Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150105

Global Stability of a Delayed Fractional-Order *SEI* Epidemic Model with Logistic Growth

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Received: 3 May 2020, Revised: 10 Oct. 2020, Accepted: 17 Oct. 2020
Published online: 1 Jan. 2021

Abstract: In this paper, we investigate the qualitative behavior of a fractional-order susceptible-exposed-infected (SEI) model with logistic growth and time delay. In the proposed epidemic model, we assumed that the susceptible individuals grow logistically and introduced time delay in latent infected individuals equation. The basic reproduction number R_0 is derived using next generation matrix to study the dynamics of the disease free and endemic equilibrium points of the system. Based on the characteristic equations and conditions of the stability of fractional-order differential systems, local stability of the two equilibrium points is discussed. Furthermore, a suitable Lyapunov function is proposed to investigate the global stability of equilibrium points. The results demonstrated that the fractional-order derivatives enriched the dynamical behavior of the epidemic system. Moreover, in the fractional-order case, the stability region of the equilibrium points increased. The theoretical results are verified by numerical simulations.

Keywords: Epidemic model, Exposed state, Fractional-order, Global stability, Time-delay.

1 Introduction

Mathematical models have been adopted in epidemiology to analyze the dynamics and control of many infectious diseases [1, 2] based on a strong understanding of biological interactions. It provides non-intuitive perspectives into the dynamics of host responses to pathogens agents and can suggest new avenues for experimentation. Infections caused by a variety of infectious organisms, such as parasites, viruses, bacteria and fungi, are a significant risk to health. Infectious diseases continue to be a major cause of the suffering and death of the kingdom of plants and animals with new and re-emerging pathogens [3–7]. Most mathematical models for transmission of infectious diseases are derived from the classical (SIR) model of Kermack and McKendrick [8] and have been defined as integer-order differential equations. Physical, engineering and biological issues can be described as fractional-order differential equations [9–12]. Modeling using fraction differential equations is an appropriate tool to get a better understanding of the complicated mechanism of diseases [13–17]. It is used to help us identify the history of the disease, so we can remove the weaknesses emerging from the neglected parameters in modeling real-life phenomena.

Numerous diseases, such as chickenpox, acquired immune deficiency syndrome and tuberculosis, have an exposed or latent stage comprised of tainted individuals but not infectious. In some cases, the pathogen remains in an infected host in a latent phase for many years and can be reactivated to cause the disease, so exposed state is considered an essential obstacle to recovery [18, 19]. In this paper, we address these diseases, then we present a *SEI* model. Also, in population dynamics, a maturation time delay is used to explain observed oscillations and reflect some biological facts, such as immunity period and the latent period of the disease, which make the model more realistic. Time delay is a certain amount of time that will elapse before infection in the infected individuals or vector reaches a sufficiently high level for further transmission of the infection.

This article is structured, as follows: In the next section, we will present a brief summary of fractional calculus and some basic definitions that will be used in the following sections. We define and formulate the

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delayed fractional-order (SEI) epidemic model with logistic growth and bilinear incidence rate in section 3. Section 4 covers obtaining the equilibrium points and calculating the basic reproduction number R_0 which controls the infection. Local and global stability is addressed in section 5. In section 6, we introduce some numerical simulation by Matlab program to enhance the theoretical results. Results and conclusion are presented in the last section.

2 Preliminaries

In this section, we introduce the basic definitions and lemma of fraction calculus which is an important tool in modeling processes of biological systems and can provide an exact description not only of the current state of the disease but also of all its historical states.

Definition 2.1. Define a function $f : [0,\infty) \longrightarrow R$, then its fractional integral of order $\alpha \in (0,1]$ is given, as follows:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx,$$

where $\Gamma(.)$ is the gamma function [20], and the Caputo fractional derivative of order α is given by:

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t), \tag{1}$$

where $n-1 < \alpha \leq n$ and $f(t)$ is a continuous function [21]. In particular, when $0 < \alpha \leq 1$, one has

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(x)}{(t-x)^{\alpha}} dx,$$

For more properties of the fraction order derivatives, (see e.g. [22] and [23]).

Lemma 2.1. Consider a fraction order system

$$D^{\alpha}(x) = f(x), \qquad x(0) = x_0, \tag{2}$$

with $0 < \alpha \leq 1$ and $x \in \{R^n\}$, evaluate the equilibrium points of the system (2) in case $D^{\alpha}(x) = 0$, then these points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix of the system evaluated at the equilibrium points satisfy the following conditions: [24]

$$|\arg(\lambda_i)| > \alpha \frac{\pi}{2}. \tag{3}$$

3 Model Formulation

Based on Caputo derivative, we propose a fractional-order susceptible-exposed-infected SEI epidemiological model. We consider SEI model with logistic growth and the time delay describing latent period of the infected individuals, so the model will be, as follows:

$$\begin{aligned} D^{\alpha} S(t) &= S(t)(r - aS(t)) - bS(t)I(t), \\ D^{\alpha} E(t) &= be^{-m\tau} S(t-\tau) I(t-\tau) - \gamma E(t) - \mu E(t), \\ D^{\alpha} I(t) &= \gamma E(t) - \mu I(t), \end{aligned} \tag{4}$$

where D^{α} is the Caputo fractional derivative with $0 < \alpha \leq 1$ which is the order of differential operator, $S(t)$ is the number of susceptible individuals of transmitted disease at time t, $E(t)$ is the number of exposed individuals at time t and the number of contagious individuals (who are infected and can spread the disease) at time t is $I(t)$. The parameters b, γ and μ are positive constants, where b is the disease transmission coefficient, $\frac{1}{\gamma}$ is the disease incubation period and μ is the death rate per capita. In system (4), we assume that the susceptible class only is capable of reproduction where it grows logistically with carrying capacity $\frac{a}{r}$ and r is the intrinsic birth rate. The initial conditions of system (4) are

$$S(\theta) = \Psi_1(\zeta), \quad E(\theta) = \Psi_2(\zeta), \quad I(\theta) = \Psi_3(\zeta), \tag{5}$$

and

$$\theta \in [-\tau, 0],$$

where $(\Psi_1(\zeta), \Psi_2(\zeta), \Psi_3(\zeta)) \in C([-\tau,0], R^3_{\geq 0})$, C is the Banach space of continuous functions mapping the interval $[-\tau,0]$ into $R_{\geq 0}$ where the delay period of the disease is represented by the parameter $\tau \geq 0$, and the term $be^{-m\tau} S(t-\tau) I(t-\tau)$ represents the individuals who survived in the latent phase and became infected at time t, where m is a positive constant. At any point of time t, the total number of population is $T(t) = S(t) + E(t) + I(t)$ and system (4) has the domain:

$$\Omega = \{(S,E,I) \in R^3_{+0}, S \geq 0, E \geq 0, I \geq 0\}.$$

3.1 Positivity and boundedness

First, we will show that all the solutions of system (4) with initial condition (5) are nonnegative and uniformly bounded motivated by the arguments of references [25, 26] as shown in the next proposition:

Proposition 3.1.1. The solutions of the system (4) are nonnegative and there exist positive constants M_1 and M_2 such that for any solution $S(t), E(t), I(t) > 0$ of the system (4) satisfies that $S(t) \leq M_1$, $E(t) \leq M_2$, and $I(t) \leq M_2$.

Proof. For the system (4), we have

$$\begin{aligned} S(t) &\geq \Psi_1(0), \\ E(t) &\geq \Psi_2(0) e^{-(\gamma+\mu)t}, \\ I(t) &\geq \Psi_3(0) e^{-\mu t}. \end{aligned}$$

Therefore, $S(t), E(t), I(t) \geq 0$ for all $t \geq 0$. To prove the boundedness, let us define

$$N(t) = e^{-m\tau} S(t-\tau) + E(t) + I(t)$$

From system (4), we find that the fractional order derivative of $N(t)$ is

$$\begin{aligned} D^{\alpha}N(t) &= e^{-m\tau}D^{\alpha}S(t-\tau)+D^{\alpha}E(t)+D^{\alpha}I(t)\\ &= e^{-m\tau}S(t-\tau)(r-aS(t-\tau))-\mu E-\mu I\\ &= e^{-m\tau}S(t-\tau)(r-aS(t-\tau)+\mu)-\mu N(t)\\ &= (-aS^2(t-\tau)+(r+\mu)S(t-\tau))e^{-m\tau}-\mu N(t)\\ &= -a\left(S(t-\tau)-\frac{r+\mu}{2a}\right)^2 e^{-m\tau}+\frac{(r+\mu)^2}{4a}e^{-m\tau}\\ &\quad -\mu N(t)\\ &\leq \frac{(r+\mu)^2}{4a}e^{-m\tau}-\mu N(t), \end{aligned}$$

which gives

$$\limsup_{t\to+\infty} N(t) \leq \frac{(r+\mu)^2}{4a\mu e^{m\tau}}. \tag{6}$$

Hence,

$$\limsup_{t\to+\infty} S(t) \leq M_1 = \frac{(r+\mu)^2}{4a\mu},$$

$$\limsup_{t\to+\infty} E(t) \leq M_2 = \frac{(r+\mu)^2}{4a\mu e^{m\tau}},$$

and

$$\limsup_{t\to+\infty} I(t) \leq M_2 = \frac{(r+\mu)^2}{4a\mu e^{m\tau}}$$

for all $t \geq 0$ and this completes the proof of Proposition 3.1.1.

4 Equilibria and Basic reproduction number

In this section, we discuss the equilibrium points and the basic reproduction number of system (4). The equilibrium points of system (4) should satisfy

$$D^{\alpha}(S)=0, \qquad D^{\alpha}(E)=0, \qquad D^{\alpha}(I)=0. \tag{7}$$

Then, system (4) has two equilibrium points, the disease free equilibrium point is $E_0=(\frac{r}{a},0,0)$ and the endemic equilibrium point which keeps the disease propagation is $E_1=(S^*,E^*,I^*)$, where

$$S^*=\frac{\mu(\mu+\gamma)e^{m\tau}}{\gamma b}, \quad I^*=\frac{r}{b}-\frac{a\mu(\mu+\gamma)e^{m\tau}}{\gamma b^2}, \quad E^*=\frac{\mu I^*}{\gamma}.$$

Letting $X=(S,E,I)^T$, system (4) can be written as

$$D^{\alpha}(X)=\varphi(X)-\psi(X),$$

where

$$\varphi(X)=\begin{pmatrix} 0\\ be^{-m\tau}SI\\ 0\end{pmatrix}, \qquad \psi(X)=\begin{pmatrix} bSI-S(r-aS)\\ \gamma E+\mu E\\ \mu I-\gamma E\end{pmatrix}.$$

Using the next generation matrix method and the notations in [27], we will evaluate basic reproduction number R_0 for system (4). The special matrices F for new infections terms and V for the other terms are given by

$$F=\begin{pmatrix} 0 & 0 & 0\\ be^{-m\tau}I & 0 & be^{-m\tau}S\\ 0 & 0 & 0\end{pmatrix},$$

$$V=\begin{pmatrix} bI-r+2aS & 0 & bS\\ 0 & \gamma+\mu & 0\\ 0 & -\gamma & \mu\end{pmatrix},$$

then, we obtain the next generation matrix FV^{-1} for system (4) at the equilibrium point $E_0=\left(\frac{r}{a},0,0\right)$, as follows:

$$FV^{-1}=\begin{pmatrix} 0 & 0 & 0\\ 0 & \frac{br\gamma}{a\mu(\mu+\gamma)}e^{-m\tau} & \frac{br}{a\mu}e^{-m\tau}\\ 0 & 0 & 0\end{pmatrix}.$$

R_0 is the spectral radius of the generation matrix, then

$$R_0=\rho(FV^{-1})=\frac{br\gamma e^{-m\tau}}{a\mu(\mu+\gamma)}.$$

Accordingly, we can reformulate the endemic equilibrium point as

$$\begin{aligned} S^*=\frac{\mu(\mu+\gamma)e^{m\tau}}{\gamma b}, \qquad I^*=\frac{r}{b}\left(1-\frac{1}{R_0}\right),\\ E^*=\frac{\mu r}{\gamma b}\left(1-\frac{1}{R_0}\right). \end{aligned} \tag{8}$$

Thus, for system (4), if $R_0 \leq 1$, then the system has only disease free equilibrium E_0. If $R_0>1$, then there exists the endemic equilibrium E_1 for system (4) in addition to E_0 and we can easily see that

$$\tau_{max}=\frac{1}{m}\ln\frac{rb\gamma}{a\mu(\mu+\gamma)}, \tag{9}$$

Hence, $R_0>1$ implies $0\leq\tau<\tau_{max}$.

5 Stability review

This section addresses local and global stability of a delayed fractional-order SEI epidemic model.

5.1 Local stability of equilibrium points

We handle the local stability of each equilibria of system (4) where the related characteristic equation is analyzed.

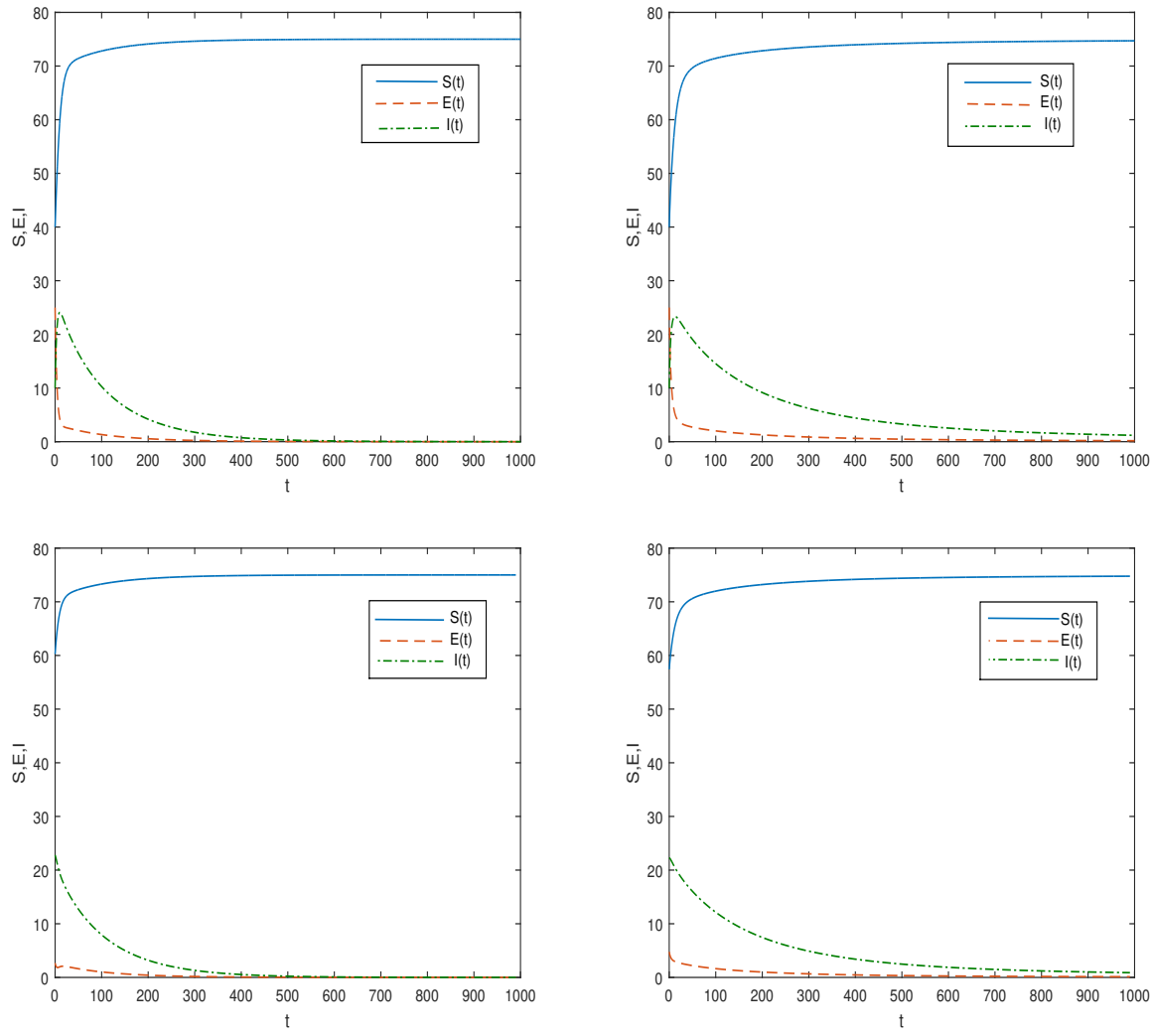

Fig. 1: Stability of disease-free equilibrium point. Left figures with $\alpha = 1$, $\tau = 0$(above) and $\alpha = 1$, $\tau = 10$(down), right figures with $\alpha = 0.9$, $\tau = 0$(above) and $\alpha = 0.9$, $\tau = 10$(down).

Theorem 5.1.1. The disease free equilibrium $E_0 = \left(\frac{r}{a}, 0, 0\right)$ of system (4) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof. The characteristic matrix of system (4) evaluated at E_0 is

$$\triangle(\lambda) = \begin{pmatrix} \lambda + r & 0 & \frac{br}{a} \\ 0 & \lambda + (\gamma + \mu) & -\frac{br}{a} e^{-m\tau} e^{-\lambda\tau} \\ 0 & -\gamma & \lambda + \mu \end{pmatrix}.$$

Then, the characteristic equation can be written as

$$(\lambda + r)\left(\lambda^2 + (2\mu + \gamma)\lambda + \mu(\mu + \gamma) - \frac{br\gamma}{a} e^{-m\tau} e^{-\lambda\tau}\right) = 0. \quad (10)$$

Equation (10) has always a negative root $\lambda = -r$. Thus, stability of E_0 depends on the following equation roots:

$$\lambda^2 + (2\mu + \gamma)\lambda + \mu(\mu + \gamma) - \frac{br\gamma}{a} e^{-m\tau} e^{-\lambda\tau} = 0. \quad (11)$$

When $\tau = 0$, (11) reduces to

$$\lambda^2 + (2\mu + \gamma)\lambda + \mu(\mu + \gamma) - \frac{br\gamma}{a} = 0. \quad (12)$$

From equation (12), we can write:

$$\lambda^2 + (2\mu + \gamma)\lambda + \mu(\mu + \gamma)(1 - R_0) = 0. \quad (13)$$

Since, $(2\mu + \gamma) > 0$ and $\mu(\mu + \gamma)(1 - R_0) > 0$ if $R_0 < 1$. Then, if $R_0 < 1$, the equation (13) roots have negative real parts.

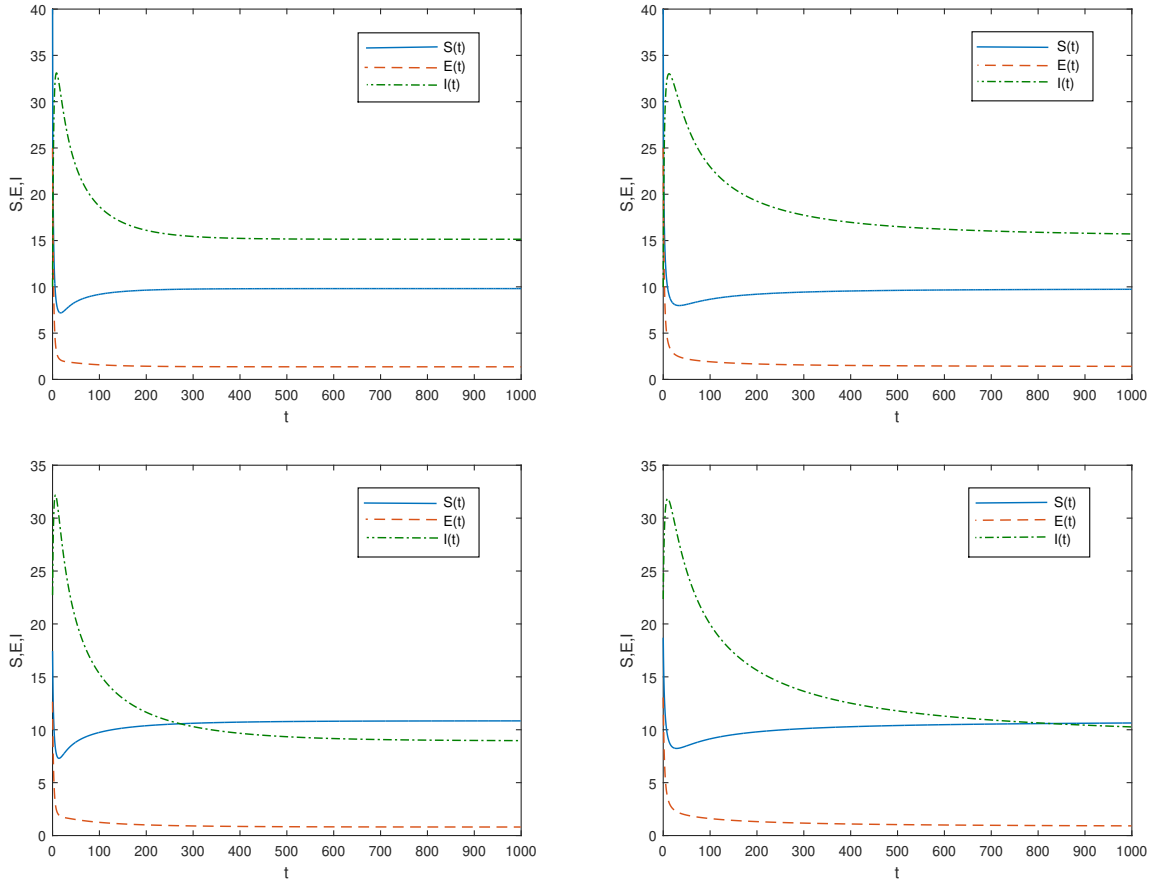

Fig. 2: Stability of endemic equilibrium point when $R_0 > 1$ and $Q > \frac{r}{2a}$. Left figures with $\alpha = 1$, $\tau = 0$(above) and $\alpha = 1$, $\tau = 1$(down), right figures with $\alpha = 0.9$, $\tau = 0$(above) and $\alpha = 0.9$, $\tau = 1$(down).

For $\tau > 0$, assume that (11) has pure imaginary roots $\lambda = \pm i\omega$ for some $\omega > 0$ and substituting $\lambda = i\omega$ in equation (11), we obtain

$$-\omega^2 + i(2\mu+\gamma)\omega + \mu(\mu+\gamma) - \frac{br\gamma}{a}e^{-m\tau}e^{-i\omega\tau} = 0. \tag{14}$$

The real and imaginary parts of (14) are:

$$\begin{aligned} \frac{br\gamma}{a}e^{-m\tau}\cos\omega\tau &= -\omega^2 + \mu(\mu+\gamma) \\ \frac{br\gamma}{a}e^{-m\tau}\sin\omega\tau &= -(2\mu+\gamma)\omega. \end{aligned} \tag{15}$$

If we square and add the two equations in (15), we get

$$\begin{aligned} &\omega^4 + (2\mu^2 + 2\mu\gamma + \gamma^2)\omega^2 + \mu^2(\mu+\gamma)^2 \\ &- \frac{b^2r^2\gamma^2}{a^2}e^{-2m\tau} = 0. \end{aligned} \tag{16}$$

Let $\omega^2 = z$, then

$$z^2 + (2\mu^2 + 2\mu\gamma + \gamma^2)z + \mu^2(\mu+\gamma)^2(1 - R_0^2) = 0. \tag{17}$$

If $R_0 < 1$, we find that $(2\mu^2 + 2\mu\gamma + \gamma^2) > 0$, and $\mu^2(\mu+\gamma)^2(1 - R_0^2) > 0$. Hence, there are no positive roots for equation (17). Furthermore, equation (11) doesn't have any purely imaginary roots for $\tau > 0$, so the roots have negative real part. From Lemma 2.1., the disease free equilibrium of system (4) is locally asymptotically stable if $R_0 < 1$ for any time delay $\tau \geq 0$, and unstable if $R_0 > 1$. This sums up the proof of Theorem 5.1.1.

Now, we address local stability of the endemic equilibrium point E_1. We assume that $R_0 > 1$ for the existence of E_1. The characteristic equation calculated at the endemic equilibrium point E_1 is

$$\begin{aligned} &\lambda^3 + p_2(\tau)\lambda^2 + (p_1(\tau) + q_1(\tau)e^{-\lambda\tau})\lambda \\ &+ p_0(\tau) + q_0(\tau)e^{-\lambda\tau} = 0, \end{aligned} \tag{18}$$

where,

$$\begin{aligned} p_2(\tau) &= aS^* + 2\mu + \gamma, \\ p_1(\tau) &= \mu(\mu+\gamma) + aS^*(2\mu+\gamma), \\ q_1(\tau) &= -b\gamma e^{-m\tau}S^*, \\ p_0(\tau) &= a\mu(\mu+\gamma)S^*, \\ q_0(\tau) &= \mu(\mu+\gamma)(bI^* - aS^*). \end{aligned} \tag{19}$$

When $\tau = 0$, equation (18) is written as

$$P(\lambda) := \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{20}$$

with

$$\begin{aligned} a_1 &= \frac{a\mu(\mu+\gamma)}{\gamma b} + 2\mu + \gamma, \\ a_2 &= \frac{a\mu(\mu+\gamma)}{\gamma b}(2\mu+\gamma), \\ a_3 &= r\mu(\mu+\gamma)\left(1 - \frac{1}{R_0}\right). \end{aligned}$$

The discriminant $D(P)$ of a polynomial $P(\lambda)$ is:

$$\begin{aligned} D(P) &= \begin{pmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 3 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3 & 2a_1 & a_2 \end{pmatrix} \\ &= 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2. \end{aligned}$$

It is clear that $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ as long as $R_0 > 1$. The results of [28] reveal that the endemic equilibrium point E_1 is asymptotically stable if either

$$\begin{aligned} &(i)\ D(P) > 0, a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0 \quad (or) \\ &(ii)\ D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 = a_3, \quad \alpha \in (2/3, 1) \end{aligned}$$

is satisfied.

For $\tau > 0$, substituting $\lambda = i\omega$ ($\omega > 0$) into (18), we get

$$\begin{aligned} &-i\omega^3 - p_2(\tau)\omega^2 + i\left[p_1(\tau) + q_1(\tau)(\cos(\omega\tau) - i\sin(\omega\tau))\right]\omega \\ &+ p_0(\tau) + q_0(\tau)(\cos(\omega\tau) - i\sin(\omega\tau)) = 0. \end{aligned} \tag{21}$$

Using the above-mentioned steps, we obtain the following equation:

$$\begin{aligned} &F(\omega,\tau) := \omega^6 + (p_2^2(\tau) - 2p_1(\tau))\omega^4 \\ &+ (p_1^2(\tau) - 2p_2(\tau)p_0(\tau) - q_1^2(\tau))\omega^2 + p_0^2(\tau) - q_0^2(\tau) = 0. \end{aligned} \tag{22}$$

Then, the root $i\omega(\omega > 0)$ of equation (18) satisfies $F(\omega,\tau) = 0$.

Let $\omega^2 = z$, then equation (22) takes this form

$$\begin{aligned} &F(z,\tau) := z^3 + (p_2^2(\tau) - 2p_1(\tau))z^2 \\ &+ (p_1^2(\tau) - 2p_2(\tau)p_0(\tau) - q_1^2(\tau))z + p_0^2(\tau) - q_0^2(\tau) = 0. \end{aligned} \tag{23}$$

We will finally obtain that if $p_0^2(\tau) - q_0^2(\tau) \geq 0$ and $p_1^2(\tau) - 2p_2(\tau)p_0(\tau) - q_1^2(\tau) > 0$ are satisfied, then there will be no positive roots of equation (23). Thus, equation (18) roots have negative real part for all $\tau > 0$.
To summarize the aforementioned analysis, we have the theorem that follows.

Theorem 5.1.2. Let $p_0^2(\tau) - q_0^2(\tau) \geq 0$, $p_1^2(\tau) - 2p_2(\tau)p_0(\tau) - q_1^2(\tau) > 0$, then the endemic equilibrium point E_1 of system (4) is asymptotically stable for any time delay $\tau \geq 0$ if either

$$\begin{aligned} &(i)\ D(P) > 0, a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0 \quad (or) \\ &(ii)\ D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 = a_3, \quad \alpha \in (2/3, 1) \end{aligned}$$

is satisfied.

5.2 *Global stability of equilibrium points*

This section covers equilibrium points global stability of the model (4) by Lyapunov direct method [29, 30]. To do this, we need the next lemma.

lemma 5.2.1 Suppose $S(t), E(t), I(t) > 0$ are system (4) solutions that satisfy the initial conditions (5), then $S(t) \geq Q$, and Q is constant. If $r - bM_2 > 0$, then Q is positive.

Proof. From Proposition 3.1.1, we have

$$\limsup_{t\to+\infty} I(t) \leq M_2 := \frac{(r+\mu)^2}{4a\mu e^{m\tau}}.$$

Thus, there is a $T_1 > 0$ such that $I(t) < M_2 + \varepsilon$ when $t > T_1$ for sufficiently small $\varepsilon > 0$. From the first equation of system (4), we find that

$$D^\alpha S(t) \geq S(t)(r - aS(t)) - bS(t)(M_2 + \varepsilon).$$

According to standard parabolic comparison theorem, we get

$$\liminf_{t\to+\infty} S(t) \geq \frac{r - b(M_2+\varepsilon)}{a}.$$

Since $\varepsilon > 0$ is arbitrarily sufficiently small, we get

$$\liminf_{t\to+\infty} S(t) \geq \frac{r - bM_2}{a} := Q. \tag{24}$$

Now, we investigate the global stability of the equilibrium points of model (4). For simplicity and mathematical convenience, we shall introduce the following function: $F : R > 0 \to R \geq 0$ as $F(u) = u - 1 - \ln(u)$, and we note that $F(u) \geq 0$ for any $u > 0$.

Theorem 5.2.1 If $R_0 \leq 1$, then the disease free equilibrium $E_0 = \left(\frac{r}{a}, 0, 0\right)$ of the model (4) is globally asymptotically stable, i.e. the disease will die out.

Proof. Define the following Lyapunov function:

$$V_1(t) = \left(S - S^0 - S^0 \ln \frac{S}{S^0}\right) + e^{m\tau}E + \left(\frac{\gamma+\mu}{\gamma}\right)e^{m\tau}I + \int_0^\tau bS(t-\theta)I(t-\theta)d\theta. \tag{25}$$

It is clear that $V_1(t) > 0$ for all $(S,E,I) > 0$, where $\theta \in [-\tau, 0]$. Also, we note that $V_1(t) = 0$ occurs only at the disease free equilibrium $E_0 = \left(\frac{r}{a}, 0, 0\right)$.

Calculating Caputo fractional derivative of $V_1(t)$, we obtain

$$\begin{aligned} D^\alpha V_1(t) &= \left(1 - \frac{S^0}{S}\right) D^\alpha S(t) + e^{m\tau} D^\alpha E(t) \\ &\quad + \frac{\gamma+\mu}{\gamma} e^{m\tau} D^\alpha I(t) + bSI - bS(t-\tau)I(t-\tau) \\ &= \left(1 - \frac{S^0}{S}\right)(rS - aS^2 - bSI) \\ &\quad + e^{m\tau}(be^{-m\tau}S(t-\tau)I(t-\tau) - (\gamma+\mu)E) \\ &\quad + \frac{\gamma+\mu}{\gamma} e^{m\tau}(\gamma E - \mu I) + bSI - bS(t-\tau)I(t-\tau) \\ &= (S - S^0)(r - aS) + bS^0 I - \frac{\mu(\mu+\gamma)}{\gamma} e^{m\tau} I. \end{aligned}$$

Substituting $S^0 = \frac{r}{a}$ yields

$$\begin{aligned} D^\alpha V_1(t) &= (S - S^0)(aS^0 - aS) + b\frac{r}{a}I - \frac{\mu(\mu+\gamma)}{\gamma} e^{m\tau} I \\ &= -a(S - S^0)^2 + \frac{\mu(\mu+\gamma)}{\gamma} e^{m\tau}(R_0 - 1)I. \end{aligned} \tag{26}$$

Obviously, $D^\alpha V_1(t) \le 0$ when $R_0 \le 1$ and $D^\alpha V_1(t) = 0$ if and only if $S = S^0$, $E = 0$, and $I = 0$. By LaSalle's Invariance Principle [31], the proof is completed.

The following theorem investigates global stability of the endemic equilibrium point of model (4).

Theorem 5.2.2 The endemic equilibrium point E_1 of the model (4) is globally asymptotically stable if $R_0 > 1$ and $Q \ge \frac{r}{2a}$, where Q is the positive constant defined in (24).

Proof. Lyapunov function is defined, as follows:

$$\begin{aligned} V_2(t) &= \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + e^{m\tau}\left(E - E^* - E^* \ln \frac{E}{E^*}\right) \\ &\quad + \frac{(\gamma+\mu)}{\gamma} e^{m\tau}\left(I - I^* - I^* \ln \frac{I}{I^*}\right) \\ &\quad + \int_0^\tau b\left(S(t-\theta)I(t-\theta) - S^*I^* \right. \\ &\quad \left. - S^*I^* \ln \frac{S(t-\theta)I(t-\theta)}{S^*I^*}\right) d\theta. \end{aligned} \tag{27}$$

Obviously, Lyapunov function is positive definite and continuous for all $S,E,I > 0$. It is easy to verify that the function $V_2(t) = 0$ at $S = S^*, E = E^*, I = I^*$. Thus, the global minimum of $V_2(t)$ occurs at the endemic equilibrium point $E_1 = (S^*, E^*, I^*)$.

From system (4), we have the endemic equilibrium conditions, as follows:

$$r = aS^* + bI^*, \quad be^{-m\tau}S^*I^* = (\gamma+\mu)E^*, \quad \gamma E^* = \mu I^*.$$

Taking the Caputo fractional derivative of $V_2(t)$, then

$$\begin{aligned} D^\alpha V_2(t) &= \left(1 - \frac{S^*}{S}\right) D^\alpha S(t) + e^{m\tau}\left(1 - \frac{E^*}{E}\right) D^\alpha E(t) \\ &\quad + \frac{\gamma+\mu}{\gamma} e^{m\tau}\left(1 - \frac{I^*}{I}\right) D^\alpha I(t) + bSI \\ &\quad - bS(t-\tau)I(t-\tau) + bS^*I^* \ln\left(\frac{S(t-\tau)I(t-\tau)}{SI}\right) \\ &= \left(1 - \frac{S^*}{S}\right)\left(rS - aS^2 - bSI\right) + e^{m\tau}\left(1 - \frac{E^*}{E}\right) \\ &\quad \times \left(be^{-m\tau}S(t-\tau)I(t-\tau) - (\gamma+\mu)E\right) \\ &\quad + \frac{\gamma+\mu}{\gamma} e^{m\tau}\left(1 - \frac{I^*}{I}\right)\left(\gamma E - \mu I\right) \\ &\quad + bSI - bS(t-\tau)I(t-\tau) \\ &\quad + bS^*I^* \ln\left(\frac{S(t-\tau)I(t-\tau)}{SI}\right). \end{aligned}$$

Cancelling and collecting terms yields

$$\begin{aligned} D^\alpha V_2(t) &= r(S - S^*) - aS(S - S^*) + bS^*I \\ &\quad - bS(t-\tau)I(t-\tau)\frac{E^*}{E} + (\gamma+\mu)e^{m\tau}E^* \\ &\quad - \frac{\mu(\gamma+\mu)}{\gamma} e^{m\tau} I - (\gamma+\mu)e^{m\tau}E\frac{I^*}{I} \\ &\quad + \frac{\mu(\gamma+\mu)}{\gamma} e^{m\tau} I^* + bS^*I^* \ln\left(\frac{S(t-\tau)I(t-\tau)}{SI}\right). \end{aligned}$$

Using endemic equilibrium conditions and collecting the same terms, we can find

$$\begin{aligned} D^\alpha V_2(t) &= (S - S^*)(r - aS) + bS^*I - bS(t-\tau)I(t-\tau)\frac{E^*}{E} \\ &\quad + 2(\gamma+\mu)e^{m\tau}E^* - \frac{\mu(\gamma+\mu)}{\gamma} e^{m\tau} I \\ &\quad - (\gamma+\mu)e^{m\tau}E\frac{I^*}{I} + bS^*I^* \ln\left(\frac{S(t-\tau)I(t-\tau)}{SI}\right) \\ &= (S - S^*)(bI^* + aS^* - aS) \\ &\quad - bS(t-\tau)I(t-\tau)\frac{E^*}{E} + 2(\gamma+\mu)e^{m\tau}E^* \\ &\quad - (\gamma+\mu)e^{m\tau}E\frac{I^*}{I} + bS^*I^* \ln\left(\frac{S(t-\tau)I(t-\tau)}{SI}\right). \end{aligned}$$

It follows

$$\begin{aligned}D^{\alpha}V_2(t) &= -a(S-S^*)^2 + bI^*(S-S^*)\\ &\quad - bS(t-\tau)I(t-\tau)\frac{E^*}{E} + 2(\gamma+\mu)e^{m\tau}E^*\\ &\quad - (\gamma+\mu)e^{m\tau}E\frac{I^*}{I} + bS^*I^*\ln\left(\frac{S(t-\tau)I(t-\tau)}{SI}\right)\\ &= -a(S-S^*)^2 + (\gamma+\mu)e^{m\tau}E^*\frac{S}{S_*}\\ &\quad - (\gamma+\mu)e^{m\tau}E^*\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\\ &\quad + (\gamma+\mu)e^{m\tau}E^* - (\gamma+\mu)e^{m\tau}E\frac{I^*}{I}\\ &\quad + (\gamma+\mu)e^{m\tau}E^*\ln\left(\frac{S(t-\tau)I(t-\tau)}{SI}\right).\end{aligned}$$

Rearranging terms in the last equation gives

$$\begin{aligned}D^{\alpha}V_2(t) &= -(\gamma+\mu)e^{m\tau}E^*\left(-\frac{S}{S^*} + \frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\right.\\ &\quad \left. - 1 + \frac{EI^*}{E^*I} - \ln\frac{S(t-\tau)I(t-\tau)}{SI}\right) - a(S-S^*)^2.\end{aligned}$$

Adding and cancelling some terms, we have

$$\begin{aligned}D^{\alpha}V_2(t) &= -(\gamma+\mu)e^{m\tau}E^*\left[\left(\frac{S^*}{S} - 1 - \ln\frac{S^*}{S}\right)\right.\\ &\quad + \left(\frac{EI^*}{E^*I} - 1 - \ln\frac{EI^*}{E^*I}\right) + \left(\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\right.\\ &\quad \left.\left. - 1 - \ln\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\right)\right] - (\gamma+\mu)e^{m\tau}E^*\\ &\quad \times\left[-\ln\left(\frac{S(t-\tau)I(t-\tau)}{SI}\right) + \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right)\right.\\ &\quad \left. + \left(\ln\frac{S^*}{S} + \ln\frac{EI^*}{E^*I} + \ln\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\right)\right]\\ &\quad - a(S-S^*)^2.\end{aligned}$$

Since

$$\begin{aligned}&\ln\frac{S^*}{S} + \ln\frac{EI^*}{E^*I} + \ln\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\\ &= \ln\frac{S(t-\tau)I(t-\tau)}{SI},\end{aligned}$$

it follows that

$$\begin{aligned}D^{\alpha}V_2(t) &= -(\gamma+\mu)e^{m\tau}E^*\left[\left(\frac{S^*}{S} - 1 - \ln\frac{S^*}{S}\right)\right.\\ &\quad + \left(\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E} - 1 - \ln\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\right)\\ &\quad \left. + \left(\frac{EI^*}{E^*I} - 1 - \ln\frac{EI^*}{E^*I}\right)\right]\\ &\quad + \left[-a(S-S^*)^2 - (\gamma+\mu)e^{m\tau}E^*\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right)\right].\end{aligned}$$

It therefore follows from endemic equilibrium conditions that

$$\begin{aligned}D^{\alpha}V_2(t) &= -(\gamma+\mu)e^{m\tau}E^*\left[F\left(\frac{S^*}{S}\right) + F\left(\frac{EI^*}{E^*I}\right)\right.\\ &\quad \left. + F\left(\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\right)\right]\\ &\quad + \left[-a(S-S^*)^2 - bI^*S^*\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right)\right]\\ &= -(\gamma+\mu)e^{m\tau}E^*\left[F\left(\frac{S^*}{S}\right) + F\left(\frac{EI^*}{E^*I}\right)\right.\\ &\quad \left. + F\left(\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\right)\right]\\ &\quad + \left[-a(S-S^*)^2 + \frac{bI^*S^*}{SS^*}(S-S^*)^2\right].\end{aligned}$$

Substituting $bI^* = r - aS^*$ into the last equation, one can get

$$\begin{aligned}D^{\alpha}V_2(t) &= -(\gamma+\mu)e^{m\tau}E^*\left[F\left(\frac{S^*}{S}\right) + F\left(\frac{EI^*}{E^*I}\right)\right.\\ &\quad \left. + F\left(\frac{S(t-\tau)I(t-\tau)E^*}{S^*I^*E}\right)\right]\\ &\quad + \frac{(S-S^*)^2}{S}\Big(r - a(S+S^*)\Big).\end{aligned} \tag{28}$$

Since the function $F(u) = u - 1 - \ln u \geq 0$ for all $u > 0$ and $F(u) = 0$ holds if $u = 1$, then $D^{\alpha}V_2(t) = 0$ is negative definite if

$$\frac{(S-S^*)^2}{S}\Big(r - a(S+S^*)\Big) \leq 0. \tag{29}$$

By similar arguments of [32], Theorem 3.1 and Lemma 5.1, when t is sufficiently large, $S(t) \geq Q$ holds. Therefore, there exists $T > 0$ where if $t > T$, $S(t) \geq Q \geq \frac{r}{2a}$, $S^* \geq Q \geq \frac{r}{2a}$. Hence, it follows that:

$$\frac{(S-S^*)^2}{S}\Big(r - a(S+S^*)\Big) \leq 0$$

with equality holds if $S(t) = S^*$. Also, we note that $D^\alpha V_2(t) = 0$ if and only if $S = S^*$, $E = E^*$, and $I = I^*$. By LaSalle's Invariance Principle [31], the endemic equilibrium point E_1 is globally asymptotically stable in the interior of R^3_+ and this completes the proof.

6 Numerical simulations

In the following, numerical simulations are introduced to investigate the theoretical results mentioned above using an Adams-type predictor corrector method mentioned in [33].

For system (4), let $r = 0.15$, $a = 0.002$, $b = 0.0004$, $\mu = 0.035$, $\gamma = 0.2$, $m = 0.01$, $\tau = 0$ with the initial condition $S(0) = 40$, $E(0) = 25$ and $I(0) = 10$. We find that $R_0 = 0.7295 < 1$ and system (4) has only a disease-free equilibrium $E_0 = (75, 0, 0)$. We note that as τ increases, R_0 decreases, and the disease-free equilibrium is globally asymptotically stable for any $\tau > 0$. The results are shown in Figure 1.

Another example: let $r = 0.37$, $a = 0.03$, $b = 0.005$, $\mu = 0.045$, $\gamma = 0.5$, $m = 0.1$ and $\tau = 0$; then we get $R_0 = 1.2572 > 1$, and a unique endemic equilibrium point $E_1 = (9.81, 1.3626, 15.14)$. By calculations, we have $M \approx 31.8935$ and $Q \approx 7.0177 > \frac{r}{2a} = 6.1667$. Thus, based on Theorem 5.2, it follows that the endemic equilibrium point E_1 is globally asymptotically stable, see Figure 2. At $\tau = 1$ with the same parameters values of the last example, we have $R_0 = 1.1376 > 1$ and $E_1 = (10.8417, 0.8055, 8.9496)$. Figure 3 indicates that the fractional order reduces the infection peak and raises the stability region. Moreover, we notice that as $\tau \in [0, \tau_{max}]$ increases, the number of infected individuals reduces. The level of the steady state of susceptible individuals is higher and the fluctuation of the trajectories of E and I is smaller during the previous period of the time as shown in Figure 4.

If we take parameters as $r = 0.5$, $a = 0.0002$, $b = 0.085$, $\mu = 0.35$, $\gamma = 0.02$, $m = 0.1$, $\tau = 0$; we find $R_0 = 32.8185 > 1$ and $Q < \frac{r}{2a}$. The endemic equilibrium point $E_1 = (76.1765, 99.8045, 5.7031)$ is stable for $\tau = 0$, see Figure 5. For $\tau > 0$, and according to a condition in Theorem 5.2, the endemic equilibrium point is unstable, a Hopf bifurcation occurs, and a family of periodic solutions bifurcate from the endemic equilibrium point E_1. We can see this property in Figure 6 and in this case, the disease will be out of control.

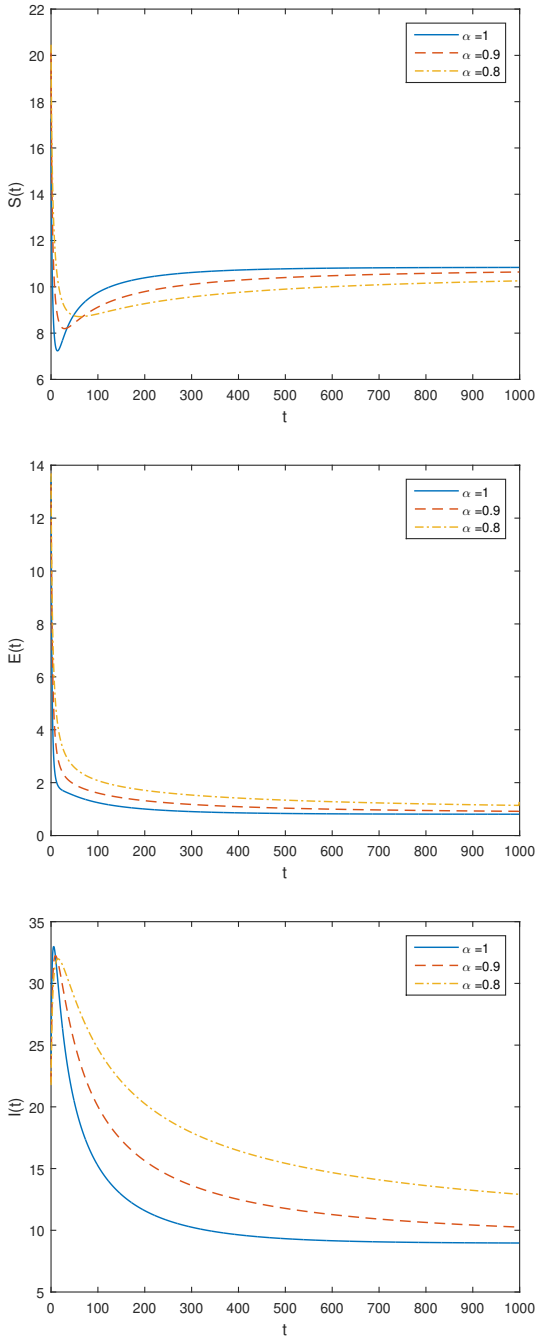

Fig. 3: $\tau = 1$, with different values of α.

7 Conclusion

The present paper investigated the stability analysis of a fractional (SEI) epidemic model with logistic growth and time delay. We calculated the basic reproduction number R_0 which depends on time delay. The local stability of both disease free equilibrium and endemic equilibrium points was investigated for the system under studying. While, we selected a suitable lyapunov function to explore the global stability of the two equilibrium points. We noticed that when $R_0 < 1$ and for any τ, the disease-free equilibrium is locally and globally asymptotically stable. If $R_0 > 1$, a sufficient conditions are given for asymptotic stability of the endemic

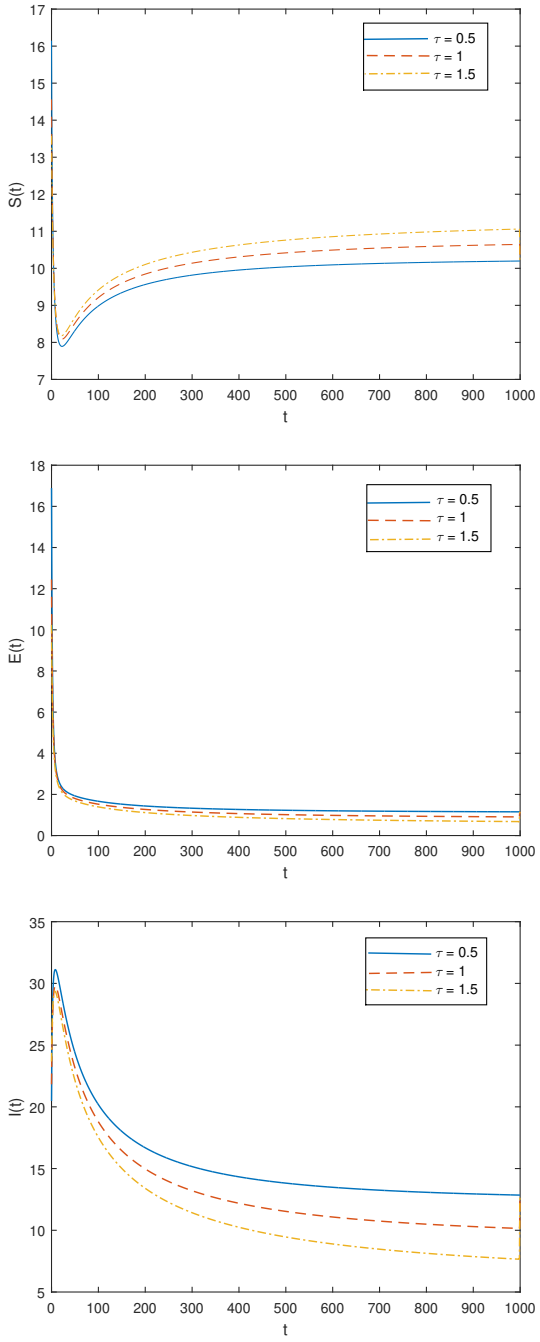

Fig. 4: $\alpha = 0.9$, with different values of τ.

equilibrium point. We have presented a numerical simulation using Matlab program and the Adams-type predictor-corrector method to verify the theoretical results. Moreover, the fractional-order derivatives showed significant changes of the stability more than integer-order derivatives. We found that in the fractional-order case, the peak of the infection reduced.

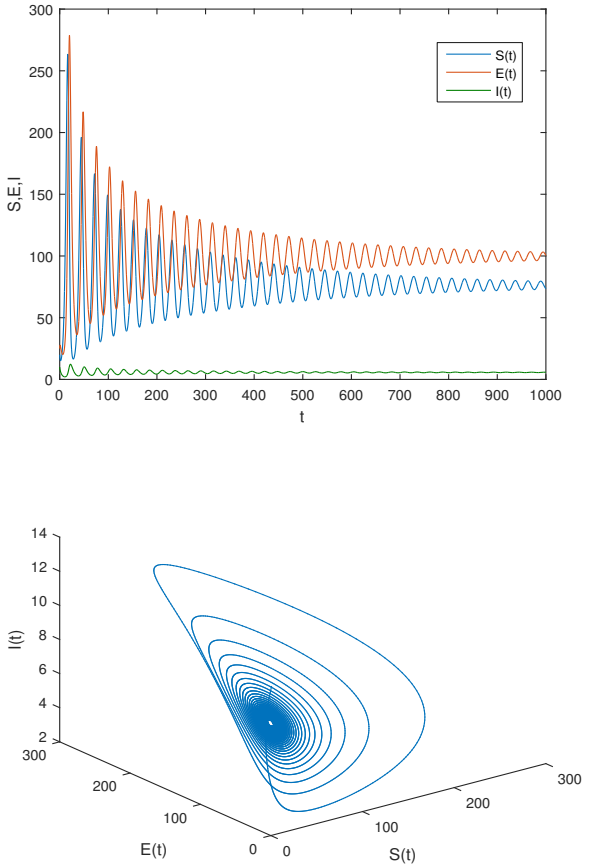

Fig. 5: $\alpha = 0.9$, $\tau = 0$, with initial values $S(0) = 20$, $E(0) = 25$, $I(0) = 10$.

However, it takes time for the disease to become stable or to approach the endemic stage as shown in Figure 3. At the same time, when the time delay τ increased, the number of $I(t)$ decreased, the number of $S(t)$ increased and the fluctuation of the trajectories of E and I was smaller during the previous period of the time as shown in Figure 4.

Acknowledgement

The authors are grateful to the anonymous referee for the careful checking of the details and the constructive comments that improved this paper.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

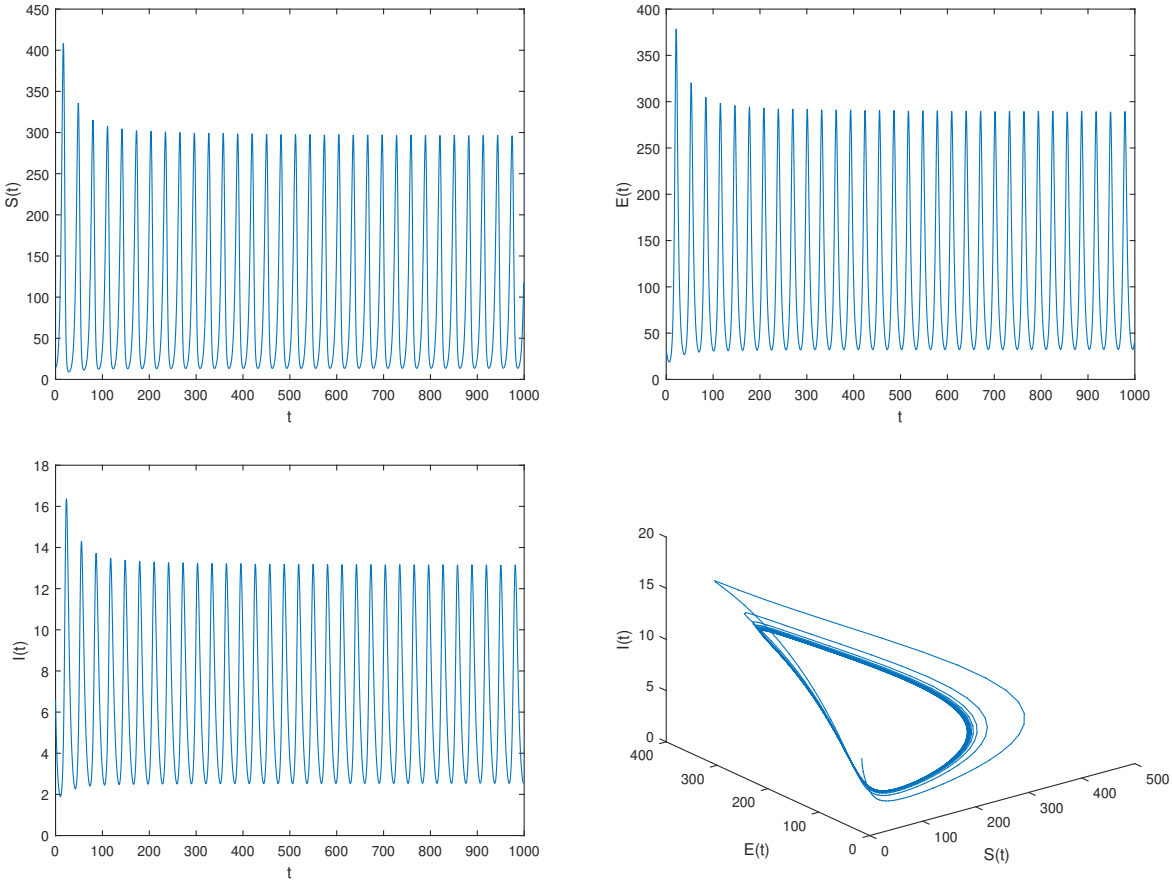

Fig. 6: $\alpha = 0.9$, $\tau = 1$, with initial values $S(0) = 20$, $E(0) = 25$, $I(0) = 10$.

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