Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150201

On Odd Average Harmonious Labeling of Graphs

Porpattama Hammachukiattikul[1], *Grienggrai Rajchakit*[2,*], *Vediyappan Govindan*[5], *S. Dhivya*[3], *Nallappan Gunasekaran*[4], *R. Vadivel*[1] *and Bussakorn Hammachukiattikul*[6]

[1]Department of Mathematics, Phuket Rajabhat University, Phuket 83000, Thailand
[2]Department of Mathematics, Faculty of Science, Maejo University, Sansai 50290, Chiang Mai, Thailand
[3]Department of Mathematics, Sri Vidya Mandir Arts & Science College, Katteri, Uthangarai, India
[4]Department of Mathematical Sciences, Shibaura Institute of Technology, Saitama 337-8570, Japan
[5]Department of Science and Humanities DMI - St John The Baptist University, P.O Box. 406, Mangochi, Malawi, Central Africa
[6]Department of Basic sciences and Mathematics, Faculty of Engineering, Thai-Nichi institute of Technology, Bangkok, 10250, Thailand

Received: 16 Oct. 2020, Revised: 12 Jan. 2021, Accepted: 23 Jan. 2021
Published online: 1 Mar. 2021

Abstract: The present paper investigates new idea of odd average harmonious labelling. A function h is known as odd average harmonious labelling of graph $G(V,E)$ with nodes of n and lines of m if $h:V(G)\to\{0,1,2,\dots,n\}$ is injective and the initiated function $h^*:E(G)\to\{0,1,2,\dots,(m-1)\}$ is characterized as

$$h^*(uv)=\left[\frac{h(u)+h(v)+1}{2}\right](mod\ m)\forall\ uv\in E(G)$$

the resultant edge names will be clearly bijective. A graph that calls for an odd average harmonious labelling is called an odd average harmonious graph.

Keywords: Average Harmonious, Even Average Harmonious, Harmonious labeling, Odd Average Harmonious.

1. Introduction

Graph theory is one of the most vivid and interesting fields of mathematics. This area has been a multifaceted field with fields spanning from the neural network to biotechnology and computational science. The latter part of last century witnessed considerable interest in the analysis of graphs. Research into optimisation methods and the advent of the digital era has accumulated the subject's development. Chart theory includes many important areas of study, such as graph enumeration, graph dominance, algorithmic theory of graphs, topological theory of graphs, fuzzy theory of graphs, graph labeling, etc. Graph Labeling is an important field of graph theory research that has grown primarily due to its multiple uses in coding theory , data network and mobile communication framework (see [1], [2]). A lot of theoretical work has been conducted on the mark of graphs in the recent history. It was introduced in 1967 by A.Rosa. Chart tagging strategies can often be traceable back to Graham and Sloane. When the meaning given to the nodes of graphs is subject to such criteria, it is regarded as graph labeling. Most of the query concerning graph marking will have three different features as follows (see [3] - [7]):

(1)A number collection from which the names of the vertex are obtained.
(2)A law assigning a value for any level.
(3)A condition that those values must satisfy.

Naturally, harmonious graphs emerged in the analysis of compact version of error correction codes and issues with channel distribution. Graham and Sloane showed that if a harmonic has an equivalent number of lines q and the degree of each vertex is divisibility by 2^{κ}, then q is divisible by $2^{\kappa+1}$. In this paper, we introduced the concept of odd average harmonious and odd average harmonious graph and proved that path, comb graph, coconut tree and stars admit odd average harmonious labeling. (see [8] - [12])

* Corresponding author e-mail: kreangkri@mju.ac.th

2. PRELIMINARIES

Definition 1.[8] *A Harmonious Labeling on a graph G is an insertion from G nodes into the graph of κ integer modules, where κ is the number of lines of G, which causes a bijection between the lines of G and the number of modules κ by taking lines of (x,y) as the sum of two nodes $x,y(mod\,\kappa)$ names.*

Definition 2. *A function h is expected to be an odd harmonious labeling of a graph G with q lines if h is an infusion from the nodes of G to the whole numbers from 0 to $2q-1$ to such a degree, that the instigated mapping $h^*(uv) = h(u)+h(v)\,(mod\ 2q)$ from the lines of G to the odd numbers between 1 to $2q-1$ is bijection.*

Definition 3.[9] *A function h is supposed to be an even harmonious labeling of a graph G with q lines if h is an injection from the nodes of G to the whole numbers from 0 to 2q to such an extent that the initiated function h^* from the lines of G to $\{0,2,\ldots,2(q-1)\}$ is characterized by $h^*(uv) = h(u)+h(v)\,(mod\ 2q)$ is bijection.*

Definition 4. *A function h is called average harmonious labelling of the graph $G(V,E)$ with n nodes and m lines if $h : V(G) \to \{0,1,\ldots,n\}$ is injected and the feature $h^* : E(G) \to \{0,1,\ldots,(m-1)\}$ is known as the function*

$$h^*(uv) = \begin{cases} \frac{h(u)+h(v)+1}{2} & if\ h(u)+h(v)\ is\ odd \\ \frac{h(u)+h(v)}{2} & if\ h(u)+h(v)\ is\ even \end{cases}$$

the resultant edge names will be clearly bijective. A graph that accepts an average harmonious labeling is considered an average harmonious graph.

Definition 5. *A function h is called even average harmonious labelling of the graph $G(V,E)$ with n nodes and m lines if $h : V(G) \to \{0,1,\ldots,n\}$ is injected and the feature $h^* : E(G) \to \{0,1,\ldots,(m-1)\}$ is known as the function*

$$h^*(uv) = \left[\frac{h(u)+h(v)}{2}\right](mod\ m)\ \forall\ uv \in E(G)$$

the resultant edge names will be clearly bijective. A graph that accepts an even average harmonious labelling is considered an even average harmonious graph.

Definition 6. *A function h is called odd average harmonious labeling of graph $G(V,E)$ with n nodes and m lines if $h : V(G) \to \{0,1,\ldots,n\}$ is injected and the feature $h^* : E(G) \to \{0,1,\ldots,(m-1)\}$ is known as the function*

$$h^*(uv) = \left[\frac{h(u)+h(v)+1}{2}\right](mod\ m)\ \ for\ all\ uv \in E(G)$$

the resultant edge names will be clearly bijective. A network diagram that accepts an odd average harmonious labeling is considered an odd average harmonious graph .

Definition 7. *A Comb is a network diagram computed by adding every vertex of a path to one edge of a pendant.*

Definition 8.*Bistar is the network diagram computed by adding to the apex nodes two versions of star $K_{1,n}$*

3. MAIN RESULT

Theorem 1. *The path $P_n(n \geq 2)$ is an odd average harmonious graph, when n is even.*

Proof. Let P_n be the path network diagram with n nodes and $m = n-1$ lines.
Let $V(P_n) = \{v_j\ : 0 \leq j \leq n-1\}$, and
$E(P_n) = \{v_j v_{j+1}\ : 1 \leq j \leq n-1\}$.
Define an injective function
$h\ :\ V(P_n) \to (0,1,2,\ldots,m+n)$ by
$h(v_j) = j-1,\ 1 \leq j \leq n.$
Then h induces the bijection
$h^* : E(P_n) \to \{0,1,2..m-1\}$.
Here are the Edge names:
$[h^*(v_j v_{j+1}) = j(mod\ m),\ 1 \leq j \leq m-1]$. Hence the path network diagram is an odd average harmonious graph.

Example 1. The path P_8 network diagram is odd average harmonious as seen in Figure 1.

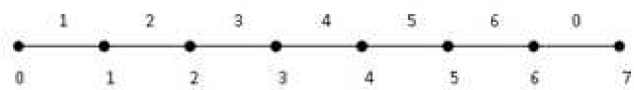

Fig. 1: Path P_8 is odd average harmonious

Theorem 2. *The star network diagram S_n is an odd average harmonious graph, when n is even.*

Proof. Let S_n be the path graph with n nodes and $m = n-1$ lines.
Consider u as the center vertex.
Let $V(S_n) = \{u, v_j\ : 1 \leq j \leq n-1\}$ and
Let $E(S_n) = \{uv_i\ : 1 \leq j \leq n-1\}$.
Define an injective function
$h : V(S_n) \to (0,1,2,\cdots,m+n\}$ by
$h(u) = 1, h(v_j) = 2j,\ 1 \leq j \leq n.$
Then h induces the bijection
$h^* : E(S_n) \to \{0,1,\cdots,m-1\}$.
Here are the Edge names:
$[h^*(uv_j) = j+1(mod\ m),\ \ 1 \leq j \leq m-1]$.
Hence the star network diagram is an odd average harmonious graph.

Example 2. The star network diagram S_5 is an odd average harmonious as shown in Figure 2.

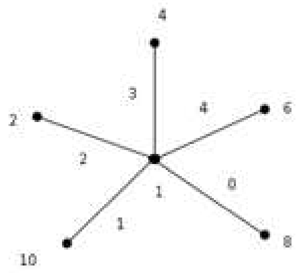

Fig. 2: Star S_5 is odd average harmonious

Theorem 3. *The comb network diagram $P_r \odot K_1$ is an odd average harmonious.*

Proof. Consider the comb network diagram with $n = 2r$ nodes and $m = 2r - 1$ lines.
Let $V(\mathrm{P_r} \odot \mathrm{K_1}) = \{v_j,\ u_j\ : 1 \le j \le r\}$ and $E(\mathrm{P_r} \odot \mathrm{K_1}) = \{u_j u_{j+1},\ u_j v_j\ : 1 \le j \le r-1,\ u_r v_r\}$.
Define an injective function
$h : V(\mathrm{P_r} \odot \mathrm{K_1}) \to (0, 1, 2, \cdots, m+n\}$ by

$$h(u_j) = j\ ,\ 1 \le j \le \left[\frac{r}{2}\right]$$
$$h(u_{j+2}) = j + 4\ ,\ 1 \le j \le \left[\frac{r}{2}\right]$$
$$h(v_j) = 3j - 3\ ,\ ,\ 1 \le j \le \left[\frac{r}{2}\right]$$
$$h(v_{j+2}) = 3j + 1\ ,\ ,\ 1 \le j \le \left[\frac{r}{2}\right].$$

Then h induces the bijection
$h^* : E(\mathrm{P_r} \odot \mathrm{K_1}) \to \{0, 1, \cdots, m-1\}$. Hence the network diagram $\mathrm{P_r} \odot \mathrm{K_1}$is an odd average harmonious graph.

Example 3. The network diagram $P_4 \odot K_1$ is odd average harmonious as shown in Figure 3.

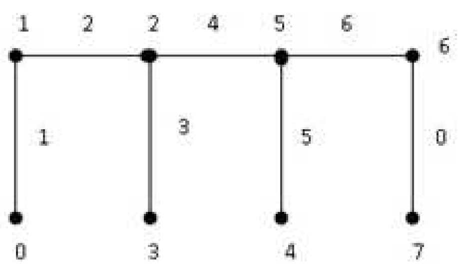

Fig. 3: Comb $P_r \odot K_1$ is odd average harmonious

Theorem 4. *Coconut tree is an odd average harmonious.*

Proof. Let G be the coconut tree network diagram obtained by recognizing the central vertex of $K_{1,j}$ with a pendant vertex of the path P_k.
Let $V(G) = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots v_j\ : 0 \le r \le k, k+1 \le j \le n\}$. Define an injective function $h : V(G) \to (0, 1, 2, \ldots, m+n\}$ by $h(u_i) = i, 0 \le i \le k-1$ $[\ h(v_i) = 2i + 5,\ 1 \le i \le j]$. Then h induces the bijection $h^* : E(G) \to \{0, 1, 2..m-1\}$.
Let $E(G) = \{u_i u_{i+1}\ : 0 \le i \le k-1\}$
$[E(G) = \{u_0 v_i\ ;; 1 \le i \le j\}]$.
Hence the coconut tree graph is an odd average harmonious.

Example 4. The Coconut tree graph is an odd average harmonious as shown in Figure 4.

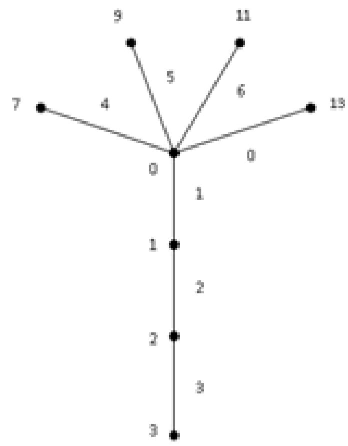

Fig. 4: Coconut tree is odd average harmonious

Theorem 5. *Bistar network diagram $B_{p,p} (p \ge 1)$ is an odd harmonious graph.*

Proof. Let the bistar network diagram $\mathrm{B_{p,p}}$ with $n = 2p + 2$ nodes and $m = 2p + 1$ lines.
Let $V(\mathrm{B_{p,p}}) = \{u, v, u_i, v_i,\ 1 \le i \le p)$ and $E(\mathrm{B_{p,p}}) = \{u_i u,\ v_i v,\ uv\ ,\ 1 \le i \le p\}$.
Define an injective function $h : V(P_n) \to (0, 1, 2, \ldots, m+n\}$

$$h(u_1) = 1,\ h(u) = 0,\ h(v) = 3$$
$$h(u_i) = 4i - 1\ ,\ 2 \le i \le p$$
$$h(v_i) = 4i - 2\ ,\ 1 \le i \le p.$$

Then h induces the bijection $h^* : E(P_n) \to \{0, 1, \cdots, m-1\}$ Hence the bistar network diagram is an odd average harmonious.

Example 5. The bistar network diagram $B_{5,5}$ is an odd average harmonious as shown in Figure 5.

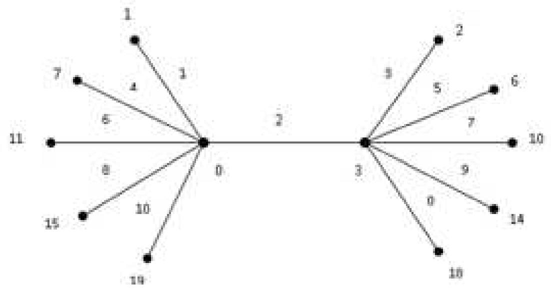

Fig. 5: Bistar $B_{5,5}$ is odd average harmonious

Theorem 6. *The Complete Bipartite graph $K_{p,p}$ is an odd average harmonious.*

Proof. A complete bipartite graph is a basic bipartite graph with a vertex bipartition set of V in X and Y in which each vertex of X is linked to each vertex of Y.
Let the nodes of the set X be u_i, $1 \leq i \leq p$ and the vertex set Y be v_i, $1 \leq i \leq p$.
The lines are $u_i v_j$, $11 \leq i \leq p,\ 1 \leq j \leq p$.
Here there are $n = 2p$ nodes and $m = p^2$ lines.
Define an injective function
$h : V\,[K_{p,p}] \to \{0, 1, 2 \ldots m+n\}$ by

$$h(u_i) = 2i - 1,\ 1 \leq i \leq p$$
$$h(v_1) = 0$$
$$h(v_i) = 2(i-1)p,\ \ 2 \leq i \leq p.$$

Example 6. The Complete Bipartite graph $K_{4,4}$ is an odd average harmonious as seen in Figure 6.

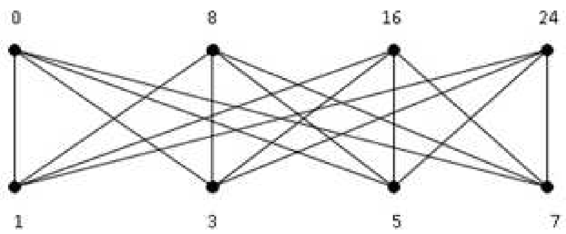

Fig. 6: Complete bipartite $k_{p,p}$ is odd average harmonious

4. Conclusion

In this paper, we investigated the odd average harmonious labeling which is one of the most important labelling techniques. As all the graphs are not odd average harmonious, it is very interesting to investigate the different types of graphs which admit the odd average harmonious. Using some mathematical derivations, we have reported the odd average harmonious labeling of various graphs. Based on the idea of the proposed paper future research work will focus on the odd average mean labeling, even average prime labeling of some other graphs.

Conflicts of Interest:
The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgement

The first and sixth authors would like to express their sincere thanks to Phuket Rajabhat University, Phuket 83000, Thailand for the financial and research support.

References

[1] J A Gallian, A dynamic survey of graph labeling, The Electronics Journal of Combinatories,17 DS6(2017).
[2] R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs, SIAM J. Alg. Discrete Meth., 1, 382-404 (1980).
[3] S. Ashokkumar and S. Maragathavalli, Prime labeling of some special graphs, IOSR Journal of Mathematics, 11, 1-5(2015).
[4] Praveen Agarwal, Shrideh khalaf Al-Omari, Some general properties of a fractional Sumudu transform in the class of Boehmians, Kuwait Journal of Science 2, 43 (2016).
[5] Praveen Agarwal, A Study of New Trends and Analysis of Special Function, LAPLAMBERT Academic publishing 1, 164, (2013).
[6] Praveen Agarwal, Monamed Ramadan, Heba S. Osheba and Yu-Ming Chu, Study of hybrid orthonormal functions method for solving second kind fuzzy Fredholm integral equations, Advances in Difference Equations 1, 1-14(2020).
[7] P. B. Sarasija and R. Binthiya, Even harmonious graphs with applications, Intern. Journal of Computer Science and Information Security, 9(7) (2011)161-163.
[8] N. Adalin Beatress1 and P. B. Sarasija, Average Harmonious Graphs, Annals of Pure and Applied Mathematics Vol. 11, 7-10 (2016).
[9] N. Adalin Beatress and P. B. Sarasija, Even-odd harmonious graphs, International Journal of Mathematics and Soft Computing, 5, 23-29 (2015) .
[10] P. Lawrence Rozario Raj and R. Lawrence Joseph Manoharan, Some Results on divisor Cordial labeling of graphs, International Journal of Innovative Science, Engineering & Technology, 10, (2014).
[11] Z-H Liang and Z-L Bai, On the odd harmonious graphs with applications, J. Appl. Math. Comput., 29, 105-116(2009).
[12] B. Liu and X. Zhang, On harmonious labeling of graphs, Ars. Combin., 36, 315-326 (1993).

Porpattama Hammachukiattikul received the Ph.D. degree in applied mathematics from the King Mongkut's University of Technology Thonburi (KMUTT), Bangkok, Thailand. She is currently working with the Department of Mathematics, Phuket Rajabhat University (PKRU), Phuket, Thailand. Her research interests include mathematic model, climate change, atmospheric model, lyapunov theory and neural networks, stability analysis of dynamical systems, synchronization, and chaos theory.

Grienggrai Rajchakit was born in 1981. He received the B.S. (Mathematics), Thammasat University, Bangkok, Thailand, in 2003. He received the M.S. (Applied Mathematics), Chiangmai University, Chiangmai, Thailand, in 2005. He was awarded Ph.D. (Applied Mathematics), King Mongkut's University of Technology Thonburi, Bangkok, Thailand, in the field of mathematics with specialized area of stability and control of neural networks. Currently, he is working as a lecturer at Department of Mathematics, Faculty of Science, Maejo University, Chiangmai, Thailand. He was the recipient of Thailand Frontier author Award by Thomson Reuters Web of Science in the year 2016 and TRF-OHEC-Scopus Researcher Awards by The Thailand Research Fund (TRF), Office of the Higher Education Commission (OHEC) and Scopus in the year 2016. His research interests are complex-valued NNs, complex dynamical networks, control theory, stability analysis, sampled-data control, multi-agent systems, T-S fuzzy theory, and cryptography, etc. Dr. Grienggrai Rajchakit serves as a reviewer for various SCI journals. He has authored and co-authored of more than 111 research articles in various SCI journals.

Vediyappan Govindan received his M.Sc degree from Sowdeswari College, Salem Affiliated with Periyar University. He was awarded Master of Philosophy in the field of mathematics with a specialized area of functional equations from Periyar University in 2009 and the Ph.D degree from Periyar University in 2017. He has more than 12 years of experience as a Assistant Professor. He is working as a Assistant Professor in DMI St.John, The Baptist University, Central Africa, Malawi. His research interests are Functional equations, Neural networks and differential equations. He has authored and co-authored of more than 73 research articles in various journals.

S. Dhivya received her M.Sc degree from Cauvery College of Arts and Science for women, Trichy affiliated with Bharathidasan university. She was awarded Master of Philosophy in 2013 in the field of mathematics with a specialized area of functional analysis from Sacred heart, Tirupattur affiliated with Thiruvalluvar University. Now she is a Research Scholar in Sri Vidya Mandir Arts and Science College, katteri, Uthangarai. She has published 6 research articles in various journals.

Nallappan Gunasekaran received the BSC from Mahendra Arts and Science College, Periyar University, Salem, Tamil Nadu, India, in 2009. He received his post-graduation from Jamal Mohamed College affiliated to Bharathidasan University, Trichy, Tamil Nadu, India, in 2012. He was awarded Master of Philosophy in 2013 from Bharathidasan University, Trichy, India, and the Ph.D. degree from Thiruvalluvar University, Vellore, India, in 2018, in mathematics. He was a Post-Doctoral Research Fellow in the Research Center for Wind Energy Systems, Kunsan National University, Gunsan, South Korea, from 2017 to 2018. Currently, he is working as a Post-Doctoral Research Fellow in Department of Mathematical Sciences, Shibaura Institute of Technology, Saitama, Japan.

R. Vadivel received the B.Sc., M.Sc., and M.Phil. degrees in Mathematics from Sri Ramakrishna Mission Vidyalaya College of Arts and Science affiliated to Bharathiar University, Coimbatore, Tamil Nadu, India, in 2007, 2010, and 2012, respectively. He was awarded the Ph.D. degree in 2018 from Department of Mathematics, Thiruvalluvar University, Vellore, Tamil Nadu, India. He was a Post-Doctoral Research Fellow in the Research Center for Wind Energy Systems, Kunsan National University, Gunsan, South Korea, from 2018 to 2019. Currently, he is working as a lecturer at Department of mathematics, Faculty of Science and technology, Phuket Rajabhat university, Thailand.

Bussakorn Hammachukiattikul was born in Trang, Thailand in 1981. She received her bachelor of science degree in statistical from Taksin University, Thailand in 2004; the master of science in applied statistics from National Institute of Development Administration, Thailand in 2010. She has been appointed as a lecture at Thai-Nichi Institute of Technology, Thailand, since 2010. She interesting in Dynamic Systems and release fields.