 ,..., $k_m \in (0,1)$, $\forall\ i = \overline{1,m-1}$, $j = \overline{2,m}$, $i < j$ such that for all $n \geq 0$*

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq k_i^n k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ij})} \tag{32}$$

$$\|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} \leq k_i^{n+1} k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ji})}$$

we consider a function $w_l \in L^\infty(\Omega_l)$ continu in $\overline{\Omega_l} \setminus (\overline{\Gamma_l} \cap \partial\Omega)$ such that

$$\begin{cases} \Delta w_l = 0 & in \quad \Omega_l, l = \overline{1,m} \\ w_l = & \begin{cases} 0 & on \quad \partial\Omega_l \backslash \overline{\Gamma_l} \\ 1 & on \quad \Gamma_l \end{cases} \end{cases}$$

and

$$k_t = sup\{w_s(x) \backslash x \in \partial\Omega_t \cap \Omega, t \neq s\} \in (0,1) \tag{33}$$

$$\forall t, s = \overline{1,m}$$

*Proof.*From the principle of the maximum

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq \|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ij})}$$

and

$$\begin{aligned} \|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\leq \|u_j - u_j^n\|_{L^\infty(\Gamma_{ij})} \leq \|w_i u_j - w_i u_j^n\|_{L^\infty(\Gamma_{ij})} \\ &\leq \|w_i u_j - w_i u_j^n\|_{L^\infty(\Omega_j)} \\ &\leq \|w_i u_j - w_i u_j^n\|_{L^\infty(\Gamma_{ji})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|u_j - u_j^n\|_{L^\infty(\Gamma_{ji})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_j - w_j u_j^n\|_{L^\infty(\Gamma_{ji})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_j u_i^n\|_{L^\infty(\Gamma_{ji})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_i u_i^n\|_{L^\infty(\Omega_i)} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_j u_i^n\|_{L^\infty(\Gamma_{ij})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_i - u_i^n\|_{L^\infty(\Gamma_{ij})} \end{aligned}$$

using (33), so

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq k_i k_j \|u_i - u_i^n\|_{L^\infty(\Gamma_{ij})}$$

By induction, we obtain

$$\begin{aligned} \|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\leq k_i^n k_j^n \|u_i - u_i^1\|_{L^\infty(\Gamma_{ij})} \\ &\leq k_i^n k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ij})} \end{aligned}$$

where $u_i^1 = u^0$ on Γ_{ij}, $u = 0$ on $\partial\Omega_i \cap \partial\Omega$
Similarly, we have

$$
\begin{aligned}
\|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} &\le \|u_j - u_j^{n+1}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Omega_i)} \\
&\le \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Gamma_{ij})} \\
&\le \|w_j\|_{L^\infty(\Gamma_{ij})}\|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ij})} \\
&\le \|w_j\|_{L^\infty(\Gamma_{ij})}\|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Gamma_{ij})} \\
&\le \|w_j\|_{L^\infty(\Gamma_{ij})}\|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Omega_j)} \\
&\le \|w_j\|_{L^\infty(\Gamma_{ij})}\|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|w_i\|_{L^\infty(\Gamma_{ji})}\|w_j\|_{L^\infty(\Gamma_{ij})}\|u_j - u_j^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\le k_i k_j \|u_i - u_j^{n}\|_{L^\infty(\Gamma_{ji})}
\end{aligned}
$$

By induction, we obtain

$$
\begin{aligned}
\|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} &\le k_i^n k_j^n \|u_j - u_i^1\|_{L^\infty(\Gamma_{ji})} \\
&\le k_i^n k_j^n \|w_j u_i - w_j u_i^1\|_{L^\infty(\Gamma_{ji})} \\
&\le k_i^n k_j^n \|u_i - u_i^1\|_{L^\infty(\Omega_i)} \\
&\le k_i^n k_j^n \|u_i - u_i^1\|_{L^\infty(\Gamma_{ij})} \\
&\le k_i^{n+1} k_j^n \|u_i - u_i^1\|_{L^\infty(\Gamma_{ij})} \\
&\le k_i^{n+1} k_j^n \|u_j - u_j^0\|_{L^\infty(\Gamma_{ij})} \\
&\le k_i^{n+1} k_j^n \|u_j - u_j^0\|_{L^\infty(\Omega_j)} \\
&\le k_i^{n+1} k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ji})}
\end{aligned}
$$

2.4 The discretization

let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i, h_i is the meshsizes. We assume that every two triangulations are mutually independent on $\Omega_i \cap \Omega_j$, in the sense that a triangle belonging to one triangulation does not necessarily belong to the other, $i = \overline{1,m}$, $j = \overline{1,m}$, $i \neq j$
Let $V_{h_{ij}} = V_{h_{ij}}(\Omega_i)$ be the space of continuous piecewise linear functions on, τ^{h_i} which vanish on $\partial\Omega \cap \Omega_i$.
For $w \in C(\bar{\tau}_i)$ we define

$$
V_{h_{ij}}^{(w)} = \{v \in V_{h_{ij}} : v = 0 \quad \text{on} \partial\Omega_i \cap \partial\Omega; v = \pi_{h_{ij}}(w) \text{on} \quad \Gamma_{ij}\}
$$

where π_{h_i} denotes a suitable interpolation operator on Γ_{ij}
We define the discrete Schwarz sequence:

$$
u_{ih}^{n+1} \in V_{h_{ij}}^{(u_{jh}^{n+1_{ij}})}
$$

solves

$$
a_i(u_{ih}^{n+1}, v - u_{ih}^{n+1}) \ge (f_i, v - u_{ih}^{n+1}) \quad \forall v \in V_{h_{ij}}^{(u_{jh}^{n+1_{ij}})}
$$

$$
u_{ih}^{n+1} \le r_h.\psi, \quad v \le r_h.\psi \tag{34}
$$

3 L^∞-error analysis

3.1 Definition of m auxilairy sequences

For $\omega_0^{ih} = u_0^{ih} = r_h\psi$; $\quad i = \overline{1,m}$, we define the sequences $\omega_{ih}^{n+1} \in V_{h_{ij}}^{(u_j^{n+1_{ij}})}$ such that

$$
a_i(\omega_{ih}^{n+1}, v - \omega_{ih}^{n+1}) \ge (f_i, v - \omega_{ih}^{n+1}) \quad \forall v \in V_{h_{ij}}^{(u_j^{n+1_{ij}})}
$$

$$
\omega_{ih}^{n+1} \le r_h.\psi, \quad v \le r_h.\psi \tag{35}
$$

To simplify the notation, we take

$$
|.|_{ij} = \|.\|_{L^\infty(\Gamma_{ij})}
$$

$$
\|.\|_i = \|.\|_{L^\infty(\Omega_i)} \quad h_{ij} = h \quad \pi_{h_{ij}} = \pi_h \tag{36}
$$

Lemma 3.1*For* $i = \overline{1,m-1}, \quad j = \overline{2,m}, \quad i < j$

$$
\|u_i^{n+1} - u_{ih}^{n+1}\|_i \le \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j \tag{37}
$$

$$
\|u_j^{n+1} - u_{jh}^{n+1}\|_j \le \sum_{p=0}^{n+1} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i
$$

*Proof.*By induction
for $n = 0$, using the discrete version of **Remark** 2.3, we get

$$
\begin{aligned}
\|u_i^1 - u_{ih}^1\|_i &\le \|u_i^1 - \omega_{ih}^1\|_i + \|\omega_{ih}^1 - u_{ih}^1\|_i \\
&\le \|u_i^1 - \omega_{ih}^1\|_i + |\pi_h u_j^0 - \pi_h u_{jh}^0|_{ij} \\
&\le \|u_i^1 - \omega_{ih}^1\|_i + |u_j^0 - u_{jh}^0|_{ij} \\
&\le \|u_i^1 - \omega_{ih}^1\|_i + \|u_j^0 - u_{jh}^0\|_j
\end{aligned}
$$

$$
\begin{aligned}
\|u_j^1 - u_{jh}^1\|_j &\le \|u_j^1 - \omega_{jh}^1\|_j + \|\omega_{jh}^1 - u_{jh}^1\|_j \\
&\le \|u_j^1 - \omega_{jh}^1\|_j + |\pi_h u_i^1 - \pi_h u_{ih}^1|_{ji} \\
&\le \|u_j^1 - \omega_{jh}^1\|_j + |u_i^1 - u_{ih}^1|_{ji} \\
&\le \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - u_{ih}^1\|_i \\
&\le \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - \omega_{ih}^1\|_i \\
&+ \|u_j^0 - u_{jh}^0\|_j
\end{aligned}
$$

So

$$
\|u_i^1 - u_{ih}^1\|_1 \le \sum_{p=1}^{1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{0} \|u_j^p - \omega_{jh}^p\|_j
$$

$$
\|u_j^1 - u_{jh}^1\|_j \le \sum_{p=0}^{1} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{1} \|u_i^p - \omega_{ih}^p\|_i
$$

For $n = 1$, using the discrete version of **Remark** 2.3:

$$
\begin{aligned}
\|u_i^2 - u_{ih}^2\|_i &\le \|u_i^2 - \omega_{ih}^2\|_i + \|\omega_{ih}^2 - u_{ih}^2\|_i \\
&\le \|u_i^2 - \omega_{ih}^2\|_i + |\pi_h u_j^1 - \pi_h u_{jh}^1|_{ij} \\
&\le \|u_i^2 - \omega_{ih}^2\|_i + |u_j^1 - u_{jh}^1|_{ij} \\
&\le \|u_i^2 - \omega_{ih}^2\|_i + \|u_j^1 - u_{jh}^1\|_j \\
&\le \|u_i^2 - \omega_{ih}^2\|_i + \|u_j^1 - \omega_{jh}^1\|_j \\
&+ \|u_i^1 - \omega_{ih}^1\|_i + \|u_j^0 - u_{jh}^0\|_j
\end{aligned}
$$

$$
\begin{aligned}
\|u_j^2 - u_{jh}^2\|_j &\le \|u_j^2 - \omega_{jh}^2\|_j + \|\omega_{jh}^2 - u_{jh}^2\|_j \\
&\le \|u_j^2 - \omega_{jh}^2\|_j + |\pi_h u_i^2 - \pi_h u_{ih}^2|_{ji} \\
&\le \|u_j^2 - \omega_{jh}^2\|_{ij} + |u_i^2 - u_{ih}^2|_{ji} \\
&\le \|u_j^2 - \omega_{jh}^2\|_j + \|u_i^2 - u_{ih}^2\|_i \\
&\le \|u_j^2 - \omega_{jh}^2\|_j + \|u_i^2 - \omega_{ih}^2\|_i \\
&+ \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - \omega_{ih}^1\|_i \\
&+ u_j^0 - u_{jh}^0\|_j
\end{aligned}
$$

So

$$
\|u_i^2 - u_{ih}^2\|_i \le \sum_{p=1}^{2} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{1} \|u_j^p - \omega_{jh}^p\|_j
$$

$$
\|u_j^2 - u_{jh}^2\|_j \le \sum_{p=0}^{2} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{2} \|u_i^p - \omega_{ih}^p\|_i
$$

We suppose that

$$
\|u_j^n - u_{jh}^n\|_j \le \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n} \|u_i^p - \omega_{ih}^p\|_i
$$

Then, using the discrete version of **Remark** 2.3 again:

$$
\begin{aligned}
\|u_i^{n+1} - u_{ih}^{n+1}\|_i &\le \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|\omega_{ih}^{n+1} - u_{ih}^{n+1}\|_i \\
&\le \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + |\pi_h u_j^n - \pi_h u_{jh}^n|_{ij} \\
&\le \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + |u_j^n - u_{jh}^n|_{ij} \\
&\le \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|u_j^n - u_{jh}^n\|_j \\
&\le \|u_i^{n+1} - u_{ih}^{n+1}\|_i + \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j \\
&+ \sum_{p=1}^{n} \|u_i^p - \omega_{ih}^p\|_i
\end{aligned}
$$

Then

$$
\|u_i^{n+1} - u_{ih}^{n+1}\|_i \le \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j
$$

$$
\begin{aligned}
\|u_j^{n+1} - u_{jh}^{n+1}\|_j &\le \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|\omega_{jh}^{n+1} - u_{jh}^{n+1}\|_j \\
&\le \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + |\pi_h u_i^{n+1} - \pi_h u_{jh}^{n+1}|_{ji} \\
&\le \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + |u_i^{n+1} - u_{ih}^{n+1}|_{ji} \\
&\le \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\
&\le \|u_j^{n+1} - u_{jh}^{n+1}\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i \\
&+ \sum_{p=0}^{n} \|u_j^p - \omega_{jh}^p\|_j
\end{aligned}
$$

Then

$$
\|u_j^{n+1} - u_{jh}^{n+1}\|_j \le \sum_{p=0}^{n+1} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i
$$

Lemma 3.2 $\forall\ i = \overline{1, m-1},\ \ j = \overline{2, m},\ \ i < j$*.Then there exists a constantc independent of h and n such that*

$$
\|u_i^{n+1} - u_{ih}^{n+1}\|_i \le 2(n+1)Ch^2|\log h|^3 \tag{38}
$$

$$\|u_j^{n+1} - u_{jh}^{n+1}\|_j \leq (2n+3)Ch^2|\log h|^3$$

*Proof.*By induction
for $n = 0$, using **Theorm 2.4**

$$\begin{aligned}\|u_i^1 - u_{ih}^1\|_i &\leq \|u_i^1 - \omega_{ih}^i\|_i + \|\omega_{ih}^1 - u_{ih}^1\|_i \\ &\leq \|u_i^1 - \omega_{ih}^1\|_i + \|u_j^0 - u_{jh}^0\|_j \\ &\leq ch^2\log|h|^2 + ch^2|\log h|^2 \\ &\leq 2ch^2|\log h|^2\end{aligned}$$

$$\begin{aligned}\|u_j^1 - u_{jh}^1\|_j &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|\omega_{jh}^1 - u_{jh}^1\|_j \\ &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - u_{ih}^1\|_i \\ &\leq ch^2|\log h|^2 + 2ch^2\log|h|^2 \\ &\leq 3ch^2|\log h|^2\end{aligned}$$

Now we suppose that

$$\|u_j^n - u_{jh}^n\|_j \leq (2n+1)Ch^2|\log h|^2$$

$$\begin{aligned}\|u_i^{n+1} - u_{ih}^{n+1}\|_i &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|\omega_{ih}^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|u_j^n - u_{jh}^n\|_j \\ &\leq ch^2|\log\ |^2 + (2n+1)ch^2|\log h|^2 \\ &\leq 2(n+1)ch^2|\log h|^2\end{aligned}$$

$$\begin{aligned}\|u_j^{n+1} - u_{jh}^{n+1}\|_j &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|\omega_{jh}^{n+1} - u_{jh}^{n+1}\|_j \\ &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq ch^2|\log h|^2 + 2(n+1)ch^2|\log h|^2 \\ &\leq (2n+3)ch^2|\log h|^2\end{aligned}$$

3.2 L^∞ error estimate

Theorem 3.3*Let $h = max(h_i, h_j)$, $i = \overline{1, m-1}$; $j = \overline{2, m}$ and $i < j$. Then, there exists a constant C independent of both h and n such that*

$$\|u_M - u_{Mh}^{n+1}\|_{L^\infty(\Omega_M)} \leq Ch^2|\log h|^3; \quad M = \overline{i, j} \tag{39}$$

*Proof.*Let us give the proof for $M = i$. The case $M = j$ is similar.
For $N = i$, let $k = max(k_i, k_j)$
Using **Theorem 2.5**, **lemma 3.2**, we obtain

$$\begin{aligned}\|u_i - u_{ih}^{n+1}\|_i &\leq \|u_i - u_i^{n+1}\|_i + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq k^{2n}|u - u^0|_{ij} + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq k^{2n}|u - u^0|_{ij} + \sum_{p=1}^{n+1}\|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^{n}\|u_j^p - \omega_{jh}^p\|_j \\ &\leq k^{2n}|u - u^0|_{ij} + 2(n+1)Ch^2|\log h|^2\end{aligned}$$

We suppose that

$$k^{2n} \leq h^2$$

we obtain

$$\|u_i - u_{ih}^{n+1}\|_i \leq Ch^2|\log h|^3$$

4 Conclusion

In this work, we have established an approach of the alternating Schwarz algorithm for m overlaping subdomains with nonmatching grids, for the class of elliptic variational inequality. This type of estimation which we have obtained relies on the geometrical convergence and the error estimate between the continuous and discrete Schwarz iterates. We contend that this result plays an important role in the study of the numerical analysis for the class of elliptic variational inequality in the context overlapping nonmatching grids using the parallel Schwarz method.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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