Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150110

Hamilton-Jacobi-Bellman Equations: An Algorithmic Contraction New Approach

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Received: 21 Feb. 2020, Revised: 22 June 2020, Accepted: 26 June 2020
Published online: 1 Jan. 2021

Abstract: In this paper, we introduce a new method to analyze the convergence of the standard finite element method for Hamilton-Jacobi-Bellman equation (HJB) with noncoercive operators. The method consists of combining Bensoussan-Lions algorithm with the characterization of the solution, in both the continuous and discrete contexts, as fixed point of contraction. Optimal error estimates are then derived, first between the continuous algorithm and its finite element counterpart, and then between the continuous solution and the approximate solution.

Keywords: Algorithm,contraction, Finite element,fixed point, Hamilton-Jacobi-Bellman equation, L^∞-error estimate.

1 Introduction

We are interested in the finite element approximation of the noncoercive problem associated with Hamilton-Jacobi-Bellman equation (HJB): find $u \in W^{2,\infty}(\Omega)$, such that:

$$\begin{cases} \max_{1\leq i\leq M}(A^i u - f^i) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma, \end{cases} \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 1$ with smooth boundary $\Gamma, A^1, ..., A^M$ denote uniformly second order elliptic operators assumed to be noncoercive, and $f^1, ..., f^M$ are M regular functions.

Problems of type (1) arise in many applications: stochastic control, management and economy, mechanics and optics, ...

The HJB equation has been analyticaly studied by [1–4]. For the numerical approximations, P.Cortey Dumont [5] investigated a finite element approximation which is used a subsolution method. M. Boulbrachene and M. Haiour [6] studied a finite element Benssoussan-Lions algorithm version. They obtained a quasi-optimal error estimate in the L^∞-norm. M. Boulbrachene and P.Cortey Dumont [7] explored a finite element method using the concept of subsolution and discrete regularity. They obtained an optimal error estimate in the L^∞-norm.

In the present paper, we instead combine, in both the continuous and discrete contexts, the Benssoussan-Lions algorithm with the characterization of the solution as a fixed point of a contraction. We first establish an error estimate between the continuous algorithm and its finite element version, and then between the exact solution and the finite element approximate.

The paper is organized, as follows: We review in Section 2 the continuous problem and in Section 3 the discrete problem. We address in Section 4 the continuous algorithm and in Section 5 the discrete algorithm and we establish, in both the continuous and discrete cases, the geometrical convergence of this algorithms. Finally, in Section 6, we present the finite element error analysis.

2 The continuous problem

We are concerned the noncoercive problem associated with Hamilton-Jacobi-Bellman equation (HJB): find $u \in W^{2,\infty}(\Omega)$, such that:

$$\begin{cases} \max_{1\leq i\leq M}(A^i u - f^i) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma, \end{cases} \tag{2}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ with smooth boundary $\Gamma, A^1, ..., A^M$ denote uniformly second order

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elliptic operators assumed to be noncoercive defined by:

$$A^i = \sum_{1\leq j,k\leq N} a^i_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{1\leq k\leq N} b^i_k(x)\frac{\partial}{\partial x_k} + a^i_0(x),$$

such that:

$$a^i_{jk}(x), b^i_k(x), a^i_0(x) \in C^2(\overline{\Omega}),$$
$$a^i_{jk}(x) = a^i_{kj}(x); a^i_0(x) \geq \beta > 0, x \in \overline{\Omega},$$
$$\sum_{1\leq j,k\leq N} a^i_{jk}(x)\xi_j\xi_k \geq \alpha|\xi|^2, \forall \xi \in \mathbb{R}^N, x \in \overline{\Omega}, \alpha > 0,$$

and the operators

$$B^i = \sum_{1\leq j,k\leq N} a^i_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{1\leq k\leq N} b^i_k(x)\frac{\partial}{\partial x_k} + (a^i_0(x)+\lambda),$$

where $\lambda > 0$ is large enough so that $B^i = A^i + \lambda I$ are strongly coercive on $H^1(\Omega)$.

We also define the associated bilinear forms

$$a^i(u,v) = \int_\Omega \Big(\sum_{1\leq j,k\leq N} a^i_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} + \sum_{1\leq k\leq N} b^i_k(x)\frac{\partial u}{\partial x_k} v, \tag{3}$$
$$+ a^i_0(x)uv\Big)dx,$$

and

$$b^i(u,v) = a^i(u,v) + \lambda(u,v), \tag{4}$$

where $(.,.)$ is the inner product in $L^2(\Omega)$.

Finally, let $f^1,...,f^M$ be nonnegative right-hand sides in $W^{2,\infty}(\Omega)$.

We are concerned with the coercive HJB equation:

$$\begin{cases} \max_{1\leq i\leq M}(B^i u - F^i(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{5}$$

where $F^i(u) = f^i + \lambda u$.

It is shown in [3] that (5) can be approximated by the following weakly coupled system of QVIs

$$\begin{cases} b^i(\xi^i, v-\xi^i) \geq (F^i(u), v-\xi^i), \forall v \in H^1_0(\Omega), \\ \xi^i \leq k + \xi^{i+1}, v \leq k + \xi^{i+1}, i = 1,...,M \\ \xi^{M+1} = \xi^1 \end{cases} \tag{6}$$

where k is a positive constant. This is, precisely, stated in the following theorem.

Theorem 1. *[3] The system (6) has a unique solution which belongs to $(W^{2,p}(\Omega))^M, 2 \leq p < \infty$. Moreover, as $k \to 0$, each component of this system converges uniformly in $C(\overline{\Omega})$to the solution u of HJB equation (5).*

2.1 The solution of noncoercive HJB equation is the unique fixed point of a contraction

Let the mapping

$$T : L^\infty(\Omega) \to L^\infty(\Omega)$$
$$\omega \to T\omega = \xi,$$

where ξ is the unique solution of the following coercive HJB equation:

$$\begin{cases} \max_{1\leq i\leq M}(B^i\xi - F^i(\omega)) = 0 & \text{in } \Omega \\ \xi = 0 & \text{on } \Gamma. \end{cases} \tag{7}$$

From [3], (7) can be approximted by the following system of QVIs

$$\begin{cases} b^i(\xi^i, v-\xi^i) \geq (F^i(\omega), v-\xi^i), \forall v \in H^1_0(\Omega), \\ \xi^i \leq k + \xi^{i+1}, v \leq k + \xi^{i+1}, i = 1,...,M \\ \xi^{M+1} = \xi^1 \end{cases} \tag{8}$$

and we have $\lim_{k\to 0} \left\|\xi^i - \xi\right\|_{C(\Omega)} = 0, \forall i = 1,...,M.$

Lemma 1. *[8] There exists a constant c independent of k, such that*

$$\left\|\xi^i - \xi\right\|_\infty \leq ck, i = 1,....,M.$$

Lemma 2.*Let $\omega, \widetilde{\omega}$ be in $L^\infty(\Omega)$ and $(\xi^1,....,\xi^M), (\widetilde{\xi}^1,....,\widetilde{\xi}^M)$ be the corresponding solutions to system (8) with right-hand sides $F^i(\omega) = f^i + \lambda\omega$ and $F^i(\widetilde{\omega}) = f^i + \lambda\widetilde{\omega}$, $i = 1,...,M$ respectively. Then we have*

$$\max_{1\leq i\leq M}\left\|\xi^i - \widetilde{\xi}^i\right\|_\infty \leq \rho\left\|\omega - \widetilde{\omega}\right\|_\infty, \rho = \frac{\lambda}{\lambda+\beta} < 1.$$

*Proof.*Let $\phi^i = \frac{1}{\lambda+\beta}\left\|F^i(\omega) - F^i(\widetilde{\omega})\right\|_\infty, i = 1,...,M.$

Then,

$$F^i(\omega) \leq F^i(\widetilde{\omega}) + \left\|F^i(\omega) - F^i(\widetilde{\omega})\right\|_\infty$$
$$\leq F^i(\widetilde{\omega}) + \frac{a^i_0(x)+\lambda}{\lambda+\beta}\left\|F^i(\omega) - F^i(\widetilde{\omega})\right\|_\infty$$
$$\leq F^i(\widetilde{\omega}) + (a^i_0(x)+\lambda)\phi^i, i = 1,...,M.$$

Thus, making use of monotonicity result with respect to right-hand side for system of QVIs related to HJB equation (see [5]), we get:

$$\xi^i \leq \widetilde{\xi}^i + \phi^i,$$

we also get:

$$\left\|\xi^i - \widetilde{\xi}^i\right\|_\infty \leq \phi^i, i = 1,...,M,$$

which completes the proof.

Theorem 2.*Under the conditions of Lemma 2, the mapping T is a contraction, so , the solution of HJB equation (5) is its unique fixed point.*

*Proof.*Let $\xi = T\omega, \widetilde{\xi} = T\widetilde{\omega}$ be solutions of HJB equtionn (7) with right-hand sides $F^i(\omega) = f^i + \lambda\omega$ and $F^i(\widetilde{\omega}) = f^i + \lambda\widetilde{\omega}$, respectively. Then making use of both Theorem 1, Lemma1 and Lemma 2, we have

$$\begin{aligned}\|T\omega - T\widetilde{\omega}\|_\infty &\leq \left\|\xi - \widetilde{\xi}\right\|_\infty \\ &\leq \|\xi - \xi^i\|_\infty + \left\|\xi^i - \widetilde{\xi}^i\right\|_\infty + \left\|\widetilde{\xi}^i - \widetilde{\xi}\right\|_\infty, \\ &\leq ck + \left\|\xi^i - \widetilde{\xi}^i\right\|_\infty + ck, i = 1, ..., M.\end{aligned}$$

Hence, passing to the limit, as $k \to 0$,we get:

$$\|T\omega - T\widetilde{\omega}\|_\infty \leq \max_{1\leq i\leq M} \left\|\xi^i - \widetilde{\xi}^i\right\|_\infty \leq \rho \|\omega - \widetilde{\omega}\|_\infty,$$

$$\rho = \frac{\lambda}{\lambda + \beta} < 1.$$

Thus, T is a contraction.

3 The discrete problem

Let Ω be decomposed into triangles, τ_h denote the set of all those elements, and $h > 0$ be the mesh size. We assume that the family τ_h is regular and quasi-uniform. Let

$$V_h = \left\{v \in C(\overline{\Omega}) \cap H_0^1(\Omega), v|_K \in P_1\right\},$$

be the finite element space, where K is a triangle of τ_h and P_1 is the space of polynomials with degree ≤ 1. Let $\varphi_i, i = 1, ..., m(h)$ be the basis functions of the space V_h, and A^i the matrices with generic coefficients

$$(A^i)_{ls} = a^i(\varphi_l, \varphi_s), l, s = 1, ..., m(h), 1 \leq i \leq M.$$

Let us also define the discrete right-hand sides

$$F^i = (f^i, \varphi_l), l = 1, ..., m(h), 1 \leq i \leq M,$$

and the usual restriction operator r_h :

$$\forall v \in C(\overline{\Omega}) \cap H_0^1(\Omega), r_h v = \sum_{l=1}^{m(h)} v_l \varphi_l.$$

The discrete Hamilton-Jacobi-Bellman equation consists of solving the following problem: Find $u_h \in V_h$ solution to:

$$\begin{cases} \max_{1\leq i\leq M}(A^i u_h - F^i) = 0 \text{ on } \Omega \\ u_h = 0 \text{ on } \Gamma \end{cases}. \quad (9)$$

As in the continuous case, we shall handle the noncoercive problem by transforming (9) into

$$\begin{cases} \max_{1\leq i\leq M}(B^i u_h - F^i(u_h)) = 0 \text{ on } \Omega \\ u_h = 0 \text{ on } \Gamma \end{cases}, \quad (10)$$

where

$$(F^i(u_h))_l = (f^i + \lambda u_h, \varphi_l), l = 1, ..., m(h), 1 \leq i \leq M,$$

and B^i are the matrices defined by

$$(B^i)_{ls} = b^i(\varphi_l, \varphi_s), l, s = 1, ..., m(h), 1 \leq i \leq M.$$

Lemma 3. *[9] The matrices B^i ,$i = 1, ..., M$, are M-matrices.*

Theorem 3. *[5] Under the conditions of Lemma 3, the HJB equation (10) has a unique solution.*

It is shown in [5] that (10) can be approximated by the following discrete weakly coupled system of QVIs

$$\begin{cases} b^i(\xi_h^i, v - \xi_h^i) \geq (F^i(u_h), v - \xi_h^i), \forall v \in V_h, \\ \xi_h^i \leq k + \xi_h^{i+1}, v \leq k + \xi_h^{i+1}, i = 1, ..., M \\ \xi_h^{M+1} = \xi_h^1. \end{cases} \quad (11)$$

Theorem 4. *[5] Under the conditions of Lemma 3. Then, the system (11) has a unique solution. Morover, as $k \to 0$; each component of the solution of this system converges uniformly in $C(\Omega)$ to the solution u_h of (10).*

3.1 The discrete solution of noncoercive HJB equation is the unique fixed point of a contraction

Let $F^i(\omega) = f^i + \lambda\omega, i = 1, ..., M$, we introduce the mapping

$$\begin{aligned} T_h &: L^\infty(\Omega) \to V_h \\ \omega &\to T\omega = \xi_h, \end{aligned}$$

where ξ_h is the unique solution of the following discrete coercive HJB equation

$$\max_{1\leq i\leq M}(B^i \xi_h - F^i(\omega)) = 0, \quad (12)$$

with

$$(F^i(\omega))_l = (f^i + \lambda\omega, \varphi_l), l = 1, ..., m(h), 1 \leq i \leq M,$$

the discrete coercive HJB equation (12) can be approximated by the following system of QVIs

$$\begin{cases} b^i(\xi_h^i, v - \xi_h^i) \geq (F^i(\omega), v - \xi_h^i), \forall v \in V_h, \\ \xi_h^i \leq k + \xi_h^{i+1}, v \leq k + \xi_h^{i+1}, i = 1, ..., M \\ \xi_h^{M+1} = \xi_h^1. \end{cases}$$

and we have [8]:

$$\left\|\xi_h^i - \xi_h\right\|_\infty \leq ck, i = 1, ..., M.$$

Lemma 4.*Under the conditions of Lemma 3. Then, we have*

$$\max_{1\leq i\leq M} \left\|\xi^i - \widetilde{\xi}^i\right\|_\infty \leq \rho \|\omega - \widetilde{\omega}\|_\infty, \rho = \frac{\lambda}{\lambda + \beta} < 1, L^\infty(\Omega).$$

$$\forall \omega, \widetilde{\omega} \in$$

*Proof.*Exactly the same as that of Lemma 2.

Theorem 5.*Under the conditions of Lemma 3. The mapping T_h is a contraction, so the solution of discrete HJB equation (10) is its unique fixed point.*

*Proof.*Exactly the same as that of Theorem 2.

4 A continuous iterative scheme

Starting from $u^0 \in H_0^1(\Omega)$ is the unique solution of the variational equation:

$$a^1(u^0,v) = (f^1,v), \forall v \in H_0^1(\Omega).$$

We define the sequence $(u^n)_{n\geq 1}$ by:

$$u^n = Tu^{n-1}, \forall n \geq 1,$$

such that each iterate u^n solves the coercive HJB equation:

$$\begin{cases} \max_{1\leq i\leq M}(B^i u^n - F^i(u^{n-1})) = 0 \text{ in } \Omega \\ u^n = 0 \ \text{ on } \Gamma \end{cases}, \quad (13)$$

with $F^i(u^{n-1}) = f^i + \lambda u^{n-1}$.

Theorem 6. *Under the conditions of theorem 2., the sequence $(u^n)_{n\geq 0}$ converges to the unique fixed point u and we have the error bound:*

$$\|u^n - u\|_\infty \leq \frac{\rho^n}{1-\rho}\left\|u^0 - u^1\right\|_\infty.$$

Let $(\xi^{i,n})_{1\leq i\leq M}$ be the unique solution of the system of QVIs wich approximates the coercive HJB equation (13):

$$\begin{cases} b^i(\xi^{i,n}, v-\xi^{i,n}) \geq (F^i(u^{n-1}), v-\xi^{i,n}), \forall v \in H_0^1(\Omega), \\ \xi^{i,n} \leq k+\xi^{i+1,n}, v \leq k+\xi^{i+1,n}, i=1,...,M \\ \xi^{M+1,n} = \xi^{1,n}, \end{cases}$$

we have [8]:

$$\left\|\xi^{i,n} - u^n\right\|_\infty \leq ck, i=1,....,M. \quad (14)$$

5 A discrete iterative scheme

Starting from $u_h^0 \in V_h$ the unique solution of the discrete variational equation:

$$a^1(u_h^0,v) = (f^1,v), \forall v \in V_h.$$

We define the sequence $(u_h^n)_{n\geq 1}$ by:

$$u_h^n = T_h u_h^{n-1}, \forall n \geq 1,$$

such that each iterate u_h^n solves the discrete coercive HJB equation:

$$\begin{cases} \max_{1\leq i\leq M}(B^i u_h^n - F^i(u_h^{n-1})) = 0, \text{ in } \Omega \\ u_h^n = 0 \text{ on } \Gamma \end{cases}, \quad (15)$$

with $F^i(u_h^{n-1}) = f^i + \lambda u_h^{n-1}$.

Theorem 7. *Under the conditions of theorem 5, the sequence $(u_h^n)_{n\geq 0}$ converges to the unique fixed point u_h and we have the error bound:*

$$\|u_h^n - u_h\|_\infty \leq \frac{\rho^n}{1-\rho}\left\|u_h^0 - u_h^1\right\|_\infty.$$

Let $(\xi_h^{i,n})_{1\leq i\leq M}$ be the unique solution of the system of QVIs which approximates the discrete coercive HJB equation (15):

$$\begin{cases} b^i(\xi_h^{i,n}, v-\xi_h^{i,n}) \geq (F^i(u_h^{n-1}), v-\xi_h^{i,n}), \forall v \in V_h, \\ \xi_h^{i,n} \leq k+\xi_h^{i+1,n}, v \leq k+\xi_h^{i+1,n}, i=1,...,M \\ \xi_h^{M+1,n} = \xi_h^{1,n}, \end{cases}$$

we have [8]:

$$\left\|\xi_h^{i,n} - u_h^n\right\|_\infty \leq ck, i=1,....,M. \quad (16)$$

6 L^∞-Error estimate

We define the sequence $(\overline{u}_h^n)_{n\geq 0}$ such that:

$$\overline{u}_h^0 = u_h^0, \overline{u}_h^n = T_h u^{n-1}, n \geq 1,$$

where $\overline{u}_h^n$ is the unique solution of the discrete HJB equation:

$$\begin{cases} \max_{1\leq i\leq M}(B^i \overline{u}_h^n - F^i(u^{n-1})) = 0, \text{ in } \Omega \\ \overline{u}_h^n = 0 \text{ on } \Gamma \end{cases} \quad (17)$$

with $F^i(u^{n-1}) = f^i + \lambda u^{n-1}, i=1,...,M$.

Let $(\overline{\xi}_h^{i,n})_{1\leq i\leq M}$ be the unique solution of the system of QVIs which approximates the discrete coercive HJB equation (17):

$$\begin{cases} b^i(\overline{\xi}_h^{i,n}, v-\overline{\xi}_h^{i,n}) \geq (F^i(u^{n-1}), v-\overline{\xi}_h^{i,n}), \forall v \in V_h, \\ \overline{\xi}_h^{i,n} \leq k+\overline{\xi}_h^{i+1,n}, v \leq k+\overline{\xi}_h^{i+1,n}, i=1,...,M \\ \overline{\xi}_h^{M+1,n} = \overline{\xi}_h^{1,n}, \end{cases}$$

we have [8]:

$$\left\|\overline{\xi}_h^{i,n} - \overline{u}_h^n\right\|_\infty \leq ck, i=1,....,M. \quad (18)$$

Lemma 5. *[10] There exists a constant c independent of h such that*

$$\left\|u^0 - u_h^0\right\|_\infty \leq ch^2 \left|\log h\right|.$$

Lemma 6. *[11], [?] There exists a constant c independent of both h and n such that*

$$\max_{1\leq i\leq M}\left\|\xi^{i,n} - \overline{\xi}_h^{i,n}\right\|_\infty \leq ch^2 \left|\log h\right|^3.$$

Theorem 8. *We have*

$$\|u^n - u_h^n\|_\infty \leq \frac{1-\rho^{n+1}}{1-\rho} ch^2 \left|\log h\right|^3, \quad (19)$$

where c is a constant independent of both n and h.

Proof. Combining Lemma 5, Lemma 6 and (14), (18) yields:

$$\begin{aligned}
\|u^1 - u_h^1\|_\infty &\le \|u^1 - \overline{u}_h^1\|_\infty + \|\overline{u}_h^1 - u_h^1\|_\infty \\
&\le \|u^1 - \overline{u}_h^1\|_\infty + \|T_h u^0 - T_h u_h^0\|_\infty \\
&\le \|u^1 - \xi^{i,1}\|_\infty + \|\xi^{i,1} - \overline{\xi}_h^{i,1}\|_\infty + \|\xi_h^{i,1} - \overline{u}_h^1\|_\infty \\
&+ \rho \|u^0 - u_h^0\|_\infty, i = 1,..,M \\
&\le ck + \max_{1\le i\le M} \|\xi^{i,1} - \overline{\xi}_h^{i,1}\|_\infty + ck \\
&+ \rho ch^2 |\log h| \\
&\le ck + ch^2 |\log h|^3 + \rho ch^2 |\log h| \\
&\le ck + (1+\rho) ch^2 |\log h|^3 \\
&\le ck + \frac{1-\rho^2}{1-\rho} ch^2 |\log h|^3.
\end{aligned}$$

Passing to the limit as $k \to 0$, we get

$$\|u^1 - u_h^1\|_\infty \le \frac{1-\rho^2}{1-\rho} ch^2 |\log h|^3 .$$

Now, we assume:

$$\|u^{n-1} - u_h^{n-1}\|_\infty \le \frac{1-\rho^n}{1-\rho} ch^2 |\log h|^3 .$$

Then, combining Lemma 5, Lemma 6 and (14), (18), we get:

$$\begin{aligned}
\|u^n - u_h^n\|_\infty &\le \|u^n - \overline{u}_h^n\|_\infty + \|\overline{u}_h^n - u_h^n\|_\infty \\
&\le \|u^n - \overline{u}_h^n\|_\infty + \left\|T_h u^{n-1} - T_h u_h^{n-1}\right\|_\infty \\
&\le \left\|u^n - \xi^{i,n}\right\|_\infty + \left\|\xi^{i,n} - \overline{\xi}_h^{i,n}\right\|_\infty + \left\|\overline{\xi}_h^{i,n} - \overline{u}_h^n\right\|_\infty + \rho \left\|u^{n-1} - u_h^{n-1}\right\|_\infty , \\
&\quad i = 1, ..., M \\
&\le ck + \max_{1\le i\le M} \left\|\xi^{i,n} - \overline{\xi}_h^{i,n}\right\|_\infty + ck + \rho \frac{1-\rho^n}{1-\rho} ch^2 |\log h|^3 \\
&\le ck + ch^2 |\log h|^3 + \rho \frac{1-\rho^n}{1-\rho} ch^2 |\log h|^3 \\
&\le ck + (1 + \rho \frac{1-\rho^n}{1-\rho}) ch^2 |\log h|^3 \\
&\le ck + \frac{1-\rho^{n+1}}{1-\rho} ch^2 |\log h|^3 .
\end{aligned}$$

Passing to the limite as $k \to 0$, we get

$$\|u^n - u_h^n\|_\infty \le \frac{1-\rho^{n+1}}{1-\rho} ch^2 |\log h|^3 .$$

Theorem 9. *We have*

$$\|u - u_h\|_\infty \le ch^2 |\log h|^3 ,$$

where c is a constant independent of h.

Proof. Combining theorem 6, theorem 7 and theorem 8, we have:

$$\begin{aligned}
\|u - u_h\|_\infty &\le \|u - u^n\|_\infty + \|u^n - u_h^n\|_\infty + \|u_h^n - u_h\|_\infty \\
&\le \frac{\rho^n}{1-\rho} \left\|u^0 - u^1\right\|_\infty + \frac{1-\rho^{n+1}}{1-\rho} ch^2 |\log h|^3 \\
&+ \frac{\rho^n}{1-\rho} \left\|u_h^0 - u_h^1\right\|_\infty , \rho < 1.
\end{aligned}$$

Hence , passing to the limit, as $n \to \infty, \rho^n \to 0$, we get:

$$\|u - u_h\|_\infty \le \frac{c}{1-\rho} h^2 |\log h|^3 .$$

7 Conclusion

Based on the constructive Bensoussan-Lions Algorithm and the Banach fixed point principle, we omit derived error estimate in the maximum norm of the standard finite element approximation of elliptic Hamilton-Jacobi-Bellman equations (HJB) with non coercive operators. This new approach has turned out to be successful and may be extended, in a future work, to system of variational inequalities and quasi-variational inequalities related to HJB equations.

List of abbreviations

Ω : bounded open set of $\mathbb{R}^N$
Γ : smooth boundary of Ω
$\|\|_\infty$: L^∞-norm.
HJB equation: Hamilton-Jacobi-Bellman equation.
$(.,.)$: the inner product in $L^2(\Omega)$.
A^i : noncoercive operators
B^i : coercive operators
$a^i(,)$: bilinear forms of A^i
$b^i(,)$: bilinear forms of B^i
T : contraction mapping
h : mesh size
T_h : contraction mapping
$(u^n)_{n\ge1}$: sequence of continuous iterative sheme
$(u_h^n)_{n\ge1}$: sequence of descrete iterative sheme

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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