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Runge-Kutta Fehlberg Method for Solving Linear and Nonlinear Fuzzy Fredholm Integro-Differential Equations

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Abstract: A study on the parametric form of fuzzy numbers is presented in this paper. The Runge-Kutta Fehlberg method is exploited to yield the approximate solution with respect to the second type of fuzzy Fredholm integro-differential equations. Both linear and nonlinear numerical examples are provided in our analysis. The results ascertain the effectiveness and precision of the proposed method.

Keywords: Integro-differential equations, Runge-Kutta method, Fehlberg method, fuzzy

1. Introduction

Real-world dynamic systems are subject to all kinds of uncertainties, such as population growth [1, 2] and [3], tower-bell oscillation [4], friction of sliding surfaces [5], contaminant migration in porous media [6], and a human life cycle [7]. Generally in mathematics, a random variable or a fuzzy set is used for handling uncertainty. In the 1960s, the theory of fuzzy sets was introduced by Zadeh to deal with uncertainty due to imprecision or vagueness, instead of randomness. Then, studies on fuzzy numbers and associated arithmetic operations were conducted by Zadeh [8] and [9], while further enhancements were proposed by Mizumoto and Tanaka [10]. The notion of LR fuzzy numbers was initiated by Dubios and Prade [11], where a computing mechanism for dealing with fuzzy functions was provided.

The study of Integro-differential Equation, on the other hand, has gained growing interest in various physical, biological and engineering sciences [12, 13] and [14]. In the last two decades, many researchers have been into the analytical and numerical methods for the solution of Integro differential equation. Since then, fuzzy differential and integral equations have been rigorously improved from fuzzy control application perspectives. In modelling uncertainties in dynamical systems, fuzzy integro-differential equations (FIDE) play an important role. Indeed, they have been successfully used in various domains, including engineering, biology, medicine, physics, and economy. The authors in [15] introduced the existence and uniqueness of solutions pertaining to FIDE. The existing results for fuzzy delay integro-differential equations and general fuzzy volterra-fredholm integral equations have been researched by Balachandran and Kanagarajan [16] and [17]. On the other hand, application of fuzzy integral equations together with control problems and fuzzy uncertainties have been modelled by Diamond [18].

In general, except for a few linear and non-linear systems, it is very difficult to get a analytical solution for an FIDE. To solve Integro-differential equations, several numerical methods are available in the literature [19–25] and [26]. Using the variation iteration technique, Abbasbandy and Hashemi [27] solved FIDE. In [28], the numerical solutions for FIDE were provided by analyzing homotopy. Allahviranloo et al. [29] presented a new technique to tackle FIDE with generalized differentiability.

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The main contribution of this paper can be summarized as follows.

i) In this paper, we employ the Rung-Kutta Fehlberg technique to yield the numerical solution of fuzzy Fredholm integro-differential equations (FFIDE).
ii) Both linear and nonlinear numerical examples are provided in our analysis.
iii) Finally, simulation results and table values are given to show the advantage and errors obtained by Runge-Kutta Fehlberg method.

2. Problem Formulation and Preliminaries

In this section, the fuzzy number definitions and the fundamental notions used in fuzzy operations are presented.

Definition 1. *[30, 31] and [32]: A fuzzy subset, u_1, is called a fuzzy number when the universal set on which φ_{u_1} is defined is the set of all real numbers, $\mathbb{R}$, and satisfies the following conditions:*

(i) All the β-levels of u_1 are not empty for $0 \le \beta \le 1$;
(ii) all the β-levels of u_1 are closed intervals of $\mathbb{R}$;
(iii) supp $u_1 = x \in \mathbb{R} : \varphi_{u_1}(x) > 0$ is bounded.

In addition, $\hat{E}$ denotes the set of all fuzzy numbers. Kaleva [33] provided an substitute definition for producing the same $\hat{E}$. As discussed in [34], this fuzzy number space can be formulated in the Banach space $B = \overline{C}[0,1] \times \overline{C}[0,1]$.

Definition 2. *[35]: For arbitrary fuzzy numbers $\tilde{u}_1, \tilde{v}_1 \in \hat{E}$, $\beta \in [0,1]$, the distance (Hausdorff metric) is employed, i.e.,*

$$D(u_1,v_1) = \sup_{\beta\in[0,1]} max\{|\underline{u_1}(\beta) - \underline{v_1}(\beta)|, |\overline{u_1}(\beta) - \overline{v_1}(\beta)|\}.$$

It has been highlighted in [36] that $(\hat{E}, D)$ is a complete metric space, and some well-established properties are as follows:

$$D(\tilde{u}_1 \oplus \tilde{w}_1, \tilde{v}_1 \oplus \tilde{w}_1) = D(\tilde{u}_1, \tilde{v}_1), \forall \tilde{u}_1, \tilde{v}_1, w_1 \in \hat{E},$$
$$D(k \odot \tilde{u}_1, k \odot \tilde{v}_1) = |k| D(\tilde{u}_1, \tilde{v}_1),\ \forall k \in R^1, \tilde{u}_1, \tilde{v}_1 \in \hat{E},$$
$$D(\tilde{u}_1 \oplus \tilde{v}_1, \tilde{w}_1 \oplus \tilde{e}_1) \le D(\tilde{u}_1, \tilde{w}_1) + D(\tilde{v}_1, \tilde{e}_1),\ \forall \tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \tilde{e}_1 \in \hat{E}.$$

Definition 3. *[37]: Suppose $g : [a_1, a_2] \to \hat{E}$ is a fuzzy function. For arbitrary fixed $t_0 \in R^1$ and $\hat{\varepsilon} > 0, \hat{\delta} > 0$, therefore,*

$$|t - t_0| < \hat{\delta} \Rightarrow D(g(t), g(t_0)) < \hat{\varepsilon}.$$

Then, g is concluded as continuous.

Definition 4. *[35] and [37]: Suppose $g : [a_1, a_2] \to \hat{E}$. For each partition $R = \{t_0, t_1, ..., t_n\}$ of $[a_1, a_2]$ and for arbitrary $\hat{\zeta}_i \in [t_{i-1}, t_i], 1 \le i \le n$, assume*

$$P_R = \sum_{i=1}^{n} g(\hat{\zeta}_i)(t_i - t_{i-1}),\ \ \lambda = \max_{1\le i\le n}\{|t_i - t_{i-1}|\}.$$

Then, the definite integral of $g(t) \in [a_1, a_2]$ is provided as follows:

$$\int_{a_1}^{a_2} g(t)dt = \lim_{\lambda\to 0} P_R,$$

and its limit exists in metric D.
Given that the fuzzy function $g(t)$ is continuous in metric D, the definite integral exists [35], and also

$$\underline{(\int_{a_1}^{a_2} g(t;\beta)dt)} = \int_{a_1}^{a_2} \underline{g}(t;\beta)dt, \overline{(\int_{a_1}^{a_2} g(t;\beta)dt)} = \int_{a_1}^{a_2} \overline{g}(t;\beta)dt.$$

Definition 5. *[38]: Suppose x_1 and $y_1 \in \hat{E}$. There exists $z_1 \in \hat{E}$ and $x_1 = y_1 + z_1$, then z_1 is known as the H-difference of x_1 and y_1, which is represented by $x_1 \ominus y_1$. In addition, if there exists H-difference $\tilde{u}_1 \ominus \tilde{v}_1$ and $\tilde{w}_1 \ominus \tilde{e}_1$, it can be deduced that $D(\tilde{u}_1 \ominus \tilde{v}_1, \tilde{w}_1 \ominus \tilde{e}_1) = D(\tilde{u}_1 \oplus \tilde{e}_1, \tilde{w}_1 \oplus \tilde{v}_1)$, $\forall 8 u_1, \tilde{v}_1, \tilde{w}_1, \tilde{e}_1 \in \hat{E}$.*

In this paper, the "$\ominus$" sign stands always for H-difference and let us remark that $x_1 \ominus y_1 \ne x_1 + (-1)y_1$. In this study, the following definition of differentiability for fuzzy-valued functions introduced in [38] and investigated in [39] is adopted:

Definition 6. *Suppose $g : (a_1, a_2) \to \hat{E}$ and $r_0 \in (a_1, a_2)$. Then, g is strongly generalized H-differentiable at r_0, there exists an element $g^{'}(r_0) \in \hat{E}$, therefore,*

(1) for all $h > 0$ sufficiently close to 0, there exist $g(r_0 + h) \ominus g(r_0)$,
$g(r_0) \ominus g(r_0 - h)$ and the limits (in the metric D) are:

$$\lim_{h\to 0^+} \frac{g(r_0+h) \ominus g(r_0)}{h} = \lim_{h\to 0^+} \frac{g(r_0) \ominus g(r_0-h)}{h} = g^{'}(r_0),$$

(2) for all $h < 0$ sufficiently close to 0, there exists $g(r_0) \ominus g(r_0 + h)$,
$g(r_0 - h) \ominus g(r_0)$ and the limits (in the metric D) are:

$$\lim_{h\to 0^+} \frac{g(r_0) \ominus g(r_0+h)}{h} = \lim_{h\to 0^+} \frac{g(r_0-h) \ominus g(r_0)}{h} = g^{'}(r_0).$$

In the special case when g is a fuzzy-valued function, the consecutive results can be obtained.

Lemma 1. *[39]: Suppose $g : R^1 \to \hat{E}$ is a function, and denote $g(t) = (\underline{g}(t;\beta), \overline{g}(t;\beta))$, for each $\beta \in [0,1]$. Then,*

(1) if g is differentiable in the first form (1) in Definition 6, then $\underline{g}(t;\beta)$ and $\overline{g}(t;\beta)$ are differentiable functions, and $g^{'}(t) = (\underline{g}^{'}(t;\beta), \overline{g}^{'}(t;\beta))$

(2) if g is differentiable in the second form (2) in Definition 6, then $\underline{g}(t;\beta)$ and $\overline{g}(t;\beta)$ are differentiable functions, and $g'(t) = (\overline{g}'(t;\beta), \underline{g}'(t;\beta))$.

In [33], we can obtain the key properties of the gH-derivatives in the first form (1), and some of which still hold for the next form (2). In [39], we can obtain the key properties for the next form (2). Note that fuzzy-valued function g is I-differentiable if it fascinates the first form (1) in Definition 6, while g is II-differentiable if it fascinates the second form (2) in Definition 6.

3. Fuzzy Fredholm Integro-Differential Equations : Runge-Kutta Fehlberg Method

The Fredholm integro-differential equations are expressed as follows: [40] and [41]

$$\begin{cases} \theta'(t) = x(t) + \sigma \int_{a_1}^{a_2} q(t,s)\theta(s)ds \\ \theta(t_0) = \theta_0, \end{cases} \tag{1}$$

where $\sigma > 0$, q is an arbitrary given kernel function over the square $a_1 \le t,s \le a_2$. Suppose θ denotes the fuzzy function, $x(t)$ noted as given fuzzy function of $t \in [a_1, a_2]$, and θ' represents a fuzzy derivative (in reference to Definition 6) of θ; this equation only possesses a fuzzy solution. The sufficient conditions for the existing equation of the second kind has been modelled in [42].

Suppose $\theta(t) = [\underline{\theta}(t;\beta), \overline{\theta}(t;\beta)]$ is a fuzzy solution of equation (1). By Definitions 4 and 6, the equivalent model is obtained:

$$\begin{cases} \underline{\theta}'(t) = \underline{x}(t) + \sigma \int_{a_1}^{a_2} \underline{q(t,s)\theta(s)}ds, \quad \theta(t_0) = \underline{\theta}_0 \\ \overline{\theta}'(t) = \overline{x}(t) + \sigma \int_{a_1}^{a_2} \overline{q(t,s)\theta(s)}ds, \quad \overline{\theta}(t_0) = \overline{\theta}_0, \end{cases} \tag{2}$$

and it possesses a unique solution $(\underline{\theta}, \overline{\theta}) \in B$, which is a fuzzy function. Specifically, for each s, the pair $[\underline{\theta}(t;\beta), \overline{\theta}(t;\beta)]$ denotes a fuzzy number, therefore, each solution of (1) is a solution of model (2). In, reverse Model (1) and Model (2) are identical.
The parametric form of Model (2) is given by

$$\begin{cases} \underline{\theta}'(t,\beta) = \underline{x}(t,\beta) + \sigma \int_{a_1}^{a_2} \underline{q(t,s)\theta(s,\beta)}ds, \\ \underline{\theta}(t_0) = \underline{\theta}_0(\beta) \\ \overline{\theta}'(t,\beta) = \overline{x}(t,\beta) + \sigma \int_{a_1}^{a_2} \overline{q(t,s)\theta(s,\beta)}ds, \\ \overline{\theta}(t_0) = \overline{\theta}_0(\beta), \end{cases} \tag{3}$$

for $\beta \in [0,1]$. Let $q(t,s)$ be continuous in $a_1 \le t \le a_2$. For fixed t, $q(t, s)$ changes the sign in finite points as t_i, where $\theta_i \in [a_1, t_1]$. As an example, let $q(t, s)$ be non-negative over $[a_1, t_1]$ and negative over $[t_1, a_2]$. Then, we have

$$\begin{cases} \underline{\theta}'(t,\beta) = \underline{x}(t,\beta) + \sigma \int_{a_1}^{t_1} q(t,s)\underline{\theta(s,\beta)}ds \\ +\sigma \int_{t_1}^{a_2} q(t,s)\overline{\theta(s,\beta)}ds, \quad \underline{\theta}(t_0) = \underline{\theta}_0(\beta) \\ \overline{\theta}'(t,\beta) = \overline{x}(t,\beta) + \sigma \int_{a_1}^{t_1} q(t,s)\overline{\theta(s,\beta)}ds \\ +\sigma \int_{t_1}^{a_2} q(t,s)\underline{\theta(s,\beta)}ds, \quad \overline{\theta}(t_0) = \overline{\theta}_0(\beta). \end{cases} \tag{4}$$

In several cases, yet, the analytical solution for Eq. (3) cannot be available, and a numerical method has to be used. In the interval $[a_1, a_2]$, consider a set of discrete equally spaced grid points $a_1 < t_0 < t_1 < t_2 < ..... < t_N = a_2$ at which two exact solutions $\Theta(t,\beta) = [\underline{\Theta}(t,\beta), \overline{\Theta}(t,\beta)]$ are approximated by $\theta(t,\beta) = [\underline{\theta}(t,\beta), \overline{\theta}(t,\beta)]$, respectively. The grid points at which the solutions are computed are $t_n = t_0 + nh$, $h = \frac{(a_2 - a_1)}{N}$. The exact and approximate solutions at $t_n, 0 < n < N$ are represented by $\Theta_n(\beta)$ and $\theta_n(\beta)$, respectively. Based on the RKF method, the first-order approximation of $\underline{\Theta}(t,\beta), \overline{\Theta}(t,\beta)$ and $\underline{\theta}(t,\beta), \overline{\theta}(t,\beta)$ is achieved as follows:

$$\begin{cases} \underline{\theta}_{n+1}(\beta) = \underline{\theta}_n(\beta) + \sum_{i=1}^{6} w_i \underline{l_i}(t_n, [\theta(t_n)]^\beta) \\ \overline{\theta}_{n+1}(\beta) = \overline{\theta}_n(\beta) + \sum_{i=1}^{6} w_i \overline{l_i}(t_n, [\theta(t_n)]^\beta), \end{cases} \tag{5}$$

where w_i's are constants, and

$$l_i(t_n, [\theta(t_n)]^\beta) = [\underline{l_i}(t_n, [\theta(t_n)]^\beta), \overline{l_i}(t_n, [\theta(t_n)]^\beta)],$$

where

$$\underline{l_i}(t_n, [\theta(t_n)]^\beta) = h\, F\Big(t_n + \beta_i h,\ [\theta(t_n)]^\beta + \sum_{j=1}^{i-1} \gamma_{ij} \underline{l_j}(t_n, [\theta(t_n)]^\beta)\Big),$$

$$\overline{l_i}(t_n, [\theta(t_n)]^\beta) = h\, G\Big(t_n + \beta_i h,\ [\theta(t_n)]^\beta + \sum_{j=1}^{i-1} \gamma_{ij} \overline{l_j}(t_n, [\theta(t_n)]^\beta)\Big),$$

and

$$
\begin{aligned}
&\underline{l_1}(t_n,[\theta(t_n)]^\beta)=\min\left\{h\,F\left[t_n,\ \underline{\theta}_n(\beta),\overline{\theta}_n(\beta)\right]\right\},\\
&\underline{l_2}(t_n,[\theta(t_n)]^\beta)=\min\left\{h\,F\left[t_n+\frac{h}{4},\ [\theta(t_n)]^\beta+\frac{1}{4}\underline{l_1}(t,[\theta(t_n)]^\beta)\right]\right\},\\
&\underline{l_3}(t_n,[\theta(t_n)]^\beta)=\min\left\{h\,F\left[t_n+\frac{3h}{8},\ [\theta(t_n)]^\beta+\frac{3}{32}\underline{l_1}(t,[\theta(t_n)]^\beta)\right.\right.\\
&\qquad\left.\left.+\frac{9}{32}\underline{l_2}(t,[\theta(t_n)]^\beta)\right]\right\},\\
&\underline{l_4}(t_n,[\theta(t_n)]^\beta)=\min\left\{h\,F\left[t_n+\frac{12h}{13},\ [\theta(t_n)]^\beta+\frac{1932}{2197}\underline{l_1}(t,[\theta(t_n)]^\beta)\right.\right.\\
&\qquad\left.\left.-\frac{7200}{2197}\underline{l_2}(t,[\theta(t_n)]^\beta)+\frac{7296}{2197}\underline{l_3}(t,[\theta(t_n)]^\beta)\right]\right\},\\
&\underline{l_5}(t_n,[\theta(t_n)]^\beta)=\min\left\{h\,F\left[t_n+h,\ [\theta(t_n)]^\beta+\frac{439}{216}\underline{l_1}(t,[\theta(t_n)]^\beta)\right.\right.\\
&\qquad\left.\left.-8\underline{l_2}(t,[\theta(t_n)]^\beta)\ +\frac{3680}{513}\underline{l_{31}}(t,[\theta_1(t_n)]^\beta)-\frac{845}{4104}\underline{l_{41}}(t,[\theta_1(t_n)]^\beta)\right]\right\},\\
&\underline{l_6}(t_n,[\theta(t_n)]^\beta)=\min\left\{h\,F\left[t_n+\frac{h}{2},\ [\theta(t_n)]^\beta-\frac{8}{27}\underline{l_1}(t,[\theta(t_n)]^\beta)\right.\right.\\
&\qquad+2\underline{l_2}(t,[\theta(t_n)]^\beta)-\frac{3544}{2565}\underline{l_3}(t,[\theta(t_n)]^\beta)\\
&\qquad\left.\left.+\frac{1859}{4104}\underline{l_4}(t,[\theta(t_n)]^\beta)-\frac{11}{40}\underline{l_5}(t,[\theta(t_n)]^\beta)\right]\right\}.
\end{aligned}
$$

And

$$
\begin{aligned}
&\overline{l_1}(t_n,[\theta(t_n)]^\beta)=\max\left\{h\,G\left[t_n,\ \underline{\theta}_n(\beta),\overline{\theta}_n(\beta)\right]\right\},\\
&\overline{l_2}(t_n,[\theta(t_n)]^\beta)=\max\left\{h\,G\left[t_n+\frac{h}{4},\ [\theta(t_n)]^\beta+\frac{1}{4}\overline{l_1}(t,[\theta(t_n)]^\beta)\right]\right\},\\
&\overline{l_3}(t_n,[\theta(t_n)]^\beta)=\max\left\{h\,G\left[t_n+\frac{3h}{8},\ [\theta(t_n)]^\beta+\frac{3}{32}\overline{l_1}(t,[\theta(t_n)]^\beta)\right.\right.\\
&\qquad\left.\left.+\frac{9}{32}\overline{l_2}(t,[\theta(t_n)]^\beta)\right]\right\},\\
&\overline{l_4}(t_n,[\theta(t_n)]^\beta)=\max\left\{h\,G\left[t_n+\frac{12h}{13},\ [\theta(t_n)]^\beta\right.\right.\\
&\qquad+\frac{1932}{2197}\overline{l_1}(t,[\theta(t_n)]^\beta)-\frac{7200}{2197}\overline{l_2}(t,[\theta(t_n)]^\beta)\\
&\qquad\left.\left.+\frac{7296}{2197}\overline{l_3}(t,[\theta(t_n)]^\beta)\right]\right\},\\
&\overline{l_5}(t_n,[\theta(t_n)]^\beta)=\max\left\{h\,G\left[t_n+h,\ [\theta(t_n)]^\beta+\frac{439}{216}\overline{l_1}(t,[\theta(t_n)]^\beta)\right.\right.\\
&\qquad-8\overline{l_2}(t,[\theta(t_n)]^\beta)+\frac{3680}{513}\overline{l_{31}}(t,[\theta_1(t_n)]^\beta)\\
&\qquad\left.\left.-\frac{845}{4104}\overline{l_{41}}(t,[\theta_1(t_n)]^\beta)\right]\right\},\\
&\overline{l_6}(t_n,[\theta(t_n)]^\beta)=\max\left\{h\,G\left[t_n+\frac{h}{2},\ [\theta(t_n)]^\beta-\frac{8}{27}\overline{l_1}(t,[\theta(t_n)]^\beta)\right.\right.\\
&\qquad+2\overline{l_2}(t,[\theta(t_n)]^\beta)-\frac{3544}{2565}\overline{l_3}(t,[\theta(t_n)]^\beta)\\
&\qquad\left.\left.+\frac{1859}{4104}\overline{l_4}(t,[\theta(t_n)]^\beta)-\frac{11}{40}\overline{l_5}(t,[\theta(t_n)]^\beta)\right]\right\}.
\end{aligned}
$$

Define

$$
\begin{cases}
F[t_n,[\theta(t_n)]^\beta]]=\dfrac{16}{135}\underline{l_1}(t_n,[\theta(t_n)]^\beta)+\dfrac{6656}{12825}\underline{l_3}(t_n,[\theta(t_n)]^\beta)\\
+\dfrac{28561}{56430}\underline{l_4}(t_n,[\theta(t_n)]^\beta)-\dfrac{9}{50}\underline{l_5}(t_n,[\theta(t_n)]^\beta)\\
+\dfrac{2}{55}\underline{l_6}(t_n,[\theta(t_n)]^\beta)\\
G[t_n,[\theta(t_n)]^\beta]]=\dfrac{16}{135}\overline{l_1}(t_n,[\theta(t_n)]^\beta)+\dfrac{6656}{12825}\overline{l_3}(t_n,[\theta(t_n)]^\beta)\\
+\dfrac{28561}{56430}\overline{l_4}(t_n,[\theta(t_n)]^\beta)-\dfrac{9}{50}\overline{l_5}(t_n,[\theta(t_n)]^\beta)\\
+\dfrac{2}{55}\overline{l_6}(t_n,[\theta(t_n)]^\beta)
\end{cases}
$$

From the above equations, we have

$$
\begin{cases}
\underline{\theta}_{n+1}(\beta)=\underline{\theta}_n(\beta)+\ F[t_n,\underline{\theta}_n(\beta),\overline{\theta}_n(\beta)]\\
\overline{\theta}_{n+1}(\beta)=\overline{\theta}_n(\beta)+\ G[t_n,\underline{\theta}_n(\beta),\overline{\theta}_n(\beta)]\\
\underline{\theta}_0(\beta)=\underline{\theta_0}(\beta)\\
\overline{\theta}_0(\beta)=\overline{\theta_0}(\beta)
\end{cases}
\tag{6}
$$

4. Numerical Examples

In this section, in view of the conditions acquired in the previous section, we present some simulation examples to demonstrate the adequacy of the proposed methods and the merits of our approach.

Example 1. Let us consider the following fuzzy integro-differential equation:

$$
\begin{cases}
\theta'(t)=[0.96+0.04\beta,1.01-0.01\beta][e^t-\dfrac{t^2}{2}]+\dfrac{t^2}{2}\displaystyle\int_0^1 s\theta(s)ds\\
\theta(0;\beta)=[0.96+0.04\beta,1.01-0.01\beta],\quad 0\le\beta\le 1,\quad 0\le t,s\le 1.
\end{cases}
$$

The exact solution is given by

$\Theta(t;\beta)=[(0.96+0.04\beta)e^t,\ (1.01-0.01\beta)e^t]$.

The approximate solution, by using the Runge-Kutta Fehlberg method, is given by

$$
\begin{cases}
\underline{\theta}^\beta_{n+1}=\underline{\theta}^\beta_n[1+h+\dfrac{h^2}{2}+\dfrac{h^3}{6}+\dfrac{h^4}{24}+\dfrac{h^5}{120}+\dfrac{h^6}{2080}]\\
\overline{\theta}^\beta_{n+1}=\overline{\theta}^\beta_n[1+h+\dfrac{h^2}{2}+\dfrac{h^3}{6}+\dfrac{h^4}{24}+\dfrac{h^5}{120}+\dfrac{h^6}{2080}]\\
\underline{\theta}^\beta_0=\underline{\theta}_0\\
\overline{\theta}^\beta_0=\overline{\theta}_0.
\end{cases}
$$

Table 1 and Fig. 1 depict a variation among the exact and the estimated solutions at t = 1.

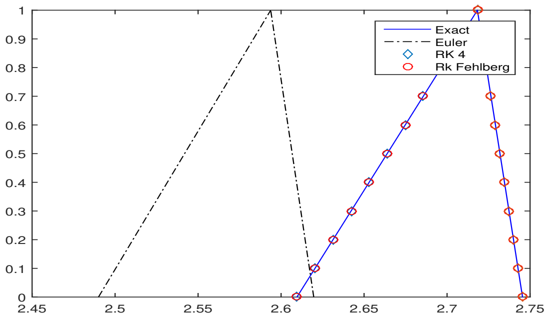

Fig. 1: At t=1 in Ex. 1

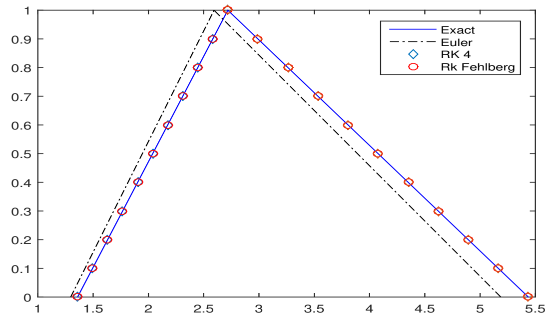

Fig. 3: At t=1 in Ex. 2

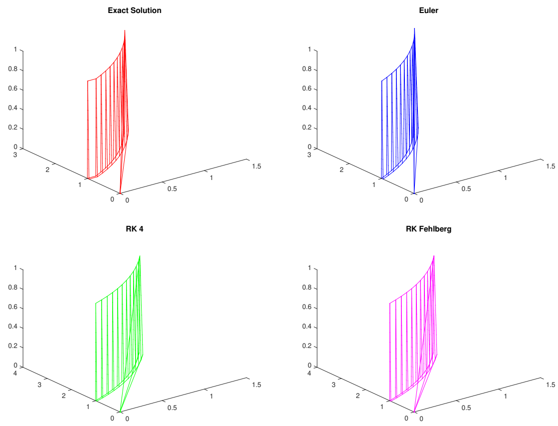

Fig. 2: At t=0.1 in Ex. 1

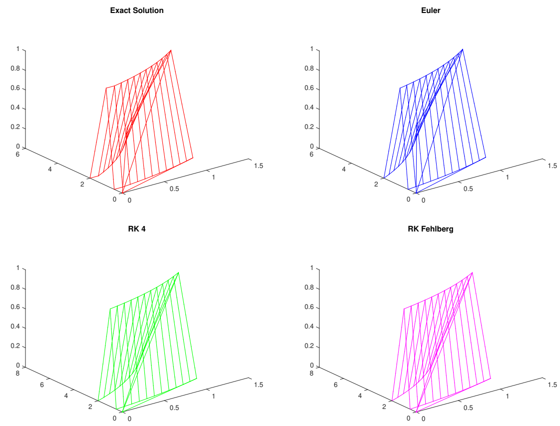

Fig. 4: At t=0.1 in Ex. 2

Example 2. Consider the following fuzzy integro-differential equation:

$$\begin{cases} \theta'(t) = [0.5+0.5\beta, 2-\beta][e^t + (\frac{1-e^2}{2})t] + \int_0^1 t\theta^2(s)ds \\ \theta(0;\beta) = [0.5+0.5\beta, 2-\beta], \quad 0 \le \beta \le 1, \quad 0 \le t, s \le 1. \end{cases}$$

The exact solution is given by

$\Theta(t;\beta) = [(0.5+0.5\beta)e^t,\ (2-\beta)e^t]$.

The approximate solution, by using RKF method, is given by

$$\begin{cases} \underline{\theta}^{\beta}_{n+1} = \underline{\theta}^{\beta}_{n}[1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{2080}] \\ \overline{\theta}^{\beta}_{n+1} = \overline{\theta}^{\beta}_{n}[1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{2080}] \\ \underline{\theta}^{\beta}_{0} = \underline{\theta}_{0} \\ \overline{\theta}^{\beta}_{0} = \overline{\theta}_{0}. \end{cases}$$

Table 2 and Fig. 3 depict a comparison between the exact and the approximate solutions at t = 1. In addition, evolution of curves are represented in Figs. 2 and 4 with $t = 0.1$, respectively. From Tables 1 and 2, it can be observed that the errors obtained by the proposed method are better than those from Euler and R-K methods of order four.

Remark 1. To the best of the authors knowledge, very few investigations on solving Fuzzy Fredholm integro-differential equations using numerical techniques [19, 26] and [32]. Therefore, in this paper, we study the Runge-Kutta Fehlberg method for solving linear and nonlinear fuzzy Fredholm integro-differential equations, as summarized in Section 3.

Table 1: Error analysis in Ex. 1

	Exact solution		Error in Euler method		Error in RK four		Error in RK Fehlberg	
β	$\underline{\Theta}(t,\beta)$	$\overline{\Theta}(t,\beta)$	$\underline{\theta_1}(t,\beta)$	$\overline{\theta_1}(t,\beta)$	$\underline{\theta_1}(t,\beta)$	$\overline{\theta_1}(t,\beta)$	$\underline{\theta_1}(t,\beta)$	$\overline{\theta_1}(t,\beta)$
0	2.609550555	2.745454547	0.1196	0.1258	0.2001e-5	0.2105e-5	0.2192e-7	0.2306e-7
0.1	2.620423683	2.742746365	0.1201	0.1257	0.2009e-5	0.2103e-5	0.2201e-7	0.2304e-7
0.2	2.631296810	2.740028083	0.1206	0.1255	0.2018e-5	0.2101e-5	0.2210e-7	0.2301e-7
0.3	2.642169937	2.737309801	0.1211	0.1254	0.2026e-5	0.2099e-5	0.2219e-7	0.2299e-7
0.4	2.653043065	2.734591519	0.1216	0.1253	0.2034e-5	0.2097e-5	0.2228e-7	0.2297e-7
0.5	2.663916192	2.731873238	0.1220	0.1252	0.2043e-5	0.2095e-5	0.2237e-7	0.2294e-7
0.6	2.674789319	2.729154956	0.1225	0.1250	0.2051e-5	0.2093e-5	0.2247e-7	0.2292e-7
0.7	2.685662447	2.726436674	0.1230	0.1249	0.2059e-5	0.2091e-5	0.2256e-7	0.2290e-7
0.8	2.696535574	2.723718392	0.1235	0.1248	0.2068e-5	0.2088e-5	0.2265e-7	0.2288e-7
0.9	2.707408701	2.72100011	0.1240	0.1247	0.2076e-5	0.2086e-5	0.2274e-7	0.2285e-7
1	2.718281828	2.718281828	0.1245	0.1245	0.2084e-5	0.2084e-5	0.2283e-7	0.2283e-7

Table 2: Error analysis in Ex. 2

	Exact solution		Error in Euler method		Error in RK four		Error in RK Fehlberg	
β	$\underline{\Theta}(t,\beta)$	$\overline{\Theta}(t,\beta)$	$\underline{\theta_1}(t,\beta)$	$\overline{\theta_1}(t,\beta)$	$\underline{\theta_1}(t,\beta)$	$\overline{\theta_1}(t,\beta)$	$\underline{\theta_1}(t,\beta)$	$\overline{\theta_1}(t,\beta)$
0	1.359140914	5.436563657	0.0623	0.2491	0.1042e-5	0.4169e-5	0.1142e-7	0.4566e-7
0.1	1.495055006	5.164735474	0.0685	0.2366	0.1146e-5	0.3960e-5	0.1256e-7	0.4338e-7
0.2	1.630969097	4.892907291	0.0747	0.2242	0.1251e-5	0.3752e-5	0.1370e-7	0.4109e-7
0.3	1.766883188	4.621079108	0.0810	0.2117	0.1355e-5	0.3543e-5	0.1484e-7	0.3881e-7
0.4	1.902797280	4.349250926	0.0872	0.1993	0.1459e-5	0.3335e-5	0.1598e-7	0.3653e-7
0.5	2.038711371	4.077411743	0.0934	0.1868	0.1563e-5	0.3126e-5	0.1712e-7	0.3425e-7
0.6	2.174625463	3.80559456	0.0996	0.1744	0.1667e-5	0.2918e-5	0.1826e-7	0.3196e-7
0.7	2.310539554	3.533766377	0.1059	0.1619	0.1772e-5	0.2710e-5	0.1941e-7	0.2968e-7
0.8	2.446453646	3.261938194	0.1121	0.1494	0.1876e-5	0.2501e-5	0.2055e-7	0.2740e-7
0.9	2.582367737	2.990110011	0.1183	0.1370	0.1980e-5	0.2293e-5	0.2169e-7	0.2511e-7
1	2.718281828	2.718281828	0.1245	0.1245	0.2084e-5	0.2084e-5	0.2283e-7	0.2283e-7

5. Conclusion

The problem of Runge-Kutta Fehlberg method for solving fuzzy integro-differential equations has been studied. We have transformed the original problem to two parametric ODEs, which are solved by the Runge-Kutta Fehlberg method. Two numerical examples have been given. The obtain error rates from the Runge-Kutta Fehlberg method, the Runge-kutta method of order four, and the Euler method are summarized in Tables 1 and 2. It can be seen that the Runge-Kutta Fehlberg method is able to produce lower error rates as compared with those from the Runge-Kutta method of order four and the Euler method.

For further work, the solutions of higher-order fuzzy integro-differential equations will be investigated for solving a variety of problems.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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