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Endpoint Approximation of Standard Three-Step Multi-Valued Iteration Algorithm for Nonexpansive Mappings

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Abstract: This paper is intended to introduce the standard three-step iteration algorithm to approximate endpoints of nonexpansive multi-valued mapping in the Banach spaces framework. Some weak and strong results under appropriate conditions for the iterative sequence generated by the proposed process convergence outcomes are discussed. In general, the results improve and unify some of Panyanak's recent analysis (J. Fixed Point Theory Appl. (2018)).

Keywords: fixed point, condition (J), endpoint, multi-valued nonexpansive mapping, standard three-step iteration algorithm, weak convergence, strong convergence

1 . Introduction

Let $\mathscr{B} = (\mathscr{B}, ||.||)$ be a Banach space and $\mathscr{C}$ be a nonempty subset of $\mathscr{B}$. For $x \in \mathscr{B}$, set

$$d(x,\mathscr{C}) = \inf\{||x-y|| : y \in \mathscr{C}\}$$

and

$$\mathscr{D}(x,\mathscr{C}) = \sup\{||x-y|| : y \in \mathscr{C}\}.$$

We shall denote the set of all nonempty and compact subsets of $\mathscr{C}$ by $\mathscr{K}(\mathscr{C})$. Assuming $\mathscr{B}$ as having two bounded subsets namely, $\mathscr{A}$ and $\mathscr{B}^*$, the Hausdorff dance between them is defined as:

$$\mathscr{H}(\mathscr{A},\mathscr{B}^*) = \max\left\{\sup_{x\in\mathscr{A}} d(x,\mathscr{B}^*),\ \sup_{y\in\mathscr{B}^*} d(\mathscr{A},y)\right\}.$$

$\mathscr{H}(\cdot,\cdot)$ is known as the Hausdorff metric on the set $\mathscr{K}(\mathscr{C})$. A multi-valued mapping $\wp : \mathscr{C} \to \mathscr{K}(\mathscr{C})$ is said to be nonexpansive if

$$\mathscr{H}(\wp x, \wp y) \le ||x-y||,$$

for each $x,y \in \mathscr{C}$. Throughout this paper, $\mathscr{N}$ stands for the set of natural numbers, and $\mathscr{R}$ stands for the set of real numbers. A point $q \in \mathscr{C}$ is said to be a fixed point of $\wp : \mathscr{C} \to \mathscr{K}(\mathscr{C})$ if $q \in \wp q$ and is said to be an endpoint (or a stationary point) of $\wp : \mathscr{C} \to \mathscr{K}(\mathscr{C})$ if $\wp q = \{q\}$. In this article, we will denote the set of all endpoints and the set of all fixed points of $\wp$ by $\mathscr{E}_\wp$ and $\mathscr{F}_\wp$ respectively. Note that, a multi-valued mapping $\wp : \mathscr{C} \to \mathscr{K}(\mathscr{C})$ is said to satisfy the endpoint condition [1] if $\mathscr{E}_\wp = \mathscr{F}_\wp$.

The existence of fixed points for nonexpansive mappings in Banach spaces was independently studied by Browder [1], Gohde [2] and Kirk [3] in 1965. They showed that every nonexpansive mapping defined on a bounded closed convex subset of a uniformly convex Banach space always has a fixed point. One of the successful iteration methods for finding fixed points of nonexpansive mappings was given by Ishikawa [4] in 1974.

Different iteration processes have been developed to approximate the fixed points of multi-valued mappings. It should be noted that Sastry and Babu [5] proved Mann and Ishikawa-type convergence results for multi-valued

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nonexpansive mappings in the framework of Hilbert spaces. Panyanak [6] extended the results of Sastry and Babu to the framework of uniformly convex Banach spaces. Actually, Panyanak showed some results using Ishikawa-type iteration process without the endpoint condition. Song and Wang [7] proved convergence for Mann and Ishikawa iterates of multivalued nonexpansive mapping $\wp$ under some appropriate conditions, which revises a gap in Panyanak [6] and gave an affirmative answer to Panyanak's open question . Song and Wang [7] reconstructed the iteration process to overcome the limitations in Panyanak's Results. After this, Shahzad and Zegeye [8] constructed an iteration scheme which removes a restrictive condition in Song and Wang results.

Shahzad and Zegeye [8] also relaxed compactness of the domain of $\wp$ and constructed an iteration scheme which removes the restriction of $\wp$ namely, "$\wp(\upsilon) = \upsilon$ for any $\upsilon \in \mathscr{F}_{\wp}$". Note that, their first type iteration also requires the endpoint condition. For a multi-valued mapping $\wp : \mathscr{C} \to \mathscr{K}(\mathscr{C})$, if $q \in \mathscr{C}$ is an endpoint of $\wp$, then q is also a fixed point of $\wp$; but the converse is not always true (see; Example 1, [9]). We refer the reader to relevant articles ([10] - [16]) for the existence of the findings of the multi-valued mapping endpoints in the context of the Banach spaces. Panyanak [17] recently used the Ishikawa-type iterative procedure to estimate the endpoints of multi-valued, nonexpansive mappings in the Banach space. Agarwal et al. [18] have introduced an iteration process called $S-$iteration process, which is independent of both Mann [19] and Ishikawa iterations, for single-valued mappings in Banach spaces. They proved that the rate of convergence of iteration process is the same as Picard iteration process and faster than Mann iteration process for the class of contraction mappings. Later, it was observed that this scheme also converges faster than Ishikawa iteration process. For more details and some recent literature on $S-$iteration process (see; [20] - [26]).

2 Preliminaries

Definition 1. *A Banach space $\mathscr{B}$ is said to be uniformly convex if for each $\alpha \in (0,2]$, there is an existence of $\beta > 0$ such that for $a,b \in \mathscr{B}$ with $\|a\| \leq 1$, $\|b\| \leq 1$ and $\|a-b\| \geq \alpha$, we have*

$$\left|\left|\frac{a+b}{2}\right|\right| \leq 1-\beta.$$

Definition 2. ([27]) A Banach space $\mathscr{B}$ is said to have Opial's property if for each sequence $\{x_\eta\} \in \mathscr{B}$ which weakly converges to $x \in \mathscr{B}$ and for every $y \in \mathscr{B} - \{x\}$, it follows that

$$\limsup_{\eta\to\infty} ||x_\eta - x|| \leq \limsup_{\eta\to\infty} ||x_\eta - y||.$$

Definition 3. *Let $\mathscr{C}$ be a nonempty subset of a Banach space $\mathscr{B}$. A mapping $\wp : \mathscr{C} \to \mathscr{K}(\mathscr{C})$ is said to satisfy condition (J) if there is a non-decreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $g(t) \geq 0$ for $t \in (0,\infty)$ such that*

$$\mathscr{D}(x,\wp x) \geq g(d(x,\mathscr{E}_\wp)),$$

for each $x \in \mathscr{C}$.

The mapping $\wp$ is called *semicompact* if for any sequence $\{x_\eta\}$ in $\mathscr{C}$ such that

$$\lim_{\eta\to\infty} \mathscr{D}(x_\eta, \wp(x_\eta)) = 0$$

There is an existence of a subsequence $\{x_{\eta_k}\}$ of $\{x_\eta\}$ and $s \in \mathscr{C}$ such that $\lim_{k\to\infty} x_{\eta_k} = 0$.

Definition 4. *Let $\mathscr{C}$ be a nonempty subset of a Banach space $\mathscr{B}$. A sequence $\{x_\eta\}$ in $\mathscr{B}$ is called Fejer-monotone with respect to $\mathscr{C}$ if*

$$||x_{\eta+1} - c|| \geq ||x_\eta - c||,$$

for each $c \in \mathscr{C}$ and $n \in \mathscr{N}$.

Lemma 1. A Banach space $\mathscr{B}$ is uniformly convex if and only if for any number $k > 0$, and there is a strictly increasing and continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$||\alpha x + (1-\alpha)y||^2 \leq \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)\psi(||x-y||),$$

for each $x,y \in \mathscr{B}$ with $||x|| \leq k$, $||y|| \leq k$, and $\alpha \in [0,1]$.

Definition 5. *If $\wp$ is a multi-valued mapping defined from $\mathscr{C}$ to $\mathscr{K}(\mathscr{C})$, then the following statements hold.*

1. $d(x,\wp x) = 0 \iff x$ is a fixed point of $\wp$.
2. $\mathscr{D}(x,\wp x) = 0 \iff x$ is an endpoint of $\wp$.
3. If $\wp$ is nonexpansive, then the mapping $h : \mathscr{C} \to \mathscr{R}$ defined by $h(x) = \mathscr{D}(x,\wp x)$ is continuous.

Lemma 2. ([28]) Let $\wp$ be a nonempty closed and convex subset of a uniformly convex Banach space and $\wp : \mathscr{C} \to \mathscr{K}(\mathscr{C})$ be a multi-valued nonexpansive mapping. Then, we have

$$\{x_\eta\} \subset \mathscr{C},$$
$$x_\eta \rightharpoonup x$$
$$\mathscr{D}(x_\eta, \wp x_\eta) \to 0 \Longrightarrow x \in \mathscr{E}_\wp.$$

The following fact is needed and can be found in [29].

Proposition 1. Let $\mathscr{C}$ be a nonempty closed subset of a Banach space. Let $\{x_\eta\}$ be a Fejer-monotone sequence with respect to $\mathscr{C}$. Then, $\{x_\eta\}$ converges (strongly) to the point of $\mathscr{C}$ if and only if

$$\lim_{\eta\to\infty} d(x_\eta,\mathscr{C})=0.$$

The following Lemma will be useful in our subsequent discussion and are easy to establish.

Lemma 3. ([8]) Considering $\{\eta^0_\eta\}$ and $\{\eta^1_\eta\}$ being real sequences, wherein

1. $0\leq \eta^0_\eta\ \eta^1_\eta<1$,
2. $\eta^1_\eta\to 0$ as $n\to\infty$,
3. $\sum \eta^0_\eta\eta^1_\eta=\infty$.

Let there is a real sequence $\{\eta^2_\eta\}$ which is non negative and exists in such a manner that $\sum \eta^0_\eta\eta^1_\eta(1-\eta^1_\eta)\eta^2_\eta$ is bounded, then the sequence $\{\eta^2_\eta\}$ has a null sub-sequence.

3 Results

It is already proved that Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators [1]. Apart from Mann and Ishikawa, there is existence of many iteration algorithms with better convergence rate. Also many iteration algorithms are defined in the setting of more generalized mappings. It is also important to note that many iteration algorithms are special case of some pre-exiting schemes. Inspired and motivated by the results of existing three-step iteration algorithms, we introduce standard three-step iteration algorithm namely, n_v iteration algorithm which is a general defining of many existing iteration algorithms. The standard three-step scheme is defined as follows:

Let $\mathscr{B}$ be a normed linear space, $\mathscr{C}$ be a nonempty convex subset of $\mathscr{B}$ and $\wp:\mathscr{C}\to\mathscr{K}(\mathscr{C})$ be a map. For any $\upsilon_0\in\mathscr{C}$, we have

$$\begin{aligned}\ell_\eta &= \varepsilon^0_\eta\upsilon_\eta+\tau^0_\eta\upsilon'_\eta+\delta^0_\eta\tau_\eta+\varsigma^0_\eta\tau'_\eta; \\ \tau_\eta &= \varepsilon^1_\eta\upsilon_\eta+\tau^1_\eta\upsilon'_\eta+\delta^1_\eta\ell_\eta+\varsigma^1_\eta\ell'_\eta; \\ \upsilon_{\eta+1} &= \varepsilon^2_\eta\upsilon_\eta+\tau^2_\eta\upsilon'_\eta+\delta^2_\eta\tau_\eta+\varsigma^2_\eta\tau'_\eta+\omega_\eta\ell_\eta+\kappa_\eta\ell'_\eta,\end{aligned} \quad (1)$$

such that $\varepsilon^0_\eta+\tau^0_\eta+\delta^0_\eta+\varsigma^0_\eta=1$, $\varepsilon^1_\eta+\tau^1_\eta+\delta^1_\eta+\varsigma^1_\eta=1$ and $\varepsilon^2_\eta+\tau^2_\eta+\delta^2_\eta+\omega_\eta+\kappa_\eta=1$. Also $\upsilon'_\eta\in\wp\upsilon_\eta$ such that $||\upsilon_\eta-\upsilon'_\eta||=\mathscr{D}(\upsilon_\eta,\wp\upsilon_\eta)$, $\tau'_\eta\in\wp\tau_\eta$ such that $||\tau_\eta-\tau'_\eta||=\mathscr{D}(\tau_\eta,\wp\tau_\eta)$ and $\ell'_\eta\in\wp\ell_\eta$ such that $||\ell_\eta-\ell'_\eta||=\mathscr{D}(\ell_\eta,\wp\ell_\eta)$.

In this section, we will study convergence analysis of sequence generated by a standard three-step iteration process for Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces.

Lemma 4. Let $\mathscr{C}$ be a nonempty closed convex subset of a Banach space $\mathscr{B}$, and let mapping $\wp:\mathscr{C}\to\mathscr{K}(\mathscr{C})$ be a multi-valued nonexpansive mapping with $\mathscr{E}_\wp\neq\emptyset$. For arbitrarily chosen $\upsilon_0\in\mathscr{C}$, let the sequence $\{\upsilon_\eta\}$ be generated by n_v iteration algorithm (1) with the condition that

$$\left(\frac{(\varepsilon^1_\eta+\tau^1_\eta)+(\delta^1_\eta+\varsigma^1_\eta)(\varepsilon^0_\eta+\tau^0_\eta)}{1-(\delta^1_\eta+\varsigma^1_\eta)(\delta^0_\eta+\varsigma^0_\eta)}\right)\leq 1 \quad (2)$$

and

$$(\varepsilon^2_\eta+\tau^2_\eta+\delta^2_\eta+\varsigma^2_\eta+(\omega_\eta+\kappa_\eta)(\varepsilon^0_\eta+\tau^0_\eta+\delta^0_\eta+\varsigma^0_\eta))\leq 1,$$

then $\lim_{\eta\to\infty}||\upsilon_\eta-\upsilon_*||$ exists for any $\upsilon_*\in\mathscr{E}_\wp$.

Proof. Let $\upsilon_*\in\mathscr{E}_\wp$ and $n\in\mathscr{N}$, we have

$$\begin{aligned}||\ell_\eta-\upsilon_*|| &= ||\varepsilon^0_\eta\upsilon_\eta+\tau^0_\eta\upsilon'_\eta+\delta^0_\eta\tau_\eta+\varsigma^0_\eta\tau'_\eta-\upsilon_*|| \\ &\leq \varepsilon^0_\eta||\upsilon_\eta-\upsilon_*||+\tau^0_\eta d(\upsilon'_\eta,\wp\upsilon_*) \\ &+\delta^0_\eta||\tau_\eta-\upsilon_*||+\varsigma^0_\eta d(\tau'_\eta,\wp\upsilon_*) \\ &\leq \varepsilon^0_\eta||\upsilon_\eta-\upsilon_*||+\tau^0_\eta\mathscr{H}(\wp\upsilon_\eta,\wp\upsilon_*) \\ &+\delta^0_\eta||\tau_\eta-\upsilon_*||+\varsigma^0_\eta\mathscr{H}(\wp\tau_\eta,\wp\upsilon_*) \\ &\leq \varepsilon^0_\eta||\upsilon_\eta-\upsilon_*||+\tau^0_\eta||\upsilon_\eta-\upsilon_*|| \\ &+\delta^0_\eta||\tau_\eta-\upsilon_*||+\varsigma^0_\eta||\tau_\eta-\upsilon_*|| \\ &= (\varepsilon^0_\eta+\tau^0_\eta)||\upsilon_\eta-\upsilon_*||+(\delta^0_\eta+\varsigma^0_\eta) \\ &\times(||\tau_\eta-\upsilon_*||).\end{aligned}$$

Also,

$$\begin{aligned}||\tau_\eta-\upsilon_*|| &= ||\varepsilon^1_\eta\upsilon_\eta+\tau^1_\eta\upsilon'_\eta+\delta^1_\eta\ell_\eta+\varsigma^1_\eta\ell'_\eta-\upsilon_*|| \\ &\leq \varepsilon^1_\eta||\upsilon_\eta-\upsilon*||+\tau^1_\eta d(\upsilon'_\eta,\wp\upsilon_*) \\ &+\delta^1_\eta||\ell_\eta-\upsilon_*||+\varsigma^1_\eta d(\ell'_\eta,\wp\upsilon_*) \\ &\leq \varepsilon^1_\eta||\upsilon_\eta-\upsilon*||+\tau^1_\eta\mathscr{H}(\wp\upsilon_\eta,\wp\upsilon_*) \\ &+\delta^1_\eta||\ell_\eta-\upsilon_*||+\varsigma^1_\eta\mathscr{H}(\wp\ell_\eta,\wp\upsilon_*) \\ &\leq \varepsilon^1_\eta||\upsilon_\eta-\upsilon_*||+\tau^1_\eta||\upsilon_\eta-\upsilon_*|| \\ &+\delta^1_\eta||\ell_\eta-\upsilon_*||+\varsigma^1_\eta||\ell_\eta-\upsilon_*|| \\ &= (\varepsilon^1_\eta+\tau^1_\eta)||\upsilon_\eta-\upsilon_*||+(\delta^1_\eta+\varsigma^1_\eta) \\ &\times(||\ell_\eta-\upsilon_*||).\end{aligned}$$

Using the value of $||\ell_\eta - \upsilon_*||$, we have

$$\begin{aligned}||\tau_\eta - \upsilon_*|| &\leq (\varepsilon_\eta^1 + \tau_\eta^1)||\upsilon_\eta - \upsilon_*|| + (\delta_\eta^1 + \varsigma_\eta^1) \\ &\times ((\varepsilon_\eta^0 + \tau_\eta^0)||\upsilon_\eta - \upsilon_*|| + (\delta_\eta^0 + \varsigma_\eta^0) \\ &\times ||\tau_\eta - \upsilon_*||) \\ &\leq ((\varepsilon_\eta^1 + \tau_\eta^1) + (\delta_\eta^1 + \varsigma_\eta^1)(\varepsilon_\eta^0 + \tau_\eta^0)) \\ &\times ||\upsilon_\eta - \upsilon_*|| + (\delta_\eta^1 + \varsigma_\eta^1)(\delta_\eta^0 + \varsigma_\eta^0) \\ &\times ||\tau_\eta - \upsilon_*|| \\ ||\tau_\eta - \upsilon_*|| &\leq \left(\frac{(\varepsilon_\eta^1 + \tau_\eta^1) + (\delta_\eta^1 + \varsigma_\eta^1)(\varepsilon_\eta^0 + \tau_\eta^0)}{1 - (\delta_\eta^1 + \varsigma_\eta^1)(\delta_\eta^0 + \varsigma_\eta^0)}\right) \\ &\times (||\upsilon_\eta - \upsilon_*||).\end{aligned}$$

Since

$$\left(\frac{(\varepsilon_\eta^1 + \tau_\eta^1) + (\delta_\eta^1 + \varsigma_\eta^1)(\varepsilon_\eta^0 + \tau_\eta^0)}{1 - (\delta_\eta^1 + \varsigma_\eta^1)(\delta_\eta^0 + \varsigma_\eta^0)}\right) \leq 1,$$

we have

$$||\tau_\eta - \upsilon_*|| \leq ||\upsilon_\eta - \upsilon_*||.$$

Now,

$$\begin{aligned}||\upsilon_{\eta+1} - \upsilon_*|| &\leq ||\varepsilon_\eta^2 \upsilon_\eta + \tau_\eta^2 \upsilon_\eta' + \delta_\eta^2 \tau_\eta + \varsigma_\eta^2 \tau_\eta' \\ &+ \omega_\eta \ell_\eta + \kappa_\eta \ell_\eta' - \upsilon_*|| \\ &\leq \varepsilon_\eta^2 ||\upsilon_\eta - \upsilon_*|| + \tau_\eta^2 d(\upsilon_\eta', \wp\upsilon_*) \\ &+ \delta_\eta^2 ||\tau_\eta - \upsilon_*|| + \varsigma_\eta^2 d(\tau_\eta', \wp\upsilon_*) \\ &+ \omega_\eta ||\ell_\eta - \upsilon_*|| + \kappa_\eta d(\ell_\eta', \wp\upsilon_*) \\ &\leq \varepsilon_\eta^2 ||\upsilon_\eta - \upsilon_*|| + \tau_\eta^2 \mathscr{H}(\wp\upsilon_\eta, \wp\upsilon_*) \\ &+ \delta_\eta^2 ||\tau_\eta - \upsilon_*|| + \varsigma_\eta^2 \mathscr{H}(\wp\tau_\eta, \wp\upsilon_*) \\ &+ \omega_\eta ||\ell_\eta - \upsilon_*|| + \kappa_\eta \mathscr{H}(\wp\ell_\eta, \wp\upsilon_*) \\ &\leq \varepsilon_\eta^2 ||\upsilon_\eta - \upsilon_*|| + \tau_\eta^2 ||\upsilon_\eta - \upsilon_*|| \\ &+ \delta_\eta^2 ||\tau_\eta - \upsilon_*|| + \varsigma_\eta^2 ||\tau_\eta - \upsilon_*|| \\ &+ \omega_\eta ||\ell_\eta - \upsilon_*|| + \kappa_\eta ||\ell_\eta - \upsilon_*|| \\ &\leq (\varepsilon_\eta^2 + \tau_\eta^2)||\upsilon_\eta - \upsilon_*|| \\ &+ (\delta_\eta^2 + \varsigma_\eta^2)||\tau_\eta - \upsilon_*|| \\ &+ (\omega_\eta + \kappa_\eta)||\ell_\eta - \upsilon_*||.\end{aligned}$$

Since

$$||\tau_\eta - \upsilon_*|| \leq ||\upsilon_\eta - \upsilon_*||,$$

we have

$$\begin{aligned}||\upsilon_{\eta+1} - \upsilon_*|| &\leq (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2) \\ &\times (||\upsilon_\eta - \upsilon_*||) + (\omega_\eta + \kappa_\eta) \\ &\times (||\ell_\eta - \upsilon_*||),\end{aligned}$$

on substituting

$$\begin{aligned}||\ell_\eta - \upsilon_*|| &= (\varepsilon_\eta^0 + \tau_\eta^0)||\upsilon_\eta - \upsilon_*|| \\ &+ (\delta_\eta^0 + \varsigma_\eta^0)||\tau_\eta - \upsilon_*||,\end{aligned}$$

we have

$$\begin{aligned}||\upsilon_{\eta+1} - \upsilon_*|| &\leq (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2)||\upsilon_\eta - \upsilon_*|| \\ &+ (\omega_\eta + \kappa_\eta)((\varepsilon_\eta^0 + \tau_\eta^0)||\upsilon_\eta - \upsilon_*|| \\ &+ (\delta_\eta^0 + \varsigma_\eta^0)||\tau_\eta - \upsilon_*||) \\ &\leq (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2)||\upsilon_\eta - \upsilon_*|| \\ &+ (\omega_\eta + \kappa_\eta)((\varepsilon_\eta^0 + \tau_\eta^0) + (\delta_\eta^0 + \varsigma_\eta^0)) \\ &\times ||\upsilon_\eta - \upsilon_*|| \\ &= (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2 + (\omega_\eta + \kappa_\eta) \\ &\times (\varepsilon_\eta^0 + \tau_\eta^0 + \delta_\eta^0 + \varsigma_\eta^0))||\upsilon_\eta - \upsilon_*||.\end{aligned}$$

Also, it is given that

$$\begin{aligned}(\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2 &+ (\omega_\eta + \kappa_\eta) \\ &\times (\varepsilon_\eta^0 + \tau_\eta^0 + \delta_\eta^0 + \varsigma_\eta^0)) \leq 1,\end{aligned}$$

we have

$$||\upsilon_{\eta+1} - \upsilon_*|| \leq ||\upsilon_\eta - \upsilon_*||.$$

This implies that $\{||\upsilon_\eta - \upsilon_*||\}$ is bounded and non-increasing for all $\upsilon_* \in f_\wp$. Hence, $\lim_{\eta\to\infty} ||\upsilon_\eta - \upsilon_*||$ exists, as required.

Theorem 1. Let $\mathscr{C}$ be a uniformly closed and convex subset satisfying Opial's property of a uniformly convex Banach space $\mathscr{B}$, and let a mapping $\wp : \mathscr{C} \to \mathscr{K}(\mathscr{C})$ be a multi-valued nonexpansive mapping. For arbitrarily chosen $\upsilon_0 \in \mathscr{C}$, let the sequence $\{\upsilon_\eta\}$ be generated by n_v iteration algorithm (1) for all $\eta \geq 1$, where $\{\varepsilon_\eta^i\}$,$\{\tau_\eta^i\}$,$\{\delta_\eta^i\}$, $\{\varsigma_\eta^i\}$ for $i = 0,1,2$ also ω_η and κ_η are sequences of real numbers in $[a,b]$, for some a,b with $0 < a \leq b < 1$. Then, for $\mathscr{E}_\wp \neq \emptyset$, $\{\upsilon_\eta\}$ converges weakly to an element of $\mathscr{E}_\wp$.

Proof. Since $\mathscr{E}_\wp \neq \emptyset$, let $\upsilon_* \in \mathscr{E}_\wp$. Using Lemma 2.1, there is an existence of $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = (0)$ such that for $\upsilon_* \in \mathscr{E}_{\wp}$ and $n \in \mathscr{N}$, we have

$$\begin{aligned}||\ell_\eta - \upsilon_*||^2 &= ||\varepsilon_\eta^0 \upsilon_\eta + \tau_\eta^0 \upsilon_\eta' + \delta_\eta^0 \tau_\eta + \varsigma_\eta^0 \tau_\eta' - \upsilon_*||^2 \\ &\leq \varepsilon_\eta^0 ||^2 \upsilon_\eta - \upsilon_*||^2 + \tau_\eta^0 d^2(\upsilon_\eta', \wp \upsilon_*) \\ &+ \delta_\eta^0 ||\tau_\eta - \upsilon_*||^2 + \varsigma_\eta^0 d^2(\tau_\eta', \wp \upsilon_*) \varepsilon_\eta^0 \tau_\eta^0 \delta^0 \\ &\times (1 - (\varepsilon_\eta^0 + \tau_\eta^0 + \delta_\eta^0)) \\ &\times \psi(||(\upsilon_\eta + \upsilon_\eta' + \tau_\eta) - \tau_\eta'||) \\ &\leq \varepsilon_\eta^0 ||\upsilon_\eta - \upsilon_*|| + \tau_\eta^0 \mathscr{H}^2(\wp \upsilon_\eta, \wp \upsilon_*) \\ &+ \delta_\eta^0 ||\tau_\eta - \upsilon_*|| + \varsigma_\eta^0 \mathscr{H}^2(\wp \tau_\eta, \wp \upsilon_*) \\ &\leq \varepsilon_\eta^0 ||\upsilon_\eta - \upsilon_*||^2 + \tau_\eta^0 ||\upsilon_\eta - \upsilon_*||^2 \\ &+ \delta_\eta^0 ||\tau_\eta - \upsilon_*||^2 + \varsigma_\eta^0 ||\tau_\eta - \upsilon_*||^2\end{aligned}$$

which implies

$$\begin{aligned}||\ell_\eta - \upsilon_*||^2 &= (\varepsilon_\eta^0 + \tau_\eta^0) ||\upsilon_\eta - \upsilon_*||^2 \\ &+ (\delta_\eta^0 + \varsigma_\eta^0) ||\tau_\eta - \upsilon_*||^2.\end{aligned}$$

Also,

$$\begin{aligned}||\tau_\eta - \upsilon_*||^2 &= ||\varepsilon_\eta^1 \upsilon_\eta + \tau_\eta^1 \upsilon_\eta' + \delta_\eta^1 \ell_\eta + \varsigma_\eta^1 \ell_\eta' - \upsilon_*||^2 \\ &\leq \varepsilon_\eta^1 ||\upsilon_\eta - \upsilon_*||^2 + \tau_\eta^1 d^2(\upsilon_\eta', \wp \upsilon_*) \\ &+ \delta_\eta^1 ||\ell_\eta - \upsilon_*||^2 + \varsigma_\eta^1 d^2(\ell_\eta', \wp \upsilon_*) \\ &- \varepsilon_\eta^1 \tau_\eta^1 \delta^1 (1 - (\varepsilon_\eta^1 + \tau_\eta^1 + \delta_\eta^1)) \\ &\times \psi(||(\upsilon_\eta + \upsilon_\eta' + \ell_\eta) - \ell_\eta'||) \\ &\leq \varepsilon_\eta^1 ||\upsilon_\eta - \upsilon_*||^2 + \tau_\eta^1 d^2(\upsilon_\eta', \wp \upsilon_*) \\ &+ \delta_\eta^1 ||\ell_\eta - \upsilon_*||^2 + \varsigma_\eta^1 d^2(\ell_\eta', \wp \upsilon_*) \\ &\leq \varepsilon_\eta^1 ||\upsilon_\eta - \upsilon_*||^2 + \tau_\eta^1 \mathscr{H}^2(\wp \upsilon_\eta, \wp \upsilon_*) \\ &+ \delta_\eta^1 ||\ell_\eta - \upsilon_*||^2 + \varsigma_\eta^1 \mathscr{H}^2(\wp \ell_\eta, \wp \upsilon_*) \\ &\leq \varepsilon_\eta^1 ||\upsilon_\eta - \upsilon_*||^2 + \tau_\eta^1 ||\upsilon_\eta - \upsilon_*||^2 \\ &+ \delta_\eta^1 ||\ell_\eta - \upsilon_*||^2 + \varsigma_\eta^1 ||\ell_\eta - \upsilon_*||^2 \\ &= (\varepsilon_\eta^1 + \tau_\eta^1) ||\upsilon_\eta - \upsilon_*||^2 + (\delta_\eta^1 + \varsigma_\eta^1) ||\ell_\eta - \upsilon_*||^2.\end{aligned}$$

Using the value of $||\ell_\eta - \upsilon_*||$, we have

$$\begin{aligned}||\tau_\eta - \upsilon_*||^2 &\leq (\varepsilon_\eta^1 + \tau_\eta^1) ||\upsilon_\eta - \upsilon_*||^2 \\ &+ (\delta_\eta^1 + \varsigma_\eta^1) \times ((\varepsilon_\eta^0 + \tau_\eta^0) ||\upsilon_\eta - \upsilon_*||^2 \\ &+ (\delta_\eta^0 + \varsigma_\eta^0 \times ||\tau_\eta - \upsilon_*||^2) \\ &\leq ((\varepsilon_\eta^1 + \tau_\eta^1) + (\delta_\eta^1 + \varsigma_\eta^1)(\varepsilon_\eta^0 + \tau_\eta^0)) \\ &\times ||\upsilon_\eta - \upsilon_*||^2 + (\delta_\eta^1 + \varsigma_\eta^1)(\delta_\eta^0 + \varsigma_\eta^0) \\ &\times ||\tau_\eta - \upsilon_*||^2 \\ ||\tau_\eta - \upsilon_*||^2 &\leq \left(\frac{\varepsilon_\eta^1 + \tau_\eta^1 + (\delta_\eta^1 + \varsigma_\eta^1)(\varepsilon_\eta^0 + \tau_\eta^0)}{1 - (\delta_\eta^1 + \varsigma_\eta^1)(\delta_\eta^0 + \varsigma_\eta^0)} \right) \\ &\times ||\upsilon_\eta - \upsilon_*||^2.\end{aligned}$$

Since

$$\left(\frac{\varepsilon_\eta^1 + \tau_\eta^1 + (\delta_\eta^1 + \varsigma_\eta^1)(\varepsilon_\eta^0 + \tau_\eta^0)}{1 - (\delta_\eta^1 + \varsigma_\eta^1)(\delta_\eta^0 + \varsigma_\eta^0)} \right) \leq 1,$$

we have

$$||\tau_\eta - \upsilon_*|| \leq ||\upsilon_\eta - \upsilon_*||.$$

Now,

$$\begin{aligned}||\upsilon_{\eta+1} - \upsilon_*||^2 &\leq ||\varepsilon_\eta^2 \upsilon_\eta + \tau_\eta^2 \upsilon_\eta' + \delta_\eta^2 \tau_\eta \\ &+ \varsigma_\eta^2 \tau_\eta' + \omega_\eta \ell_\eta + \kappa_\eta \ell_\eta' - \upsilon_*||^2 \\ &\leq \varepsilon_\eta^2 ||\upsilon_\eta - \upsilon_*||^2 + \tau_\eta^2 d(\upsilon_\eta', \wp \upsilon_*|| \\ &+ \delta_\eta^2 ||_\eta^\tau - \upsilon_*||^2 + \varsigma_\eta^2 d(\tau_\eta', \wp \upsilon_*) \\ &+ \omega_\eta ||\ell_\eta - \upsilon_*||^2 + \kappa_\eta d^2(\ell_\eta', \wp \upsilon_*) \\ &- \varepsilon_\eta^2 \tau_\eta^2 \delta_\eta^2 \varsigma_\eta^2 \omega_\eta \\ &\times (1 - (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2 + \omega_\eta)) \\ &\times \psi(||(\upsilon_\eta + \upsilon_\eta' + \tau_\eta + \tau_\eta' + \ell_\eta) - \ell_\eta'||) \\ &\leq \varepsilon_\eta^2 ||\upsilon_\eta - \upsilon_*||^2 + \tau_\eta^2 \mathscr{H}^2(\wp \upsilon_\eta, \wp \upsilon_*) \\ &+ \delta_\eta^2 ||\tau_\eta - \upsilon_*||^2 + \varsigma_\eta^2 \mathscr{H}^2(\wp \tau_\eta, \wp \upsilon_*) \\ &+ \omega_\eta ||\ell_\eta - \upsilon_*||^2 + \kappa_\eta \mathscr{H}^2(\wp \ell_\eta, \wp \upsilon_*) \\ &- \varepsilon_\eta^2 \tau_\eta^2 \delta_\eta^2 \varsigma_\eta^2 \omega_\eta \\ &\times (1 - (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2 + \omega_\eta)) \\ &\times \psi(||(\upsilon_\eta + \upsilon_\eta' + \tau_\eta + \tau_\eta' + \ell_\eta) - \ell_\eta'||) \\ &\leq \varepsilon_\eta^2 ||\upsilon_\eta - \upsilon_*||^2 + \tau_\eta^2 ||\upsilon_\eta - \upsilon_*||^2 \\ &+ \delta_\eta^2 ||\tau_\eta - \upsilon_*|| + \varsigma_\eta^2 ||\tau_\eta - \upsilon_*||^2 \\ &+ \omega_\eta ||\ell_\eta - \upsilon_*||^2 + \kappa_\eta ||\ell_\eta - \upsilon_*||^2 \\ &- \varepsilon_\eta^2 \tau_\eta^2 \delta_\eta^2 \varsigma_\eta^2 \omega_\eta \\ &\times (1 - (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2 + \omega_\eta)) \times \\ &\psi(||(\upsilon_\eta + \upsilon_\eta' + \tau_\eta + \tau_\eta' + \ell_\eta) - \ell_\eta'||) \\ &\leq (\varepsilon_\eta^2 + \tau_\eta^2) \\ &(||\upsilon_\eta - \upsilon_*||^2) + (\delta_\eta^2 + \varsigma_\eta^2) \\ &\times ||\tau_\eta - \upsilon_*||^2 + (\omega_\eta + \kappa_\eta) ||\ell_\eta - \upsilon_*||^2 \\ &- (1 - (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2 + \omega_\eta)) \times \\ &\psi(||(\upsilon_\eta + \upsilon_\eta' + \tau_\eta + \tau_\eta' + \ell_\eta) - \ell_\eta'||).\end{aligned}$$

Since

$$||\tau_\eta - \upsilon_*||^2 \leq ||\upsilon_\eta - \upsilon_*||^2,$$

we have

$$\begin{aligned}||\upsilon_{\eta+1} - \upsilon_*||^2 &\leq (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2) \times (||\upsilon_\eta - \upsilon_*||^2) \\ &+ (\omega_\eta + \kappa_\eta) ||\ell_\eta - \upsilon_*||^2 - \varepsilon_\eta^2 \tau_\eta^2 \delta_\eta^2 \varsigma_\eta^2 \omega_\eta \\ &\times (1 - (\varepsilon_\eta^2 + \tau_\eta^2 + \delta_\eta^2 + \varsigma_\eta^2 + \omega_\eta)) \\ &\times \psi(||(\upsilon_\eta + \upsilon_\eta' + \tau_\eta + \tau_\eta' + \ell_\eta) - \ell_\eta'||)\end{aligned}$$

on substituting

$$||\ell_\eta - \upsilon_*||^2 = (\varepsilon^0_\eta + \tau^0_\eta)||\upsilon_\eta - \upsilon_*||^2 + (\delta^0_\eta + \varsigma^0_\eta)||\tau_\eta - \upsilon_*||^2,$$

we have

$$\begin{aligned}||\upsilon_{\eta+1} - \upsilon_*||^2 &\le (\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta + \varsigma^2_\eta)||\upsilon_\eta - \upsilon_*||^2 \\ &+ (\omega_\eta + \kappa_\eta)((\varepsilon^0_\eta + \tau^0_\eta)||\upsilon_\eta - \upsilon_*||^2 \\ &+ (\delta^0_\eta + \varsigma^0_\eta) \times ||\tau_\eta - \upsilon_*||^2) \\ &- \varepsilon^2_\eta \tau^2_\eta \delta^2_\eta \varsigma^2_\eta \omega_\eta \\ &\times (1 - (\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta + \varsigma^2_\eta + \omega_\eta)) \times \\ &\psi(||(\upsilon_\eta + \upsilon'_\eta + \tau_\eta + \tau'_\eta + \ell_\eta) - \ell'_\eta||) \\ &\le (\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta + \varsigma^2_\eta)||\upsilon_\eta - \upsilon_*||^2 \\ &+ (\omega_\eta + \kappa_\eta)((\varepsilon^0_\eta + \tau^0_\eta) + (\delta^0_\eta \\ &+ \varsigma^0_\eta)) \times ||\upsilon_\eta - \upsilon_*||^2\end{aligned}$$

and hence, we have

$$\begin{aligned}&= (\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta + \varsigma^2_\eta + (\omega_\eta + \kappa_\eta) \\ &\times (\varepsilon^0_\eta + \tau^0_\eta + \delta^0_\eta + \varsigma^0_\eta))||\upsilon_\eta - \upsilon_*||^2 - \varepsilon^2_\eta \tau^2_\eta \delta^2_\eta \varsigma^2_\eta \omega_\eta \\ &\times (1 - (\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta + \varsigma^2_\eta + \omega_\eta)) \\ &\times \psi(||(\upsilon_\eta + \upsilon'_\eta + \tau_\eta + \tau'_\eta + \ell_\eta) - \ell'_\eta||).\end{aligned}$$

Also, it is given that

$$(\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta + \varsigma^2_\eta + ((\omega_\eta + \kappa_\eta) \times (\varepsilon^0_\eta + \tau^0_\eta + \delta^0_\eta + \varsigma^0_\eta)) \le 1$$

we have

$$\begin{aligned}||\upsilon_{\eta+1} - \upsilon_*||^2 &\le ||\upsilon_\eta - \upsilon_*||^2 - \varepsilon^2_\eta \tau^2_\eta \delta^2_\eta \varsigma^2_\eta \omega_\eta \\ &\times (1 - (\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta + \varsigma^2_\eta + \omega_\eta)) \\ &\times \psi(||(\upsilon_\eta + \upsilon'_\eta + \tau_\eta + \tau'_\eta + \ell_\eta) - \ell'_\eta||).\end{aligned}$$

This follows

$$\begin{aligned}&\sum_{\eta=1}^{\infty} \xi^5(1 - (\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta + \varsigma^2_\eta + \omega_\eta)) \\ &\times \psi||(\upsilon_\eta + \upsilon'_\eta + \tau_\eta + \tau'_\eta + \ell_\eta) - \ell'_\eta|| \\ &\le \sum_{\eta=1}^{\infty} (\varepsilon^2_\eta \tau^2_\eta \delta^2_\eta \varsigma^2_\eta \omega_\eta \times (1 - (\varepsilon^2_\eta + \tau^2_\eta + \delta^2_\eta \\ &+ \varsigma^2_\eta + \omega_\eta))) \times \psi(||(\upsilon_\eta + \upsilon'_\eta + \tau_\eta + \tau'_\eta + \ell_\eta) - \ell'_\eta||).\end{aligned}$$

Thus,

$$\lim_{\eta\to\infty} \psi(||(\upsilon_\eta + \upsilon'_\eta + \tau_\eta + \tau'_\eta + \ell_\eta) - \ell'_\eta||) = 0.$$

Also, it is given that ψ is strictly increasing and continuous function, we have

$$\lim_{\eta\to\infty} ||(\upsilon_\eta + \upsilon'_\eta + \tau_\eta + \tau'_\eta + \ell_\eta) - \ell'_\eta|| = 0.$$

Hence,

$$\lim_{\eta\to\infty} \mathscr{D}(\upsilon_\eta, \wp\upsilon_\eta) = \lim_{\eta\to\infty} ||\upsilon_* - \upsilon_\eta|| = 0.$$

This implies that $\{||\upsilon_\eta - \upsilon_*||\}$ is bounded and non-increasing for all $\upsilon_* \in f_\wp$. Hence, $\lim_{\eta\to\infty} ||\upsilon_\eta - \upsilon_*||$ exists, as required.

Now, we are in the position to prove weak convergence theorem.

Since $f_\wp \neq \emptyset$. By Lemma 2.1 $\{\upsilon_\eta\}$ is bounded and $\lim_{\eta\to\infty} ||\wp\upsilon_\eta - \upsilon_\eta|| = 0$. To show that $\{\upsilon_\eta\}$ converges weakly to an element of $\mathscr{E}_\wp$, it suffices to show that $\{\upsilon_\eta\}$ has a unique weak sub-sequential limit in $\mathscr{E}_\wp$. For this purpose, we assume that there are sub-sequences $\{\upsilon_\eta \varepsilon^\tau\}$ and $\{\upsilon_\eta \varepsilon^\ell\}$ of $\{\upsilon_\eta\}$ such that $\{\upsilon_\eta \varepsilon^\tau\} \rightharpoonup v^1_*$ and $\{\upsilon_\eta \varepsilon^\ell\} \rightharpoonup v^2_*$.

Since, $\lim_{\eta\to\infty} \mathscr{D}(\upsilon_\eta, \wp\upsilon_\eta) = \lim_{\eta\to\infty} ||\upsilon_* - \upsilon_\eta|| = 0$. By Lemma 4, $v^1_* \in \mathscr{E}_\wp$. Similarly, it can be shown that $v^2_* \in \mathscr{E}_\wp$. Next, we prove $v^1_* = v^2_*$. On the contrary, suppose $v^1_* \neq v^2_*$, then by Lemma 4 together with Opial's property, we have

$$\begin{aligned}\lim_{\eta\to\infty} ||\upsilon_\eta - \upsilon_*|| &= \lim_{\eta\to\infty} ||\upsilon_\eta \varepsilon^\tau - \upsilon^1_*|| \\ &< \lim_{\eta\to\infty} ||\upsilon_\eta \varepsilon^\tau - \upsilon^2_*|| \\ &= \lim_{\eta\to\infty} ||\upsilon_\eta - \upsilon^2_*|| \\ &= \lim_{\eta\to\infty} ||\upsilon_\eta \varepsilon^\ell - \upsilon^2_*|| \\ &< \lim_{\eta\to\infty} ||\upsilon_\eta \varepsilon^\ell - \upsilon^1_*|| \\ &= \lim_{\eta\to\infty} ||\upsilon_\eta - \upsilon^1_*||\end{aligned}$$

which is a contradiction. So, $\upsilon^1_* = \upsilon^2_*$. This implies that $\{\upsilon_\eta\}$ converges weakly to a fixed point of $\wp$.

Next, we show significant convergence theorems in Banach, uniformly convex. Opial's property is not essential, but it is appropriate to include some additional conditions.

Theorem 2. Let $\mathscr{C}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{B}$ and $\wp : \mathscr{K} \to \mathscr{K}(\mathscr{C})$ be a multi-valued nonexpansive mapping with $\mathscr{E}_\wp \neq \emptyset$. For arbitrarily chosen $\upsilon_0 \in \mathscr{C}$, let the sequence $\{\upsilon_\eta\}$ be generated by n_v iteration algorithm (1) for all $\eta \ge 1$, where $\{\varepsilon^i_\eta\}$,$\{\tau^i_\eta\}$,$\{\delta^i_\eta\}$, $\{\varsigma^i_\eta\}$ for

$i = 0,1,2$ also ω_η and κ_η are sequences of real numbers in $[a,b]$ for some a and b with $0 < a \leq b < 1$. If $\wp$ satisfies condition (J), then $\{\upsilon_\eta\}$ converges strongly to an endpoint of $\wp$.

Proof. It follows from the nonexpansiveness of $\wp$ that $\mathscr{E}_\wp$ is closed. Since $\wp$ satisfies condition (J),

$$\lim_{\eta\to\infty} d(\upsilon_\eta, \mathscr{E}_\wp).$$

Lemma 3.1 implies that υ_η is Fejer monotone with respect to $\mathscr{E}_\wp$. The conclusion follows from Proposition 2.7.

Conclusion

Standard three-step iteration process namely, n_v iteration process (1) is introduced to find endpoints of nonexpansive multi-valued mapping. n_v iteration scheme unifies most of the existing iteration schemes. For different values of $\varepsilon_\eta^i, \tau_\eta^i, \delta_\eta^i, \zeta_\eta^i, \omega_\eta, \kappa_\eta^i$ for $i = 0,1,2$ iteration schemes like S, CR, Picard-S, Noor, SP and many more can be achieved. Weak and strong convergence results of n_v iteration scheme are also attained.

Authors' Contributions

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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