Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150104

Mathematical Model for Hyperbolic Two Temperature Fractional-Order Thermoelastic Materials Subjected to Thermal Loading

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Received: 2 Oct. 2020, Revised: 2 Nov. 2020, Accepted: 6 Dec. 2020
Published online: 1 Jan. 2021

Abstract: The behaviour of a homogeneous and isotropic thermoelastic semi-infinite material is investigated based on the accelerations of the conductive and thermodynamical temperature. A half-space $x > 0$ under stress free boundary condition at x = 0 and subjected to a thermal loading represented by a heavy sidestep function is considered. A one- dimensional system of equations in the framework of fractional order generalised thermoelasticity theory is considered as well. Laplace transform is used to get the solution in the Laplace domain. Thermally induced temperature, stress and strain distribution functions are determined in the Laplace domain. The Riemann-sum approximation method is used to obtain the different inverse field functions numerically. The behaviour of the stress, strain and the heat conductive temperature with the fractional-order parameter and time are investigated and presented graphically. Comparisons with the classical two-temperature models are discussed.

Keywords: Fractional-order strain, Fractional-order equation of motion, Generalized Thermoelasticity, Hyperbolic two-temperature, Thermal Loading

Nomenclature:
The following notations will be used throughout the present work:
φ^h: Conductive temperature in the hyperbolic two-temperature model
φ^p: Conductive temperature in the classical two-temperature model
σ^h: Principal stress component in the hyperbolic mode
σ^p: Principle stress component in the classical model
e^h: Cubic dilatation in the hyperbolic two-temperature model
e^p: Cubic dilatation in the classical two-temperature model
C_E: Specific heat at constant strain
c_o: Longitudinal wave speed
T: Absolute temperature
T_o: Reference temperature
t: Time
u_i: Components of displacement vector
α: Two-temperature parameter.
α_T: Coefficient of linear thermal expansion
ε: dimensionless mechanical coupling constant
λ, μ: Lame's constants
ρ: Mass density
τ_o Relaxation time
β: Fractional-order parameter
Γ: Gamma function
$\theta = T - T_0$: Thermodynamic temperature increment such that $\theta = T - T_o$

1 Introduction

In Thermoelasticity the heat conduction in deformable bodies arises from the conductive and thermodynamical temperatures [1], [2]. It is noticeable that in case of time dependent situation when there is no supply of heat the two-temperatures are the same. Where as in case of time dependent situation the two temperatures are different. Some more details of such studies can be found in [3],[4]. Youssef has defined the variance theory and the

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uniqueness of the initial boundary value problem in the generalized thermoelasticity with two-temperatures in separate situations [5]-[7]. To remove the paradox of heat conduction present in the two-temperature thermoelasticity theory which admits infinite speed of signals, Youssef enhanced the thermoelasticity theory based on conductive and thermodynamic temperature by assuming a hyperbolic form of the two-temperature relation [8]. The concept of derivative and integral has been generalized to a non-integer order and multiple researchers addressed it [9]-[15]. Various physical processes and models have been implemented through the application of fractional-order derivatives. Applications of the fractional-order theory and other contributions have been published by many researchers [16]- [24] The fractional-order thermoelasticity becomes more realistic when it relies on the fractional-order operator because the presence of the fractional-order derivatives permits the differential equations of the system to consider the effects of the intermediate as well as the previous states to express the present and the next states of the medium. One of the most famous definitions of fractional-order was introduced by Riemann-Liouville [14] as follows:

$$ {}_{RL}D_t^{\beta} f(t) = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} f(\tau) d\tau\right], \quad n-1<\beta<n \tag{1} $$

The second definition was presented in [14] and given by:

$$ {}_{C}D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} \frac{d^n f(\tau)}{d\tau^n} d\tau \quad n-1<\beta<n \tag{2} $$

These two definitions are the same if $f(0)=0$. More details about the comparison between the two definitions of the fractional-order time derivative introduced by Riemann-Liouville and that of Caputo as well as various definitions and works of fractional- order derivatives were reported in [22]. Based on the new theory of the hyperbolic two-temperature generalized thermoelasticity introduced by Youssef [8] the present work can be considered as a generalization of the application studied in [8] and more realistic as the present model contains fractional- order derivatives in both equations of motion as well as the heat equation. In the present work we will use the following equation:

$$ L_C D_t^{\beta} f(t) = s^{(\beta-n)} L f^n(t), \qquad n-1<\beta<n, \tag{3} $$

as in [21] to investigate the behaviours of a thermoelastic isotropic and homogeneous half-space subjected to a thermal loading represented by a heavy side step function at the end $x=0$. In Eq. (3), s is the complex parameter connected to Laplace transform.

2 One Dimensional Thermoelastic Model

For the present model we presume the following one-dimensional fractional-order system of equations which can describe the overall behaviour of a semi-infinite one-dimensional homogeneous isotropic material occupying the half- space $x \geq 0$ and subjected to thermal loading at the end $x=0$. The three- dimensional forms of this system are present in Youssef [8]. We assume that the material is subjected to thermal loading and stress-free at the end $x=0$. All the field functions are initially set at zero. We also presume that no body force is applied to the medium. When no inner heat sources and charges are present, the generalized thermoelastic one dimensional system of differential fractional-order equation assumes the following equations:
The conductive heat equation:

$$ K\left(\frac{\partial^2 \varphi(x,t)}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)\left(\rho C_E\, \theta(x,t) + T_o \gamma (1+\tau^{\beta} D_t^{\beta}) e(x,t)\right) \tag{4} $$

Equation of motion:

$$ \rho \frac{\partial^2 e(x,t)}{\partial t^2} = (\lambda+2\mu)(1+\tau^{\beta} D_t^{\beta}) \frac{\partial^2 e(x,t)}{\partial x^2} - \gamma \frac{\partial^2 \theta(x,t)}{\partial x^2} \tag{5} $$

and the stress-strain constitutive equations take the forms:

$$ \sigma(x,t) = (1+\tau^{\beta} D_t^{\beta})(\lambda+2\mu) e(x,t) - \gamma \theta(x,t) \tag{6} $$

and

$$ e(x,t) = \frac{\partial u(x,t)}{\partial x}. \tag{7} $$

Instead of the classical two-temperature relation between the heat conduction φ and the thermodynamical temperature θ given by:

$$ \theta = \varphi - \alpha \frac{\partial^2 \varphi}{\partial x^2} \tag{8} $$

we used the following hyperbolic relation as given in[8]:

$$ \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial^2 \varphi}{\partial t^2} - \alpha \frac{\partial^2 \varphi}{\partial x^2} \tag{9} $$

3 Dimensionless System of Equations in Laplace Domain

For converting the previous system of Eqs.(4)-(9) into dimensionless system we used the set of dimensionless variables as in [8] and dropping the primes for convenience, we get the following non-dimensional system of equations:

$$\frac{\partial^2\varphi(x,t)}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2}\right)(\theta(x,t) + \xi\varepsilon_1(1+\tau^\beta D_t^\beta)e(x,t)) \tag{10}$$

The dimensionless fractional-order differential equation of strain takes the form:

$$\frac{\partial^2 e(x,t)}{\partial t^2} = (1+\tau^\beta D_t^\beta)\frac{\partial^2 e(x,t)}{\partial x^2} - \omega\frac{\partial^2\theta(x,t)}{\partial x^2} \tag{11}$$

The constitutive equations take the following forms:

$$\sigma(x,t) = (1+\tau^\beta D_t^\beta)e(x,t) - \omega\theta(x,t) \tag{12}$$

and

$$e(x,t) = \frac{\partial u(x,t)}{\partial x} \tag{13}$$

The hyperbolic two-temperature non-dimensional equation becomes:

$$\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\theta}{\partial t^2} - \alpha\frac{\partial^2\varphi}{\partial x^2} \tag{14}$$

Applying the Laplace transform defined by:

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \tag{15}$$

together with Caputo definition (3) to the system of Eqs. (10)-(14) we get the following none dimensional system of equations in Laplace domain:

$$\frac{\partial^2\overline{\varphi(x,s)}}{\partial x^2} = s(1+s\tau_o)[\overline{\theta}(x,s) + \varepsilon_1\xi(1+s^\beta\tau^\beta\beta)\overline{e}(x,s)] \tag{16}$$

$$\frac{\partial^2\overline{e(x,s)}}{\partial x^2} = \frac{1}{s^2(1+s^\beta\tau^\beta)}[s^4\overline{e(x,s)} + \omega(s^2\frac{\partial^2\overline{\varphi(x,s)}}{\partial x^2} - \alpha\frac{\partial^4\overline{\varphi(x,s)}}{\partial x^4})] \tag{17}$$

$$\overline{\sigma(x,s)} = (1+s^\beta\tau^\beta)\overline{e}(x,s) - \frac{\omega}{s^2}(s^{-2}\overline{\varphi(x,s)} - \alpha\frac{\partial^2\overline{\varphi(x,s)}}{\partial x^2}) \tag{18}$$

and the relation between the two types of temperature:

$$\overline{\theta(x,ps)} = \overline{\varphi(x,s)} - s^{-2}\alpha\frac{\partial^2\overline{\varphi(x,s)}}{\partial x^2}) \tag{19}$$

Combining Eqs. (17) and (18) gives:

$$\frac{\partial^2\overline{\sigma(x,s)}}{\partial x^2} = s^2\overline{e(x,s)} \tag{20}$$

where $\varepsilon_1 = \frac{\varepsilon_2}{s+\Omega}$.
Eqs. (16)-(20) represent the non-dimensionless governing equations of the present one - dimensional fractional-order thermoelastic model in the light of generalized fractional-order thermoelasticity with hyperbolic two-temperature equation.

4 The Solutions in the Laplace Domain

Eliminating $\overline{e}(x,s)$ between the Eqs. (16) and (17), we get the following fourth order non-homogeneous differential equation;

$$N\,\overline{\varphi(x,s)} - M\,\frac{\partial^2\overline{\varphi(x,s)}}{\partial x^2} + \frac{\partial^4\overline{\varphi(x,s)}}{\partial x^4} = 0, \tag{21}$$

where

$$M = \frac{s^2\left(\alpha + \left(\tau^\beta s^\beta(s\tau_o+1)+1\right)(\xi\omega\varepsilon_1+1) + s\,L\right)}{\alpha\left(\tau^\beta s^\beta+1\right)(s\tau_o+1)(\xi\omega\varepsilon_1+1)}$$

$$N = \frac{s^4(s\tau_o+1)}{\alpha\left(\tau^\beta s^\beta+1\right)(s\tau_o+1)(\xi\omega\varepsilon_1+1)},$$

where $L = \tau_o(\alpha+\xi\omega\varepsilon_1+1)+1)$ The most general solution of (21) according to the current formulation of the problem takes the form;

$$\overline{\varphi(x,s)} = \sum_{i=1}^{2} C_i e^{-k_i x}, \tag{22}$$

where C_i are coefficients depending on s whose values can be evaluated using the boundary conditions, $\pm k_i$ are the roots of the characteristic equations corresponding to Eq. (21), which is;

$$N - M\,k^2 + k^4 = 0.$$

After some manipulations to the system of Eqs. (16)- (19) we get the following general solutions of the physical quantities of the present model in the domain of Laplace; The thermodynamical temperature assumes the form:

$$\overline{\theta(x,s)} = \sum_{i=1}^{2} C_i\,e^{-k_i x}(1 - \frac{\alpha k_i^2}{s^2}) \tag{23}$$

The strain and the stress in the domain of Laplace takes the form:

$$\overline{e(x,s)} = \sum_{i=1}^{2}\frac{C_i e^{-k_i x}}{\varepsilon}(-1 + k^2{}_i(\frac{s^{-\xi}}{1+s\tau_o} + \frac{\omega}{s^2})) \tag{24}$$

$$\overline{\sigma(x,s)} = \sum_{i=1}^{2} C_i e^{-k_i x}[\beta(-1+\frac{k_i^2}{s^2}) + \frac{-1+k_i^2(\frac{s^{-\xi}}{1+s\tau_o}+\frac{\omega}{s^2})}{\varepsilon}] \tag{25}$$

Eqs. (22) -(25) represent the complete solution of the system (16)-(20) in the Laplace transform domain.

5 Determination of the Parameters

To define the previous parameters, the following initial conditions have been provided as well as the medium is

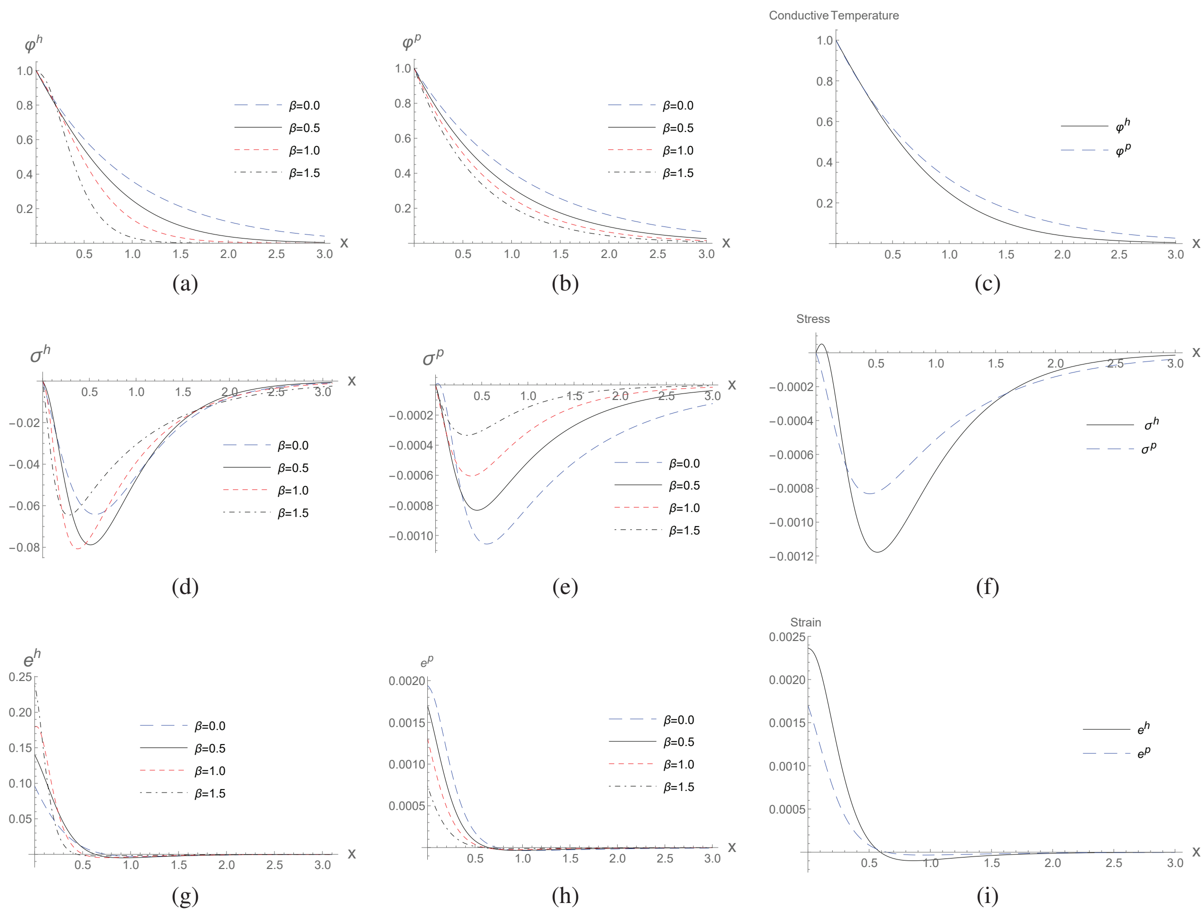

Fig. 1: Effect of fractional-order parameter β on φ, σ and e at $t = 0.2$, $\tau_o = 0.1$ (a) hyperbolic conductive temperature; (b) parabolic conductive temperature; (c) hyperbolic and parabolic conductive temperature; (d) stress in hyperbolic case; (e) stress in parabolic case; (f) hyperbolic and parabolic stress; (g) strain in hyperbolic case; (h) strain in parabolic case; (i) hyperbolic and parabolic strain.

set at rest initially and has reference temperature T_o, so the initial conditions are given by;

$$\theta(x,0) = 0, \quad \sigma(x,0) = 0, \quad \partial\theta(x,0)/\partial t = 0, \quad \partial\sigma(x,0)/\partial t = 0, \tag{26}$$

we also presume that the medium undergoes the following boundary conditions at the close end $x = 0$;

$$\varphi(0,t) = \varphi_o H(t), \quad \sigma(0,t) = 0, \tag{27}$$

where $H(t)$ is the Heaviside step function and φ_o is the strength of the thermal loading, while at $x = \infty$, the boundary conditions take the form:

$$\varphi(\infty,t) = 0, \quad , \quad \sigma(\infty,t) = 0, \qquad 0 < t < \infty, \tag{28}$$

Applying the Laplace transform to Eqs. (27) and (28) we obtain the following dimensionless form of the boundary conditions:

$$\overline{\varphi}(0,s) = \varphi_o/s, \qquad \overline{\sigma}(0,s) = 0,$$
$$\overline{\theta}(\infty,s) = 0, \qquad \overline{\sigma}(\infty,s) = 0, \tag{29}$$

Similarly the dimensionless initial conditions in the domain of Laplace can be obtained. By applying these conditions to (22)-(25), the constants C_i, e_i can be obtained as given below:

$$C_1 = \frac{\varphi_o(\beta\ \varepsilon+1)s^{\xi}(s\tau_o+1)\left(s^2-k_2^2\ \omega\right)-k_2^2\ s^2\varphi_o}{(k_1^2-k_2^2)\left(\omega\ (\beta\ \varepsilon+1)\ s^{\xi}(s\ \tau_o+1)+s^2\right)};$$
$$C_2 = \frac{\varphi_o(\beta\ \varepsilon+1)s^{\xi}(s\tau_o+1)\left(s^2-k_1^2\ \omega\right)-k_1^2\ s^2\varphi_o}{(k_1^2-k_2^2)\left(\omega\ (\beta\ \varepsilon+1)\ s^{\xi}(s\ \tau_o+1)+s^2\right)}; \tag{30}$$

After substituting with the constants given by Eq. (30) into Eqs. (22)-(25), we obtain the complete solution in the Laplace domain of the non-dimensional field functions; temperature, stress and strain respectively.

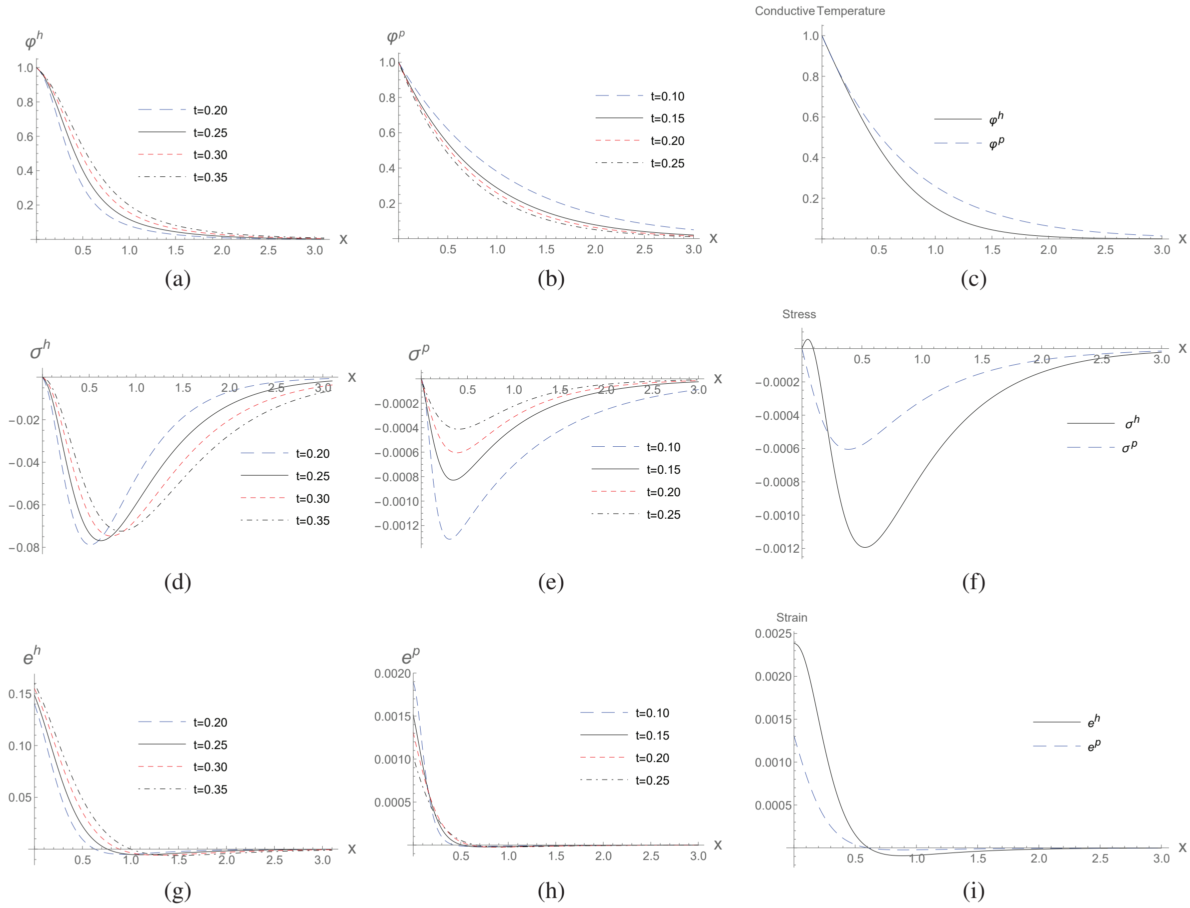

Fig. 2: Effect of time t on φ, σ and e at $\beta = 0.5$, $\tau_o = 0.1$
(a) hyperbolic conductive temperature; (b) parabolic conductive temperature; (c) hyperbolic and parabolic conductive temperature; (d) stress in hyperbolic case; (e) stress in parabolic case; (f) hyperbolic and parabolic stress; (g) strain in hyperbolic case; (h) strain in parabolic case; (i) hyperbolic and parabolic strain.

6 Numerical Form of the Inversion of the Laplace Transform

The physical quantities $\varphi(x,t)$, $\sigma(x,t)$ and $e(x,t)$ can be obtained by inverted the system of Eqs.(22)-(25) back to the time domain. Therefore, we use a numerical equation based on the expansion of Fourier. In this technique any function $\overline{f}(s)$ is inverted back to the original function $f(t)$ in the time domain as given below:

$$f(t) = \frac{\exp(ct)}{t_1}[\frac{1}{2}\overline{f}(c)+\Re(\sum_{1}^{N}\overline{f}(c+\frac{ik\pi}{t_1}\exp(\frac{ik\pi}{t_1})], \quad 0 < t_1 < 2t, \tag{31}$$

where $\Re$ is the real part, i is imaginary number unit and N is a sufficiently large integer representing the number of terms in the truncated Fourier series chosen such that:

$$\exp(ct)\Re\left[\overline{f}(c+\frac{iN\pi}{t_1})\exp(\frac{iN\pi t}{t_1})\right] \leq \varepsilon_1, \tag{32}$$

where ε_1 is a small positive number that represents the degree of accuracy required. The parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\overline{f}(s)$. The optimal choice of c was obtained according to the criteria described in Honig and Hirdes [25]. Details about the analysis of the formula (31) are present in [26].

7 Discussion of The Results

For numerical computations, we used the physical constants of the Copper material used in [21]. We investigate the distributions of the field functions; (i.e. φ, σ and e) for different values of the parameters δ , β and t

and compare their behaviour with the corresponding physical quantities φ^c, σ^c and e^c. The results are collected in groups of figures; each group presents the effect of one of the mentioned parameters on the physical quantities.

Figure (1) illustrates the effects of the fractional-order parameter β on the field functions. In Fig.1 (a) and (b) we noticed that both of the hyperbolic φ^h and parabolic conductive temperature φ^c are inversely proportional to the variation of the fractional-order parameter β. Figure 1(c) shows the comparison between the two types of the heat conduction; φ^h and φ^P; where it is noticeable that the hyperbolic heat conduction φ^h is asymptotically stable, while the heat conduction in a model with classical two-temperature φ^P has a local asymptotic stability. Figures 1 (d), (e) and (f) illustrate the effects of fractional-order parameter β on the stresses σ^h and σ^P. We noticed that the field functions changes significantly with the variation of the fractional-order parameter. Figures 1(d) and 1(e) indicate that the absolute value of the stress magnitude varies inversely with the variance of the fractional-order parameter. In Fig. 1(f), it is noticed that the absolute value of the stress amplitude σ^h is less than the stress in the classical two-temperature model σ^P and the stress σ^h attains its equilibrium state before σ^P. Figures 1(g), (h) and (i) illustrate the effects of the fractional-order parameter on the strain. It is noticeable that the strain in the two forms of heat conduction varies inversely with the variance of the fractional-order parameter.

Figures (2) represent the variations of the field function under the changes of time t. Figure 3(a) and 3(b) shows inverse proportionality between the two types of heat conduction with the variance of time t. Figure 2(c) shows that the hyperbolic heat conduction φ^h is asymptotically stable while the the heat conduction φ^P has a local stability. Figures 2(d), 2(e) and 2(f) show the variation of the stress with the variation of time t. It is noticed that the stress under the two types of heat conduction resembles that under the variation of the fractional-order parameter β. Figures 2(g) and 2(h) illustrate the variation of the strain with the variation of time t. It is noticeable that the strain under the two models of two-temperature varies inversely with the variation of time unlike the variance of the strain with the variance of the fractional- order parameter β.

In comparison with the model presented in [8] our model is more acceptable because [8] involves a sudden drop of temperature to zero degree in figures (1) and (2) which is impossible. However, in the present model the temperature drops to zero degree asymptotically.

8 Conclusion

We noticed that the field function φ, σ and e has an asymptotic stability in the hyperbolic case while they have local stability in the parabolic two-temperature model. The field functions in the hyperbolic two-temperature model attained their equilibrium state faster than the field function in the parabolic two-temperature model. Moreover, all field functions were inversely proportional to the variance of the fractional-order parameter and attained the equilibrium state at the same point $\simeq 3.0$.

Acknowledgment

The author is grateful to the Deanship of Scientific Research, Prince Sattam Bin Abdulaziz University for extending support and facilities that helped accomplish this paper.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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