Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150112

Calderon-Zygmund Operators and Singular Integrals

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Received: 21 Sep. 2020, Revised: 21 Nov. 2020, Accepted: 3 Dec. 2020
Published online: 1 Jan. 2021

Abstract: In this article, we establish conditions on continuous restrictively bounded linear mapping T from S to S' associated with the kernel K under which the operator T extends to a bounded operator $T\ :\ L^p\left(R^l\right)\to L^p\left(R^l\right)$. Next, we generalize the interpolation theorem for new functional classes, we show that bounded operator T defined, whose kernel satisfies the standard conditions, is bounded with respect to convex seminorm, so, an inequality $\tilde{M}_1{}^{-1}\left(\left\langle \tilde{M}_1\left(\left|T\left(f\right)\right|\right)\right\rangle_\mu\right)\le A_1\tilde{M}_1{}^{-1}\left(\left\langle \tilde{M}_1\left(\left|f\right|\right)\right\rangle_\mu\right)$ holds for the constant A_1 that depends only on A, M_1, M_2.

Keywords: harmonic analysis, singular integrals, convex seminorms, interpolation theorem, Calderon-Zygmund decomposition

1 Introduction

The important tool of the theory of integral transforms is the Calderon-Zygmund decomposition theorem, which states that a given function f can be represented as a sum of two functions one is "good" another is "bad" but on the sets of small measure. Application of this approach to the research of pseudodifferential operators yields the Calderon-Zygmund requirements on the kernels of the corresponded integral transforms that provide its limitation on the functional spaces where these operators can be defined [1 -10].

In the present paper, we establish some properties of the pseudodifferential operators and attempt to generalize the Calderon-Zygmund theory in the case of functional spaces with convex norms. We consider the Calderon-Zygmund operators in Lebesgue spaces and generalize on functions with the convex norms [10 - 23].

Pseudodifferential operators can be associated with a symbols class $S^m_{\rho,\delta}$ that defined by the inequality

$$\left|\partial_x^\beta\partial_\xi^\alpha a\left(x,y\right)\right|\le A_{\alpha,\beta}\left(1+\left|\xi\right|\right)^{m-\rho|\alpha|+\delta|\beta|}.$$

The Calderon-Zygmund operator T is a mapping from test function to distributions that can be defined by the formula

$$T\left(f\right)=\left\langle K\left(x,\cdot\right)f\left(\cdot\right)\right\rangle_\mu=\int_{R^l}K\left(x,y\right)f\left(y\right)d\mu\left(y\right),$$

where associated kernel K is well defined for $x\neq y$ and the inequality

$$\left|\partial_x^\alpha\partial_y^\beta K\left(x,y\right)\right|\le A\left|x-y\right|^{-l-|\alpha|-|\beta|}$$

holds for such $x,\ y$.

Let us assume functions $\mu\left(t\right)\ \eta\left(s\right)\ t,\ s\in[0,\ +\infty)$ are monotonous strictly increasing functions and the function $\eta\left(s\right)$ is an inverse to $\mu\left(t\right)$, and the function$\mu\left(t\right)$ is an inverse to $\eta\left(s\right)$, so

$$s=\mu\left(t\right)=\mu\left(\eta\left(s\right)\right),\quad \mu\left(0\right)=0;$$

$$t=\eta\left(s\right)=\eta\left(\mu\left(t\right)\right),\quad \eta\left(0\right)=0.$$

and let

$$\tilde{M}\left(\tau\right)=\int_0^\tau\mu\left(t\right)dt,$$

$$\tilde{N}\left(\tau\right)=\int_0^\tau\eta\left(s\right)ds\quad and\quad \tau\in[0,\ +\infty)$$

be convex functions.

We establish that if T is a continuous bounded linear mapping from S to S' and its kernel K satisfying inequalities,

$$\left|K\left(x,y\right)\right|\le A\left|x-y\right|^{-l},$$

$$\left|K\left(x,y\right)-K\left(\hat{x},y\right)\right|\le A\frac{\left|x-\hat{x}\right|^\gamma}{\left|x-y\right|^{l+\gamma}},$$

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$$|x-\hat{x}| \leq \frac{1}{2}|x-y|,$$

$$|K(x,y)-K(x,\hat{y})| \leq A\frac{|y-\hat{y}|^{\gamma}}{|x-y|^{l+\gamma}}$$

and

$$|y-\hat{y}| \leq \frac{1}{2}|x-y|,$$

then the operator T extends to a bounded operator $T : L^p\left(R^l\right) \to L^p\left(R^l\right)$ if and only if the operator

$$T^*(g) = \langle \bar{K}^*(\cdot,y)\, g(\cdot)\rangle_\mu = \int_{R^l} \bar{K}^*(x,y)\, g(x)\, d\mu(x)$$

is bounded.

Next, we prove that if the integral operators with well defined and measurable function-kernel $K(x,y)$ that satisfies the regularity condition: there are constants A and $\delta > 1$ such that for all $\breve{y} \in B(y,\varepsilon)$ there is

$$\int_{R^l\setminus B(y,\delta\varepsilon)} |K(x,y)-K(x,\breve{y})|\, d\mu(x) \leq A$$

for all $y \in R^l$ and all $\varepsilon > 0$. And if

$$\tilde{M}_2{}^{-1}\left(\left\langle \tilde{M}_2(|T(f)|)\right\rangle_\mu\right) \leq A\tilde{M}_2{}^{-1}\left(\left\langle \tilde{M}_2(|f|)\right\rangle_\mu\right)$$

for functions $f : \left\langle \tilde{M}_2(|f|)\right\rangle_\mu < \infty$, then

$$\tilde{M}_1{}^{-1}\left(\left\langle \tilde{M}_1(|T(f)|)\right\rangle_\mu\right) \leq A_1\tilde{M}_1{}^{-1}\left(\left\langle \tilde{M}_1(|f|)\right\rangle_\mu\right)$$

holds for all functions $f : \left\langle \tilde{M}_1(|f|)\right\rangle_\mu < \infty$ *and* $\left\langle \tilde{M}_2(|f|)\right\rangle_\mu < \infty$, where A_1 is a constant.

2 Correlation Between Pseudodifferential Operators and Singular Integrals

In this paragraph, we are going to consider a standard integral operator defined on the functional space with the positive Borel measure $d\mu$, which we define by the formula

$$T(f) = \langle K(x,\cdot)\, f(\cdot)\rangle_\mu = \int_{R^l} K(x,y)\, f(y)\, d\mu(y), \quad (1)$$

where the singular kernel $K(x,y)$ satisfies certain regularity conditions, and an adjoint operator is given by

$$T^*(g) = \langle \bar{K}^*(\cdot,y)\, g(\cdot)\rangle_\mu = \int_{R^l} \bar{K}^*(x,y)\, g(x)\, d\mu(x).$$

Let the integral operators be expressed in formula (1), where the singular kernel is such that this integral is well defined and the measurable function-kernel $K(x,y)$ satisfies the regularity condition: here are constants A and $\delta > 1$ such that for all $\breve{y} \in B(y,\varepsilon)$ there is

$$\int_{R^l\setminus B(y,\delta\varepsilon)} |K(x,y)-K(x,\breve{y})|\, d\mu(x) \leq A, \quad (2)$$

for all $y \in R^l$, and all $\varepsilon > 0$. Similarly, for the adjoint kernel, we have constants $\bar{A}$ and $\delta > 1$ such that for all $\breve{x} \in B(x,\varepsilon)$ there is

$$\int_{R^l\setminus B(x,\delta\varepsilon)} |\bar{K}^*(x,y)-\bar{K}^*(\breve{x},y)|\, d\mu(y) \leq \bar{A}, \quad (3)$$

for all $x \in R^l$ and all $\varepsilon > 0$.

Statement 1. *Assume $K(x,y)$ is a function given for $x \neq y$ such that $K(x,y) \geq c|x-y|^{-l}$, $c > 0$. Then there does not exist the operator T that satisfies the next equality, $\left\langle \tilde{M}(|T(f(\cdot))|)\right\rangle \leq C\left\langle \tilde{M}(|f(\cdot)|)\right\rangle$.*

Proof. Let us assume the opposite. Let Q be a cube of side $\frac{1}{2}$, in which its center coincides with the origin. Let us consider the set S of cubes Q_i, which are cube Q translated, so the center is $i \in Z$, namely $Q_i = Q + i$, so we have

$$S = \bigcup_{|i|\leq 2R,\quad i\in Z} Q_i.$$

Let us take $f = \chi_S$, if $x \notin S$, then

$$T(f) = \langle K(x,\cdot)\, f(\cdot)\rangle \geq c\sum_{|i|\leq 2R}\left\langle \frac{\cdot}{|x-\cdot|^l}\right\rangle_{Q_i}.$$

We have

$$T(f) \geq c_1 \log R,$$

for $|x| \leq R$, $R \geq 1$.

So, we obtain

$$\left\langle \tilde{M}|T(f)|\right\rangle \geq \left\langle \tilde{M}|T(f)|\right\rangle_{^C S\bigcap\{x:\,|x|<R\}} \geq$$
$$\geq \tilde{M}(c_1\log R)\, mes\left({}^C S\bigcap\{x:\ |x|<R\}\right) \geq \tilde{M}(c_1\log R)\, R^l,$$

which contradicts with $\left\langle \tilde{M}|f|\right\rangle \leq c_f R^l$. This contradiction proves statement 1.

Let us consider integral representations of pseudo-differential operator T_a presented as following:

$$T_a f = \langle k(x,\cdot)\, f(x-\cdot)\rangle,$$

where $k(x,\cdot)$ is inverse Fourier transform of the function $a(x,\eta)$ such that

$$a(x,\eta) = \langle k(x,\cdot)\exp(-2\pi i\eta\cdot)\rangle.$$

If we assume $K(x,x-y) = k(x,x-y)$, we have

$$T_a f = \langle K(x,\cdot)\, f(\cdot)\rangle.$$

Statement 2. *Assume $a \in S^m$, then $k(x,y) \in C^\infty\left(R^l \times \left(R^l\setminus\{0\}\right)\right)$ and inequality*

$$\left|\partial_x^\beta \partial_y^\alpha k(x,y)\right| \leq A_{\alpha,\beta,L}|y|^{-l-m-|\alpha|-L}, \quad y \neq 0$$

holds for all multi-indices α, β *and all* $L \geq 0$, *so that* $l + m + |\alpha| + L > 0$.

Statement 3. *Assume* $a \in S^0_{1,1}$ *and*

$$T_a f = \left\langle a(x,\cdot)\exp(2\pi i x\cdot)\hat{f}(\cdot)\right\rangle, \quad f \in S$$

then there is the kernel K that satisfies the next inequality:

$$\left|\partial_x^\beta \partial_y^\alpha K(x,y)\right| \leq A_{\alpha,\beta}|x-y|^{-l-|\alpha|-|\beta|}$$

and

$$T_a f = \langle K(x,\cdot) f(\cdot)\rangle$$

for all $x \notin \operatorname{supp}(f)$.

Proof. Let us assume

$$T_a = \sum_{k=0}^{\infty} T_{ak},$$

where $T_{a0} = TS_0$ and $T_{ak} = T_a \Delta_k, \quad k \geq 1$, we have

$$a_0(x,\eta) = a(x,\eta)\hat{\Phi}(\eta)$$

and

$$a_k(x,\eta) = a(x,\eta)\hat{\Psi}\left(2^{-k}\eta\right), \quad k \geq 1,$$

where $\hat{\Phi}(\eta)$ is the Fourier transform of function η, and function Ψ is defined as $\hat{\Psi} = \xi(\eta) - \xi(2\eta)$.

Now, let ξ be a fixed infinitely differentiable function of compact support defined in the η- space R^l such that $\xi(\eta) = 1$ for $|\eta| \leq 1$, and $\xi(\eta) = 0$ for $|\eta| \geq 2$. So, we obtain that

$$1 = \xi(\eta) + \sum_{k=1}^{\infty} \xi\left(2^{-k}\eta\right) - \xi\left(2^{-k+1}\eta\right),$$

and

$$1 = \sum_{k=-\infty}^{\infty} \xi\left(2^{-k}\eta\right) - \xi\left(2^{-k+1}\eta\right), \quad \eta \neq 0.$$

So, we have function $\Phi \in S$ and integrals $\langle\Phi\rangle = 1$ and $\langle\Psi\rangle = 0$.

Next, we have

$$T_{aj} f = \left\langle k_j(x,\cdot) f(x-\cdot)\right\rangle,$$

where kernel k_j satisfies the estimations

$$\left|\partial_x^\beta \partial_y^\alpha k_j(x,y)\right| \leq A_{\alpha,\beta,L}|y|^{-M} 2^{j(l+m+|\alpha|+|\beta|)}, \quad M \geq 0.$$

Applying statement 1 we obtain that $T_a f = \langle k(x,\cdot) f(x-\cdot)\rangle$, where $K(x,y) = k(x,x-y)$.

Let function $f \in S$ has a compact support, then

$$(Tf)(x) = \langle K(x,\cdot) f(\cdot)\rangle$$

for all $x \notin \operatorname{supp}(f)$.

The operator T^* that is an adjoint of T can be defined by the formula

$$\langle Tf, g\rangle = \langle f, T^* g\rangle.$$

The operator T^* is associated with the kernel $K^*(x,y) = \bar{K}(y,x)$. Now, we can prove the following theorem.

Theorem. ***Let*** T ***be a continuous restrictedly bounded linear mapping from*** S ***to*** S' ***and let kernel*** K ***satisfy inequalities***

$$|K(x,y)| \leq A|x-y|^{-l},$$

$$|K(x,y) - K(\hat{x},y)| \leq A\frac{|x-\hat{x}|^\gamma}{|x-y|^{l+\gamma}}, \quad |x-\hat{x}| \leq \frac{1}{2}|x-y|;$$

$$|K(x,y) - K(x,\hat{y})| \leq A\frac{|y-\hat{y}|^\gamma}{|x-y|^{l+\gamma}}, \quad |y-\hat{y}| \leq \frac{1}{2}|x-y|.$$

Then, in order for the operator T ***to extend to a bounded operator to operator*** $T : L^p\left(R^l\right) \to L^p\left(R^l\right)$***, it has been necessary and sufficient that the operator*** T^* ***was restrictedly bounded.***

Proof.

Let φ_{R,x_0} be a normalized test function for the ball $B(x_0,R)$. The restrictedly boundedness of the operator T means that the estimate

$$\left\langle\left(T\varphi_{R,x_0}\right)^p\right\rangle \leq AR^{lp}$$

holds for $T\left(\varphi_{R,x_0}\right) \in L^p$ and all $x_0 \in R^l, \quad R > 0$. Similarly, constant $A > 0$, so that

$$\left\langle\left(T^*\varphi_{R,x_0}\right)^q\right\rangle \leq AR^{lq}$$

for $T^*\left(\varphi_{R,x_0}\right) \in L^q$ and $x_0 \in R^l, \quad R > 0$. So, from the boundedness of T follows an inequality,

$$\left\langle\left(T\varphi_{R,x_0}\right)^p\right\rangle \leq \hat{A}\left\langle\left(\varphi_{R,x_0}\right)^p\right\rangle \leq AR^{lp},$$

and for T, we have

$$\left\langle\left(T^*\varphi_{R,x_0}\right)^q\right\rangle \leq \hat{A}\left\langle\left(\varphi_{R,x_0}\right)^q\right\rangle \leq AR^{lq}.$$

Let us suppose that

$$\langle Tf\rangle = 0, \langle T^* g\rangle = 0,$$

where f and g are smooth functions of compact supports.

Let us denote the partial sum operator S_j by the formula

$$S_j(f) = f * \Phi_{2^{-j}}$$

and

$$\Delta_j(f) = S_j(f) - S_{j-1}(f) = f * \Phi_{2^{-j}}.$$

Supposing $\Phi \in C^\infty$ is supported in the unit ball $|x| < 1$ and $\langle\Phi\rangle = 1$, we have the equality

$$T = \sum_{j=m_1,\ldots,m_1} \left(S_j T S_j - S_{j-1} T S_{j-1}\right)$$

and

$$\sum_{j=m_1,\ldots,m_1}\left(S_jTS_j-S_{j-1}TS_{j-1}\right)=$$

$$=\sum_{j=m_1,\ldots,m_1}\Delta_jTS_j+_{j-1}\sum_{j=m_1,\ldots,m_1}S_{j-1}T\Delta_j.$$

The operator Δ_jTS_j is associated with a smooth kernel K_j as

$$\left(\Delta_jTS_j(f)\right)(x)=\left\langle K_j(x,\cdot)f(\cdot)\right\rangle,$$

where the kernel K_j is such that

$$\left|\partial_x^\alpha\partial_y^\beta K_j(x,y)\right|\leq$$

$$\leq A_{\alpha\beta}2^{(l+|\alpha|+|\beta|)j}\min\left\{1,\left(2^j|x-y|\right)^{-l-\gamma}\right\}.$$

It is easy to see that

$$\left(T(f*\Phi)*\Psi\right)=\left\langle\left\langle T(\Phi(\cdot-y)),\bar{\Psi}(x-\cdot)\right\rangle f(y)\right\rangle_y,$$

so that implies

$$K_j(x,y)=\left\langle T\left(\Phi_{2^{-j}}(\cdot-y)\right),\bar{\Psi}_{2^{-j+1}}(x-\cdot)\right\rangle,$$

that yields an inequality

$$\left|K_j(x,y)\right|\leq A_{\alpha\beta}2^{lj}.$$

Next, from the notation

$$K_j(x,y)=\left\langle\left\langle\Phi_{2^{-j}}(\circ-y)K(\cdot,\circ)\bar{\Psi}_{2^{-j+1}}(x-\cdot)\right\rangle\right\rangle,$$

we have an estimate

$$|K(z,s)-K(x,s)|\leq A2^{j\gamma}\frac{1}{|x-y|^{l+\gamma}},$$

for $\sup p\left(\Phi_{2^{-j}}\right)$ is in the ball $|z-x|\leq\frac{1}{2^{j+1}}$, and $\sup p\left(\bar{\Psi}_{2^{-j+1}}\right)$ is the ball $|s-y|\leq\frac{1}{2^j}$.

Thus, we have estimate

$$\left|\partial_x^\alpha\partial_y^\beta K_j(x,y)\right|\leq$$

$$\leq A_{\alpha\beta}2^{(l+|\alpha|+|\beta|)j}\min\left\{1,\left(2^j|x-y|\right)^{-l-\gamma}\right\}.$$

Applying estimates

$$\left|K_j(z,y)-K_j(x,y)\right|\leq$$

$$\leq A2^{(l+\tilde{\gamma})j}|z-x|^{\tilde{\gamma}}$$

$$\min\left\{1,\left(2^j|x-y|\right)^{-l-\gamma}+\left(2^j|z-y|\right)^{-l-\gamma}\right\}$$

and

$$\left|K_j(z,y)\right|\leq A2^{lj}\left(1+2^j|z-y|\right)^{-l-\gamma},$$

we obtain

$$\left|k_j(x,y)\right|\leq A2^{lj+\tilde{\gamma}(j-i)}\left(1+2^j|z-y|\right)^{-l-\gamma}$$

for $0<\tilde{\gamma}<\gamma$.

Lemma. ***Let us assume that kernel $N(x,y)$ is such that***

$$\sup_x\langle|N(x,\cdot)|\rangle\leq1 \textbf{ and } \sup_y\langle|N(\cdot,y)|\rangle\leq1,$$

and the associated operator S is given by the formula

$$Sf=\langle N(x,\cdot)f(\cdot)\rangle.$$

Then,

$$\|S\|_{L^p\to L^p}\leq1.$$

Proof. Indeed, we can estimate

$$\sup_{f\in L^p;\,g\in L^q}|\langle Sf,g\rangle|$$
$$=\sup_{f\in L^p;\,g\in L^q}|\langle\langle N(\circ,\cdot)f(\cdot)g(\circ)\rangle\rangle|\leq$$
$$\sup_{f\in L^p;\,g\in L^q}\left|\frac{1}{p}\langle\langle N(\circ,\cdot)|f(\cdot)|^p\rangle\rangle+\frac{1}{q}\langle\langle N(\circ,\cdot)|g(\circ)|^q\rangle\rangle\right|.$$

Changing the order of integration and applying our assumption, we obtain the statement of our Lemma.

As a result of this lemma, we have

$$\left\|T_i^*T_j\right\|_{L^q\to L^q}\leq2^{-\widehat{\gamma}|i-j|}.$$

So, we have proven the theorem for such T that $\langle Tf\rangle=0$, $\langle T^*g\rangle=0$ on smooth functions with compact supports.

Now, let us consider the integral

$$\left\langle\left|T\varphi_{R,x_0}-\langle K(\tilde{x},\cdot)f(\cdot)\rangle\right|^p\right\rangle_{\tilde{B}}\leq\hat{A}mes\left(\tilde{B}\right),$$

where $\tilde{B}=B\left(\tilde{x},\tilde{R}\right)$ be any ball, $\tilde{B}_2=B\left(\tilde{x},2\tilde{R}\right)$, and $\tilde{B}_3=B\left(\tilde{x},3\tilde{R}\right)$.

Let $\theta\in C^\infty$ such that

$$\theta=\begin{cases}1, & |x|\leq2\\ 0, & |x|\geq3.\end{cases}$$

We write

$$\varphi_{R,x_0}(x)=\varphi_{R,x_0}(x)\,\theta\left(\frac{x-\tilde{x}}{R}\right)+$$

$$+\varphi_{R,x_0}(x)\left(1-\theta\left(\frac{x-\tilde{x}}{R}\right)\right),$$

then

$$\left\langle\left|T\varphi_{R,x_0}(x)\,\theta\left(\frac{x-\tilde{x}}{R}\right)\right|^p\right\rangle_{\tilde{B}}\leq A\min\left\{R^l,\left(3\tilde{R}\right)^l\right\},$$

and

$$\begin{aligned}&\left|T\left(\varphi_{R,x_0}(x)\left(1-\theta\left(\tfrac{x-\tilde{x}}{R}\right)\right)\right)-\right.\\&\left.-\left\langle K(\tilde{x},\cdot)\varphi_{R,x_0}(x)\left(1-\theta\left(\tfrac{x-\tilde{x}}{R}\right)\right)\right\rangle_{\tilde{B}}\right|\leq\\&\leq\langle|K(x,\cdot)-K(\tilde{x},\cdot)|\rangle_{|y-\tilde{x}|\geq2\tilde{R}}.\end{aligned}$$

So,

$$\left\|T\left(\varphi_{R,x_0}(x)\right)\right\|_{BMO}\leq A.$$

Let us denote operator $S_a = \sum_{j=-\infty}^{\infty} \left(S_{j+m}(a) - S_{j+m-1}(a) \right) S_j$, where $a = T(1)$.

We define the additive set function $d\mu$ defined on ${R_+}^{l+1}$ by the formula

$$d\mu = \sum_{j=-\infty}^{\infty} \left| S_{j+m}(a) - S_{j+m-1}(a) \right| dx \delta_{2^{-j}}(t),$$

where $\delta_{2^{-j}}$ is unit Dirac measure at $t = 2^{-j}$. We have

$$\langle |\langle f(\cdot)\, \Phi_t(x-\cdot) \rangle|^p \rangle_{{R_+}^{l+1}} \le A^p \|f\|_p^p,$$

so

$$\sum_{j=-\infty}^{\infty} \left\| \left(S_{j+m}(a) - S_{j+m-1}(a) \right) S_j(f) \right\|_p^p \le A^p \|f\|_p^p.$$

As a result, we can see that the kernel $K_n(x,y)$ of the operator S_a^n can be estimated from above by

$$\sum_j 2^{lj} \left(1 + 2^j |x-y| \right)^{-l-1} \le A |x-y|^{-l}.$$

Let $\nu_\varepsilon(x) = \nu(\varepsilon x),\ \nu(0) = 1$ be an arbitrary smooth function with compact support. So, we have

$$S_a(\nu_\varepsilon)(x) = S_a(\varphi \nu_\varepsilon)(x)$$

$$+ \langle (K(x,\cdot) - K(0,\cdot))(1 - \varphi(\cdot))\, \nu_\varepsilon(\cdot) \rangle;$$

taking a limit when $\varepsilon \to 0$, we obtain

$$S_a(1)(x) = S_a(\varphi)(x) + \langle (K(x,\cdot) - K(0,\cdot))(1 - \varphi(\cdot)) \rangle.$$

Now, we are streaming $n \to \infty$ obtaining

$$S_{a,n}(1) = \sum_{j=-n}^{n} S_{j+m}(a) - S_{j+m-1}(a) \to a.$$

Similarly, we have

$$S_a{}^*(\nu_\varepsilon)(x) = S_a{}^*(\varphi \nu_\varepsilon)(x)$$

$$+ \langle (\bar{K}(\cdot,y) - \bar{K}(\cdot,0))(1 - \varphi(\cdot))\, \nu_\varepsilon(\cdot) \rangle,$$

and

$$S_{a,n}{}^*(1) = \sum_{j=-n}^{n} S_j{}^* \left(S_{j+m}{}^*(a) - S_{j+m-1}{}^*(a) \right) \to 0.$$

Thus, the general case of the theorem is reduced to the special case with the operator S such that

$$T = T' + S_a + S_b{}^*.$$

Let φ_{R,x_0} be a normalized test function, we have

$$\begin{aligned} &\left(T_\varepsilon \varphi_{R,x_0} \right)(x) = \left\langle K_\varepsilon(x,\cdot)\, \varphi_{R,x_0}(\cdot) \right\rangle \\ &= \left\langle K_\varepsilon(x,\cdot) \left(\varphi_{R,x_0}(\cdot) - \varphi_{R,x_0}(x) \right) \chi_{3R,x_0}(\cdot) \right\rangle \\ &+ \varphi_{R,x_0}(x) \left\langle K_\varepsilon(x,\cdot) \right\rangle_{|y-x|<3R}, \end{aligned}$$

where $K_\varepsilon(x,y) = \nu_\varepsilon\left(\frac{x-y}{\varepsilon}\right) K(x,y)$, which implies the estimate

$$\left| T_\varepsilon\left(\varphi_{R,x_0}(x) \right) \right| \le \frac{AR^l}{|x-x_0|^l}, \quad |x-x_0| \ge 2R.$$

Since the norm of T_ε is uniformly bounded, there is a subsequence $T_{\varepsilon(k)}$, that converges to T weakly in the L^p topology. Invoking estimations for kernels $K_{\varepsilon(k)}(x,y)$, we have $K_{\varepsilon(k)}(x,y)$ converges pointwise to $K(x,y)$, so (1) holds. So, the theorem has been proven.

The singular integrals and convex norm

Let $\mu(t)$ be a monotonous strictly increasing function of the real argument, then this function has an inverse $\eta(s)$, so

$$s = \mu(t) = \mu(\eta(s)), \quad \mu(0) = 0,\ \ \mu(+\infty) = +\infty,$$

and

$$t = \eta(s) = \eta(\mu(t)), \quad \eta(0) = 0,\ \ \eta(+\infty) = +\infty.$$

Then convex functions $\tilde{M}(\tau)$ and $\tilde{N}(\tau)$ can be obtained as $\tilde{M}(\tau) = \int_0^\tau \mu(t)\, dt, \quad \tau \in [0, +\infty)$ and $\tilde{N}(\tau) = \int_0^\tau \eta(s)\, ds, \quad \tau \in [0, +\infty)$, respectively.

There is an integral inequality

$$|\langle f, g \rangle| \le \int_0^\infty \mu(\lambda)\, mes\left\{ x \in R^l : \quad |f| > \lambda \right\} d\lambda +$$

$$+ \int_0^\infty \mu^{-1}(\lambda)\, mes\left\{ x \in R^l : \quad |g| > \lambda \right\} d\lambda$$

for the arbitrary monotonous strictly increasing function $\mu(t)$ of the real argument $\tau \in [0, +\infty)$.

The essential properties of monotone norms:

An integral operator $M(f)(x)$ is defined for an arbitrary locally integrable function f by the formula

$$M(f)(x) = \sup_{r>0} \frac{1}{mes(B(r))} \int_{|y|<r} |f(x-y)|\, dy$$

is called the maximal operator, this operator is well defined on the space of all locally integrable functions.

The grand maximal operator $M_{\mathfrak{Z}}(f)(x)$ can be defined as

$$M_{\mathfrak{Z}}(f)(x) = \sup_{t>0} |(f * \mathfrak{I}(t))(x)|,$$

We are going to describe the nonnegative measure with the weight ω that satisfies the following inequality

$$\left\langle \tilde{M}(M(|f|)\, \omega) \right\rangle \le A \left\langle \tilde{M}(|f|)\, \omega \right\rangle$$

and define the functional class $A_{\tilde{M}}$ as the class of all weights such that

$$\tilde{M}\left(\frac{1}{mes(B)} \langle |f| \rangle_B \right) \le \frac{const}{\langle \omega \rangle_B} \left\langle \omega \tilde{M}(|f|) \right\rangle_B$$

holds for an arbitrary locally integrable function f and any ball B. The smallest constant for which this inequality is valid will be called $A_{\tilde{M}}$ bounds of the weight ω.

From harmonic analysis it is a well-known result that if the measure leads to a weighted maximal inequality, then the weight of the measure belongs to the functional class $A_{\tilde{M}}$.

Let us assume that $f(x) \equiv f\left(x^1,...,x^l\right)$ and $g(x) \equiv g\left(x^1,...,x^l\right)$, $x \in R^l$, $l \in N$ are well defined measurable functions such that $\Phi(u) \equiv \left\langle \tilde{M}(f)\right\rangle < \infty$ and $\Upsilon(f) \equiv \left\langle \tilde{N}(f)\right\rangle < \infty$.

Applying properties of Lebesgue integrals, we can write

$$\langle |f|\rangle = \int_0^\infty mes\left\{x \in R^l : \quad |f| > \lambda\right\} d\lambda,$$

and

$$\left\langle \tilde{M}(|f|)\right\rangle = \int_0^\infty \mu(\lambda)\, mes\left\{x \in R^l : \quad |f| > \lambda\right\} d\lambda.$$

Theorem. ***Let f be a real positive integrable function. Then we have***

$$\left\langle \tilde{M}(f)\right\rangle_B \leq \tilde{M}\left(\langle f\rangle_B\right),$$

where the B is a ball with a measure that equals one.

This theorem can be proven directly as a consequence of Jensen's inequality applied to Lebesgue's sums. We are going to prove this theorem using geometrical arguments, let us rewrite the inequality of the theorem as

$$\left\langle \tilde{M}(f)\right\rangle_B = \int_0^\infty \mu(\lambda)\, mes\{x \in B : \quad f > \lambda\}\, d\lambda$$

$$\leq \tilde{M}\left(\langle f\rangle_B\right) = \int_0^{\int_0^\infty mes\{x\in B: \quad f>\lambda\}d\lambda} \mu(t)\, dt.$$

It is easy to see that the Riemann improper integrals here converge, so for any natural numbers n and i, we can consider the Riemann partition of the real axis λ $[0, n]$ as

$$0 = \lambda_1 < \lambda_2 < ... < \lambda_i = n,$$

the Riemann sums are

$$\sum_k \mu(\lambda_{k+1})\, mes\{x \in B : \quad f > \lambda_{k+1}\}(\lambda_{k+1} - \lambda_k),$$

and

$$\int_0^{\sum_k mes\{x\in B: \quad f>\lambda_{k+1}\}(\lambda_{k+1}-\lambda_k)} \mu(t)\, dt$$

applying strictly the monotony of the function μ, we have

$$\sum_k \mu(\lambda_{k+1})\, mes\{x \in B : \quad f > \lambda_{k+1}\}(\lambda_{k+1} - \lambda_k)$$

$$\leq \int_0^{\sum_k mes\{x\in B: \quad f>\lambda_{k+1}\}(\lambda_{k+1}-\lambda_k)} \mu(t)\, dt.$$

Passing to the limit as $i \to \infty$, we obtain

$$\int_0^n \mu(\lambda)\, mes\{x \in B : \quad f > \lambda\}\, d\lambda \leq$$

$$\leq \int_0^{\int_0^n mes\{x\in B: \quad f>\lambda\}d\lambda} \mu(t)\, dt$$

for any natural numbers n. Since these integrals are convergent we can pass to the limit as $n \to \infty$ and obtain

$$\int_0^\infty \mu(\lambda)\, mes\{x \in B : \quad f > \lambda\}\, d\lambda$$

$$\leq \int_0^{\int_0^\infty mes\{x\in B: \quad f>\lambda\}d\lambda} \mu(t)\, dt.$$

Statement. ***Since the function $\tilde{M}$ is convex, there are several simple correlations:***

1. There is a linear function $a\tau$, such that

$$a\tau < \tilde{M}(\tau);$$

2. for $0 < p < 1$ there is an estimation

$$\tilde{M}(p\tau) < p\tilde{M}(\tau),$$

and for $1 < p$

$$p\tilde{M}(\tau) < \tilde{M}(p\tau);$$

3. let us assume that function f is locally integrable and function φ is positive and locally integrable, then

$$\tilde{M}\left(\frac{\int_a^\tau f(t)\varphi(\tau)\, d\tau}{\int_a^\tau \varphi(\tau)\, \tau}\right) \leq \frac{\int_a^\tau \tilde{M}(f(t))\varphi(\tau)\, d\tau}{\int_a^\tau \varphi(\tau)\, \tau}.$$

3 Generalization of the Interpolation Theorem

Let us consider the singular integral (1) which defines an operator on functional space; here, the singular kernel $K(x,y)$ is such that this integral is well defined in the sense of distribution, and the measurable function-kernel $K(x,y)$ satisfies the regularity condition: there are constants A and $\delta > 1$ such that for all $\check{y} \in B(y,\varepsilon)$ there is inequality (3) for all $y \in R^l$ and all $\varepsilon > 0$. The integral converges absolutely on the complement to the support of f almost everywhere x.

Next, let us introduce a pair of strictly monotonously increasing functions of real argument $\mu_1(t)$ and $\mu_2(t)$, $t \in [0, +\infty)$ such that $t_0 \in (0, \infty)$, so

$$\mu_1(t) \leq \mu_2(t)$$

for all $t > t_0$.

We denote two functions,

$$\tilde{M}_1(\tau) = \int_0^\tau \mu_1(t)\, dt, \quad \tau \in [0, +\infty);$$

$$\tilde{M}_2(\tau) = \int_0^\tau \mu_2(t)\,dt, \quad \tau \in [0, +\infty),$$

these two functions are convex over $\tau \in [0, +\infty)$ and have the property: there is $\tau_0 \in (0, \infty)$ such that

$$\tilde{M}_1(\tau) \le \tilde{M}_2(\tau)$$

for $\tau > \tau_0$.

We can introduce the definitions of operator boundedness as follows. An operator T is said to be $\tilde{M}_1(\tau)$ weakly bounded if

$$mes\mu\{x:\ |T(f(x))| > \lambda\} \le \frac{\left\langle \tilde{M}_1(|f|)\right\rangle_\mu}{\tilde{M}_1(\lambda)},$$

and an operator T is said to be $\tilde{M}_1(\tau)$ strongly bounded if

$$\tilde{M}_1{}^{-1}\left(\left\langle \tilde{M}_1(|T(f)|)\right\rangle_\mu\right) \le \tilde{M}_1{}^{-1}\left(\left\langle \tilde{M}_1(|f|)\right\rangle_\mu\right).$$

Theorem (Interpolation) 1. ***Let us assume that condition (3) is held and the integral operator T (1) satisfies the following integral inequality:***

$$\tilde{M}_2{}^{-1}\left(\left\langle \tilde{M}_2(|T(f)|)\right\rangle_\mu\right) \le A\tilde{M}_2{}^{-1}\left(\left\langle \tilde{M}_2(|f|)\right\rangle_\mu\right)$$

for arbitrary measurable function f under assumption $\left\langle \tilde{M}_2(|f|)\right\rangle_\mu < \infty$, where constant A is the same in regularity condition (3).

Then an inequality

$$\tilde{M}_1{}^{-1}\left(\left\langle \tilde{M}_1(|T(f)|)\right\rangle_\mu\right)$$

$$\le A_1\tilde{M}_1{}^{-1}\left(\left\langle \tilde{M}_1(|f|)\right\rangle_\mu\right)$$

is true for arbitrary measurable function f such that

$$\left\langle \tilde{M}_1(|f|)\right\rangle_\mu < \infty, \quad and \quad \left\langle \tilde{M}_2(|f|)\right\rangle_\mu < \infty,$$

the constant A_1 depends only on A, M_1, and M_2.

To prove the interpolation theorem, we will need an analog of the Calderon-Zygmund decomposition theorem, which states:

Let a function $f \in L^1$ and a positive number α are given such that

$$\frac{1}{\mu(R^l)}\langle |f|\rangle_\mu < \alpha.$$

Then there is a decomposition of $f = g + b$ such that $b = \sum_k b_k$, and there is a sequence of the balls $\{B_k^*\}$, so

$$|g(x)| \le c\alpha, \quad for \quad a.e.\ x;$$

$$\langle |b_k|\rangle_\mu \le c\alpha mes\mu(B_k^*), \quad \langle b_k\rangle_\mu = 0;$$

$$\sum_k \mu(B_k^*) \le \frac{c}{\alpha}\langle |f|\rangle_\mu,$$

where the set B_k^* is the support of the function b_k.

Proof of an analog of Calderon-Zygmund theorem**.** We are going to consider the decomposition of $f = g + b$ in the form

$$mes\mu\left\{x:\ |T(g(x))| > \frac{C}{2}\alpha\right\}$$

$$+mes\mu\left\{x:\ |T(b(x))| > \frac{C}{2}\alpha\right\} \le \frac{\hat{A}}{\alpha}\langle |f|\rangle_\mu.$$

In order to establish that the function g satisfies the inequality $\left\langle \tilde{M}_2(|g|)\right\rangle_\mu < \infty$, we present $\left\langle \tilde{M}_2(|f|)\right\rangle_\mu$ as

$$\left\langle \tilde{M}_2(|f|)\right\rangle_\mu = \int_0^\infty \mu_2(\lambda)\, mes\mu\left\{x \in R^l:\quad |f| > \lambda\right\}d\lambda$$

and integral $\left\langle \tilde{M}_2(|g|)\right\rangle_\mu$ as

$$\begin{aligned}&\left\langle \tilde{M}_2(|g|)\right\rangle_\mu\\ &= \textstyle\int_0^\infty \mu_2(\lambda)\, mes\mu\left\{x \in \bigcup B_k^*:\quad |g| > \lambda\right\}d\lambda\\ &+ \textstyle\int_0^\infty \mu_2(\lambda)\, mes\mu\left\{x \in {}^C\bigcup B_k^*:\quad |g| > \lambda\right\}d\lambda.\end{aligned}$$

From the first statement of the analog of Calderon-Zygmund decomposition theorem, we obtain the inequality

$$\left\langle \tilde{M}_2(|g|)\right\rangle_{\mu({}^C\bigcup B_k^*)} \le \mu_2(c\alpha)\|f\|_{L^1},$$

applying the second statement of the Calderon-Zygmund decomposition, we obtain

$$\left\langle \tilde{M}_2(|g|)\right\rangle_{\mu(\bigcup B_k^*)} \le \frac{c}{\alpha}\tilde{M}_2(c\alpha)\|f\|_{L^1},$$

so

$$\left\langle \tilde{M}_2(|g|)\right\rangle_\mu \le \left(\mu_2(c\alpha) + \frac{c}{\alpha}\tilde{M}_2(c\alpha)\right)\|f\|_{L^1}.$$

Therefore, we have

$$\begin{aligned}&mes\mu\left\{x:\ |T(g(x))| > \tfrac{C}{2}\alpha\right\}\\ &\le \left\langle \tilde{M}_2(|T(g)|)\right\rangle_\mu \left(\tilde{M}_2\left(\tfrac{C}{2}\alpha\right)\right)^{-1}\\ &\le \hat{A}_1\left(\tilde{M}_2\left(\tfrac{C}{2}\alpha\right)\right)^{-1}\left\langle \tilde{M}_2(|g|)\right\rangle_\mu\\ &\le \tfrac{\hat{A}_1}{\alpha}\left(\alpha\mu_2(c\alpha) + c\tilde{M}_2(c\alpha)\right)\|f\|_{L^1}\\ &\le \tfrac{\hat{A}}{\alpha}\|f\|_{L^1}.\end{aligned}$$

Applying the classical approach, we consider $T(b)$. Function b can be defined as

$$b_k(x) = \chi_{Q(k)}\left(f(x) - \frac{1}{\mu(Q(k))}\langle |f|\rangle_{\mu(Q(k))}\right),$$

where $B_k \subset Q_k \subset B_k^*$, and $\chi_{Q(k)}$ is a characteristic function of cube $Q(k)$. Let $\check{y}_k$ be a common center of B_k^* and B_k^{**} be balls radium ρ and $\rho\varepsilon$, respectively. Since $T(b_k)(x) = \langle |K(x,\cdot) - K(x,\check{y}_k)|\, b_k(\cdot)\rangle_\mu$, we have

$$\left\langle \left|\langle |K(x,\cdot) - K(x,\check{y})|\, b(\cdot)\rangle_\mu\right|\right\rangle_{x\mu({}^C\bigcup B_k^{**})}$$

$$\leq \sum_k \left\langle \left| \langle |K(x,\cdot) - K(x,\check{y}_k)| \, b_k(\cdot) \rangle_\mu \right| \right\rangle_{x\mu\left({}^C\bigcup B_k^{**}\right)}$$

$$\leq \sum_k \left\langle |K(x,\cdot) - K(x,\check{y}_k)| \, \langle |b_k| \rangle_{\mu(B_k^*)} \right\rangle_{x\mu\left({}^C\bigcup B_k^{**}\right)}.$$

Applying the second and third statement of the Calderon-Zygmund decomposition, we obtain

$$\left\langle \left| \langle |K(x,\cdot) - K(x,\check{y})| \, b(\cdot) \rangle_\mu \right| \right\rangle_{x\mu\left({}^C\bigcup B_k^{**}\right)} \leq \hat{A} \langle |f| \rangle_\mu ,$$

so

$$mes\mu \left\{ x \in {}^C\bigcup B_k^{**} : \ |T(g(x))| > \frac{C}{2}\alpha \right\} \leq \frac{\hat{A}}{\alpha} \|f\|_{L^1}.$$

However, we have

$$mes\mu \left\{ \bigcup B_k^{**} \right\} \leq \sum_k mes\mu \{ B_k^{**} \}$$

$$\leq C \sum_k mes\mu \{ B_k^{*} \} \leq \frac{\hat{A}}{\alpha} \|f\|_{L^1}.$$

Thus, we have obtained

$$mes\mu \{ x : \ |T(g(x))| > \alpha \} \leq \frac{\hat{A}}{\alpha} \langle |f| \rangle_\mu .$$

Next, we are going to prove the statement about the weak interpolation: ***let an operator T satisfy the condition of subadditivity***

$$|T(f+g)| \leq |T(f)| + |T(g)|.$$

Then, if operator T is weakly bounded in $\tilde{M}_1(\tau)$ and in $\tilde{M}_2(\tau)$, it is strongly bounded for any $\tilde{M}(\tau)$, where $\tilde{M}(\tau)$ satisfies the condition: there is $\tau_0 > 0$ such that $\tilde{M}_1(\tau) \leq \tilde{M}(\tau) \leq \tilde{M}_2(\tau)$ for all $\tau > \tau_0$.

Proof. The weakly boundedness means that

$$mes\mu \{ x : \ |T(f(x))| > \lambda \} \leq \frac{\left\langle \tilde{M}_1(|f|) \right\rangle_\mu}{\tilde{M}_1(\lambda)}$$

and

$$mes\mu \{ x : \ |T(f(x))| > \lambda \} \leq \frac{\left\langle \tilde{M}_2(|f|) \right\rangle_\mu}{\tilde{M}_2(\lambda)}.$$

Let $\tilde{M}(\tau)$ be such that $\tilde{M}_1(\tau) \leq \tilde{M}(\tau) \leq \tilde{M}_2(\tau)$, then we present function $f(x)$ as

$$f(x) = h(x) + s(x),$$

where

$$h(x) = \begin{cases} f(x), & for\, x : \ |f(x)| > \lambda \\ 0 & for\, x : \ |f(x)| \leq \lambda \end{cases}$$

and

$$s(x) = \begin{cases} f(x) & for\, x : \ |f(x)| \leq \lambda \\ 0 & for\, x : \ |f(x)| > \lambda . \end{cases}$$

We can write

$$\left\langle \tilde{M}(|f|) \right\rangle = \int_0^\infty \mu(\lambda)\, mes\mu \left\{ x \in R^l : \ |f| > \lambda \right\} d\lambda,$$

applying subadditivity inequality

$$|T(f+g)| \leq |T(h)| + |T(s)|,$$

we are obtaining

$$mes\mu \{ x : \ |T(f(x))| > \lambda \}$$

$$\leq \frac{\left\langle \tilde{M}_1(|h|) \right\rangle_\mu}{\tilde{M}_1(\lambda)} + \frac{\left\langle \tilde{M}_2(|s|) \right\rangle_\mu}{\tilde{M}_2(\lambda)}$$

and

$$\left\langle \tilde{M}(|T(f)|) \right\rangle_\mu =$$
$$\int_0^\infty \mu(\lambda)\, \mu mes \left\{ x \in R^l : \ |T(f(x))| > \lambda \right\} d\lambda$$
$$\leq \int_0^\infty \mu(\lambda) \frac{\left\langle \tilde{M}_1(|h|) \right\rangle_\mu}{\tilde{M}_1(\lambda)} d\lambda + \int_0^\infty \mu(\lambda) \frac{\left\langle \tilde{M}_2(|s|) \right\rangle_\mu}{\tilde{M}_2(\lambda)} d\lambda.$$

We are estimating each term separately,

$$\int_0^\infty \mu(\lambda) \frac{\left\langle \tilde{M}_1(|h|) \right\rangle_\mu}{\tilde{M}_1(\lambda)} d\lambda$$
$$= \int_0^\infty \frac{\mu(\lambda)}{\tilde{M}_1(\lambda)} \int_0^\infty \mu_1(\sigma)\, \mu mes \left\{ x \in R^l : |h| > \sigma \right\} d\sigma d\lambda$$
$$= \int_0^\infty \frac{\mu(\lambda)}{\tilde{M}_1(\lambda)} \tilde{M}_1(\lambda)\, \mu mes \left\{ x \in R^l : \ |f| > \lambda \right\} d\lambda$$
$$= \left\langle \tilde{M}_1(|f|) \right\rangle_\mu ,$$

and the second:

$$\int_0^\infty \mu(\lambda) \frac{\left\langle \tilde{M}_2(|s|) \right\rangle_\mu}{\tilde{M}_2(\lambda)} d\lambda =$$
$$= \int_0^\infty \frac{\mu(\lambda)}{\tilde{M}_2(\lambda)} \int_0^\infty \mu_2(\sigma)\, \mu mes \left\{ x \in R^l : |s| > \sigma \right\} d\sigma d\lambda =$$
$$= \int_0^\infty \frac{\mu(\lambda)}{\tilde{M}_2(\lambda)} \tilde{M}_2(\lambda)\, \mu mes \left\{ x \in R^l : \ |f| > \lambda \right\} d\lambda =$$
$$= \left\langle \tilde{M}_1(|f|) \right\rangle_\mu .$$

Next, compounding these two integrals together, we have obtained

$$\left\langle \tilde{M}(|T(f)|) \right\rangle_\mu \leq \left\langle \tilde{M}(|f|) \right\rangle_\mu ,$$

which proves our statement about the weak interpolation.

Applying the statement of the weak interpolation, we conclude the proof of the statement of the strong interpolation theorem.

4 An Exemplar of the Calderon-Zygmund Integral

Let us consider a class of singular integrals,

$$T(f) = \langle K(x-\cdot) f(\cdot) \rangle_\mu = \int_{R^l} K(x-y) f(y)\, d\mu(y),$$

where the singular kernel is such that this integral is well defined is the sense of distribution and the measurable function-kernel $K(x,y)$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} K(x)\right| \leq A|x|^{-l-\alpha}$$

for all $x \in R^l, \quad x \neq 0, \quad |\alpha| \leq 1$.

In accordance with classical theory, the truncated approximations can be defined as

$$T_\varepsilon(f) = \langle K_\varepsilon(x-\cdot) f(\cdot)\rangle_\mu = \int_{R^l} K_\varepsilon(x-y) f(y) d\mu(y),$$

where $K_\varepsilon(x) = K(x) \quad if \quad |x| \geq \varepsilon \quad and \quad K_\varepsilon(x) = 0 \quad if \quad |x| < \varepsilon$. The function $T_\varepsilon(f)$ is continuous for all $f \in L^1$.

The maximal operator can be defined as

$$T_M(f(x)) = \sup_{\varepsilon>0} |T_\varepsilon(f(x))|$$

under the condition: There is constant C such that

$$|T(f(x))| \leq |T_M(f(x))| + C|f(x)|.$$

Since set $O = \left\{x: \quad T_M(f(x)) = \sup_{\varepsilon>0} |T_\varepsilon(f(x))| > BC\right\}$ is open, so we can apply Whitney covering lemma and obtain $O = \bigcup Q_i$. Let us consider one of these cubes $\widehat{Q}$ with a diameter d. According to Whitney covering decomposition, we can find point $\hat{x} \in {}^C O$ such that $dist(\hat{x}, O) \leq 4d$, the ball $B = B(\hat{x}, 6d)$, so we have $\widehat{Q} \subset B$.

It has to be shown that

$$mes\{x \in Q: \quad T_M(f(x)) > \alpha \quad and \quad M(f(x)) \leq c\alpha\}$$

$$\leq \frac{AC}{1-b} mes Q$$

hold for all cubes

The function f can be presented as the sum $f = f_1 + f_2$, where

$$f_1 = \begin{cases} f & x \in B \\ 0 & x \in {}^C B \end{cases}$$

and

$$f_1 = \begin{cases} 0 & x \in B \\ f & x \in {}^C B. \end{cases}$$

So, we have

$$T_M(f) \leq T_M(f_1) + T_M(f_2)$$

and

$$\{T_M(f) > \alpha\} \subset \{T_M(f_1) > b_1\alpha\} \bigcup \{T_M(f_2) > b_2\alpha\}$$

for $b_1 + b_2 = 1$.

Since, for $f \in L^1$, we have

$$mes\left\{x \in R^l: \ T_M f(x) > \alpha\right\} \leq \frac{A}{\alpha} \langle |f| \rangle,$$

then

$$mes\{x \in Q: \ T_M f_1(x) > \alpha b_1\} \leq \frac{A}{\alpha b_1} \langle |f_1| \rangle,$$

and

$$\langle |f_1| \rangle \leq AC\alpha mes(Q),$$

so

$$mes\{x \in Q: \ T_M f_1(x) > \alpha b_1\} \leq \frac{AC}{b_1} mes(Q).$$

If $x \in Q$, $y \in {}^C B$ and ${}^C B \subset \{y: \ |y - \hat{y}| \geq d\}$, then

$$|K_\varepsilon(\hat{x} - y) - K_\varepsilon(x - y)| \leq \frac{dA}{|y - \hat{y}|^{l+1}},$$

and we obtain

$$dA \left\langle \frac{|f(\cdot)|}{|\cdot - \hat{y}|^{l+1}} \right\rangle_{|\cdot - \hat{y}| \geq d} = dA \left\langle \frac{|f(\hat{y} - \cdot)|}{|\cdot|^{l+1}} \right\rangle_{|y| \geq d}$$
$$= \sum_i dA \left\langle \frac{|f(\hat{y} - \cdot)|}{|\cdot|^{l+1}} \right\rangle_{2^i d \leq |y| < 2^{i+1} d} \leq \breve{A} \sum_i 2^{-i} M(f(\hat{y})),$$

so

$$|T_M(f_2(\hat{x})) - T_M(f_1(x))| \leq AM(f(\hat{y}))$$

for all $x \in Q$. Taking a supremum over ε, we are obtaining

$$T_M(f_2(x)) - T_M(f(\hat{x})) \leq AM(f(\hat{y})) \leq \alpha(b + CA)$$

for $x \in Q$. Assuming that $b_2 \geq b + CA$, we have $T_M(f_2(x)) < \alpha b_2$.

For $b_1 = \frac{1-b}{2}$, $b_2 = 1 - b$, $0 < b < 1$ and $b_2 \geq b + CA$, we have

$$mes\left\{x \in R^l: \ T_M f(x) > \alpha, \ M(f(x)) \leq C\alpha\right\}$$

$$\leq \frac{AC}{1-b} mes\left\{x \in R^l: T_M f(x) > \alpha b\right\}$$

for all $\alpha > 0$.

Now, we are going to prove that assuming $\omega \in A_{\tilde{M}}$, then there is $0 < \tilde{a} < 1$ such that there is $C > 0$

$$\omega\left\{x \in R^l: \ T_M f(x) > \alpha, \ M(f(x)) \leq C\alpha\right\}$$

$$\leq \tilde{a}\omega\left\{x \in R^l: T_M f(x) > \alpha b\right\}$$

holds for $0 < b < 1$, for all $\alpha > 0$.

Indeed, let us take C small enough so that

$$\omega\{x \in Q: \ T_M f(x) > \alpha, \ M(f(x)) \leq C\alpha\} \leq \check{C}\omega\{Q\}$$

and summing over all cubes, we obtain

$$\omega\left\{x \in \bigcup Q_i: \ T_M f(x) > \alpha, \ M(f(x)) \leq C\alpha\right\}$$

$$\leq \check{C}\omega\left\{x\in\bigcup Q_i : T_M f(x) > \alpha b\right\},$$

which proves our statement.

Assuming $f \neq 0$ gives $M(f(x)) \geq \frac{c}{(1+|x|)^n}$ and for all smooth functions, we have inequality

$$|T_M(f(x))| \leq \frac{A}{(1+|x|)^n}.$$

Every function such that $\left\langle \tilde{M}(|f|\,\omega)\right\rangle < \infty$ can be approximated by elements of C_0^∞, more precisely, for every function f, $\left\langle \tilde{M}(|f|\,\omega)\right\rangle < \infty$, and for any $\varepsilon > 0$, there is a sequence of functions $\phi_k \in C_0^\infty, \quad k \in N$, and there is a natural number $k_0(\varepsilon)$ such that

$$\left\langle \tilde{M}(|f-\phi_k|\,\omega)\right\rangle < \varepsilon$$

for every $k > k_0$. The application of this fact concludes the proving of the following theorem.

Theorem 2. ***Assume that***

$$T(f) = \langle K(x-\cdot)f(\cdot)\rangle_\mu = \int_{R^l} K(x-y)f(y)\,d\mu(y),$$

where the singular kernel is such that this integral is well defined and the measurable function-kernel $K(x,y)$ ***such that***

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha K(x)\right| \leq A|x|^{-l-\alpha}$$

for all $x \in R^l, \quad x \neq 0, \quad |\alpha| \leq 1$***. The maximal operator*** $T_M(f(x))$ ***satisfies the condition:***

$$|T(f(x))| \leq |T_M(f(x))| + |f(x)|.$$

Then the integral estimation

$$\left\langle \tilde{M}(|T(f)|)\right\rangle_\mu \leq A\left\langle \tilde{M}(|f|)\right\rangle_\mu$$

holds for all smooth continuous functions $f \in C_0^\infty$ ***with bounded support.***

5 Conclusion

We have introduced new functional classes and established the generalized interpolation theorem for them; has been shown that bounded operator, whose kernel satisfies the standard conditions, can be extended with the preservation of its boundary constants. In our future works, we are going to generalize these results in the case of abstract Banach space."

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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