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Dynamics Behaviors of a Hybrid Switching Epidemic Model with Lévy Noise

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Abstract: The present paper investigates the dynamical behaviors of a stochastic SIRS epidemic model with telegraphic noise and Lévy noise. First, we establish the existence of a unique global positive solution for stochastic model. Furthermore, by constructing some suitable Lyapunov functions, we show that if $R_0 \leq 1$ and under some conditions on the parameters, then the solution of stochastic system fluctuates around the disease-free equilibrium, and if $R_0 > 1$ the solution of stochastic system fluctuates around the disease-endemic equilibrium of the deterministic model. Finally, we present numerical simulations to support the theoretical results.

Keywords: Lévy jumps, Markov switching, Stochastic model.

1 Introduction

The transmission and threat of infectious diseases constantly increase causing the larger population. They represent a major factor of mortality in developed and underdeveloped countries. Therefore, the interest of mathematical modeling in infectious diseases increases due to the important role it plays in public health research. This field of research helps analyze and understand the dynamics of the spread of infectious diseases. In 1927 Kermack and Mckendrick, introduced the first SIR model which divides the population into three classes: susceptible S, infectious I and recovered R with permanent acquired immunity. Thus, some recovered individuals R back to susceptible class S because they lost immunity. To model this case we need to use SIRS epidemic models (see, [1–5]). The SIRS epidemic model is represented by the following system of ordinary differential equations

$$
\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \beta S(t)I(t) - \mu S(t) + \delta R(t), \qquad (1)\\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - (\mu+\gamma)I(t),\\
\frac{dR(t)}{dt} &= \gamma I(t) - (\mu+\delta)R(t),
\end{aligned}
$$

$S(t)$, $I(t)$ and $R(t)$ represent the population densities of susceptible, infected and recovered at time t, respectively. The parameter Λ is the recruitment rate of the population, μ is the natural death rate of the population, γ is the recovery rate of the infective individuals, δ is the rate at which recovered individuals lose immunity and return to the susceptible class, β is the transmission rate, βSI represents the bilinear incidence rate [6]. The basic reproduction number of the model (1) is given by $R_0 = \frac{\beta\Lambda}{\mu(\mu+\gamma)}$ which represents the threshold that determines the extinction and the persistence of disease i.e.:

– If $R_0 \leq 1$ then model (1) has a unique disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ which is globally stable.
– If $R_0 > 1$ then model (1) has an endemic equilibrium $E^* = (S^*, I^*, R^*)$, which is globally asymptotically stable.

However, the biological systems are necessarily subject to different environmental fluctuations (climate change, nutrition, pandemic, etc). Therefore, many scientists have studied the disturbance of random environment when investigating disease dynamics [7–15]. For example, Khan et al. [16] proposed and analyzed a stochastic Hepatitis B epidemic model with varying population size. They introduced random perturbations of white noise type directly to the fluctuation of the Hepatitis B transmission

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rate. Liu and Jiang [17] analyzed a stochastic SIR epidemic model with Logistic birth. Using the stochastic Lyapunov function method. They established sufficient conditions for the existence of a stationary distribution and conditions for the extinction of the disease. Jihad et al. [18] introduced the noise effect directly to the parameters in the model. They found necessary and sufficient conditions for the extinction and persistence of the disease. Tornatore et al. [19] addressed a stochastic SIR model with or without delay. They investigated the stability of disease-free equilibrium. Then, assuming that the environmental noise is proportional to the variables and of the white noise type, we obtain the following stochastic model

$$\begin{aligned} dS(t) &= [\Lambda-\beta S(t)I(t)-\mu S(t)+\delta R(t)]dt+\sigma_1 S(t)dB_1(t),\\ dI(t) &= [\beta S(t)I(t)-(\mu+\gamma)I(t)]dt+\sigma_2 I(t)dB_2(t), \qquad (2)\\ dR(t) &= [\gamma I(t)-(\mu+\delta)R(t)]dt+\sigma_3 R(t)dB_3(t), \end{aligned}$$

where $B_1(t), B_2(t), B_3(t)$ are independent Brownian motions defined on a complete probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$, satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets), $\sigma_1,\sigma_2,\sigma_3$ representing the intensity of the white noise.

On the other hand, the universal situation in nature is that the population fluctuates around a stable mean. Nevertheless, this situation is seldom disrupted by a sudden change to an entirely different regime caused by factors such as nutrition, climatic characteristics, or sociocultural factors. Then, in this case, it is necessary to introduce telegraphic noise (or colored noise) [20–23]. Then, the telegraphic noise can be modeled by a continuous-time Markov chain $r(t)$ taking values in a finite state space $\mathbb{S}=\{1,2,...,N\}$ with the generator $\Phi=(\phi_{uv})_{1\leq u,v\leq N}$ given by

$$P(r(t+\Delta)=v\mid r(t)=u)=\begin{cases}\phi_{uv}\Delta+o(\Delta) & \text{if } u\neq v,\\ 1+\phi_{uu}\Delta+o(\Delta) & \text{if } u=v,\end{cases}$$

where $\Delta>0$, ϕ_{uv} is the transition rate from u to v and $\phi_{uv}\geq 0$ if $u\neq v$, while $\phi_{uu}=-\sum_{u\neq v}\phi_{uv}$. Assume more that Markov chain $r(t)$ is irreducible and has a unique stationary distribution $\pi=(\pi_1,\pi_2,...,\pi_N)$ which can be determined by solving the linear equation $\pi\Phi=0$, subject to $\sum_{i=1}^N\pi_i=1,\ \ and\ \ \pi_i>0,\ \ \forall i\in\mathbb{S}$.

Besides, population dynamics may be attacked with sudden and large environmental disturbances, such as pandemics, earthquakes, hurricanes, tsunamis, etc. These events may break the continuity of the solution and can not be better modeled by the white noise. Hence, under the situation introducing Lévy jumps into these systems may be a logical and significant approach (see, [24–26]). Recently, Lui et al. [27] established a stochastic SIR epidemic model with media coverage incorporating Lévy noise. Fan et al. [28] explored the effect of Lévy noise on the deterministic SIR epidemic models with delay.

Motivated by [29, 30] and the above-mentioned discussion, we consider the stochastic SIRS model under regime switching with Lévy jump, as follows

$$\begin{aligned} dS(t) &= [\Lambda(r(t))-\beta(r(t))S(t)I(t)-\mu(r(t))S(t)\\ &\quad+\delta(r(t))R(t)]\,dt+\sigma_1(r(t))S(t)dB_1(t)\\ &\quad+\int_A q_1(r(t),\alpha)S(t-)\tilde{N}(dt,d\alpha), \qquad (3)\\ dI(t) &= [\beta(r(t))S(t)I(t)-(\mu(r(t))+\gamma(r(t)))I(t)]\,dt\\ &\quad+\sigma_2(r(t))I(t)dB_2(t)\\ &\quad+\int_A q_2(r(t),\alpha)I(t-)\tilde{N}(dt,d\alpha),\\ dR(t) &= [\gamma(r(t))I(t)-(\mu(r(t))+\delta(r(t)))R(t)]\,dt\\ &\quad+\sigma_3(r(t))R(t)dB_3(t)\\ &\quad+\int_A q_3(r(t),\alpha)R(t-)\tilde{N}(dt,d\alpha), \end{aligned}$$

where $S(t-)$, $I(t-)$ and $R(t-)$ are the left limit of $S(t)$, $I(t)$ and $R(t)$ respectively, the system parameters $\beta(k)$, $\mu(k)$, $\gamma(k)$, $\delta(k)$ and $\sigma_i(k)$ $(i=1,2,3)$ are all positive constants for all $k\in\mathbb{S}$. $\tilde{N}(dt,d\alpha)=N(dt,d\alpha)-\nu(\alpha)dt$, N is a Poisson counting measure with characteristic measure ν on measurable subset A of $[0,\infty)$, with $\nu(A)<\infty$, and $q_i: A\times\Omega\longrightarrow\mathbb{R}$ $(i=1,2,3)$ represent the effects of random jumps it's bounded and continuous with respect to ν and $\mathfrak{B}(A)\times\mathscr{F}_t$-measurable.

Assume that initially, the Markov chain $r(0)=j\in\mathbb{S}$, then the model (3) satisfies

$$\begin{aligned} dS(t) &= [\Lambda(j)-\beta(j)S(t)I(t)-\mu(j)S(t)+\delta(j)R(t)]\,dt\\ &\quad+\sigma_1(j)S(t)dB_1(t)+\int_A q_1(j,\alpha)S(t-)\tilde{N}(dt,d\alpha),\\ dI(t) &= [\beta(j)S(t)I(t)-(\mu(j)+\gamma(j))I(t)]\,dt\\ &\quad+\sigma_2(j)I(t)dB_2(t)+\int_A q_2(j,\alpha)I(t-)\tilde{N}(dt,d\alpha),\\ dR(t) &= [\gamma(j)I(t)-(\mu(j)+\delta(j))R(t)]\,dt\\ &\quad+\sigma_3(j)R(t)dB_3(t)+\int_A q_3(j,\alpha)R(t-)\tilde{N}(dt,d\alpha), \end{aligned}$$

as soon as the Markov chain $r(t)$ jumps to another state, say $k\in\mathbb{S}$, model (3) becomes

$$\begin{aligned} dS(t) &= [\Lambda(k)-\beta(k)S(t)I(t)-\mu(k)S(t)+\delta(k)R(t)]dt\\ &\quad+\sigma_1(k)S(t)dB_1(t)+\int_A q_1(k,\alpha)S(t-)\tilde{N}(dt,d\alpha),\\ dI(t) &= [\beta(k)S(t)I(t)-(\mu(k)+\gamma(k))I(t)]\,dt \qquad (4)\\ &\quad+\sigma_2(k)I(t)dB_2(t)+\int_A q_2(k,\alpha)I(t-)\tilde{N}(dt,d\alpha),\\ dR(t) &= [\gamma(k)I(t)-(\mu(k)+\delta(k))R(t)]\,dt\\ &\quad+\sigma_3(k)R(t)dB_3(t)+\int_A q_3(k,\alpha)R(t-)\tilde{N}(dt,d\alpha). \end{aligned}$$

We consider the following hybrid stochastic differential equations with jumps

$$\begin{aligned} dX(t) &= f(X(t),r(t))dt+g(X(t),r(t))dB(t)\\ &\quad+\int_A h(X(t-),r(t),\alpha)\tilde{N}(dt,d\alpha), \end{aligned}$$

on $t \geq 0$, with initial data $X(0) \in \mathbb{R}^n$ and $r(0) \in \mathbb{S}$, $f : \mathbb{R}^n \times \mathbb{S} \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{S} \longrightarrow \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \times \mathbb{S} \times A \longrightarrow \mathbb{R}^n$. We assume that the Markov chain $r(t)$ is independent of the Brownian motion $B(t)$ and Poisson random measures $\tilde{N}(t,A)$.

Let $C^{1,2}(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ the family of continuous non-negative functions $Q(X,i)$ defined on $\mathbb{R}^n \times \mathbb{S}$ such that for each $i \in \mathbb{S}$, they are continuously twice differentiable in x. The differential operator $\mathscr{L}$ acts on a function $Q(X,i) \in C^{1,2}(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ given by

$$\begin{aligned}\mathscr{L}Q(X,i) &= Q_x(X,i)f(X,i)\\ &+\frac{1}{2}trace[g^T(X,i)Q_{xx}(X,i)g(X,i)]\\ &+\int_A [Q(X+h(X,i,\alpha),i)-Q(X,i)]\nu(d\alpha)\\ &+\sum_{j=1}^{N}\phi_{ij}Q(X,j),\end{aligned}$$

where $Q_x = \left(\frac{\partial Q}{\partial x_1},\ldots,\frac{\partial Q}{\partial x_n}\right)$, $Q_{xx} = \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}\right)_{n\times n}$.

The organization of this paper is as follows: In Section 2, we investigate the existence and uniqueness of the global positive solution to model (4). In Section 3, we study the behavior of the solution to (4) around the disease-free equilibrium E_0. In Section 4, we address the behavior of the solution to the system (4) around the endemic equilibrium E^*. In Section 5, the analytical results are illustrated with the support of numerical examples.

2 Global positive solution

In this section, we establish the existence of a unique global positive solution for our stochastic epidemic model with Lévy noise and telegraphic noise. Next, we impose two assumptions, Assumption 2.1 and Assumption 2.2, which are necessary to show the existence and uniqueness of a global positive solution of (4).

Assumption 2.1. For each $d > 0$ there exists $L_d > 0$ such that

$$\int_A |G_i(x,k,\alpha)-G_i(y,k,\alpha)|^2\nu(d\alpha) \leq L_d|x-y|^2$$

for $i = 1,2,3,\ \ k \in \mathbb{S}$, where
$G_1(z,\alpha) = q_1(\alpha,k)z$ for $z = S(t-)$, $G_2(z,\alpha) = q_2(\alpha,k)z$ for $z = I(t-)$, $G_3(z,\alpha) = q_3(\alpha,k)z$ for $z = R(t-)$,
with $|x| \vee |y| \leq d$.

Assumption 2.2. $|\ln(1+q_i(\alpha))| \leq C$, for $q_i(\alpha) > -1$, where C is positive constant $i = 1,2,3,\ \ k \in \mathbb{S}$.

Theorem 2.1. For any given initial value $(S(0),I(0),R(0)) \in \mathbb{R}^3_+$, there is a unique solution $(S(t),I(t),R(t))$ to Equation (4) on $t \geq 0$ and the solution will remain in $\mathbb{R}^3_+$ with probability one, namely $(S(t),I(t),R(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$ almost surely.

proof By Assumption 2.1 for any initial value $(S(0),I(0),R(0)) \in \mathbb{R}^3_+$, there is a unique local solution $(S(t),I(t),R(t))$ of system (4) on $[0,\zeta_e)$, where ζ_e is the explosion time. To show that $(S(t),I(t),R(t)) \in \mathbb{R}^3_+$ $a.s.$ for all $t \geq 0$, we need to check $\zeta_e = \infty$ $a.s.$ Let m_0 be sufficiently large so both $S(0), I(0)$ and $R(0)$ lie within the interval $\left[\frac{1}{m_0}, m_0\right]$. For each integer $m \geq m_0$, define the stopping time

$$\begin{aligned}\zeta_m &= \inf\left\{t\in[0,\zeta_e): S(t)\notin\left(\frac{1}{m},m\right) \text{ or } I(t)\notin\left(\frac{1}{m},m\right)\right.\\ &= \left.\text{ or } R(t)\notin\left(\frac{1}{m},m\right)\right\}\end{aligned}$$

Obviously, ζ_m is increasing as $m \to \infty$. Set $\zeta_\infty = \lim_{m\to\infty}\zeta_m$ and $\zeta_\infty \leq \zeta_e$ $a.s.$ So to complete the proof, if we can prove $\zeta_\infty = \infty$ $a.s.$ then $\zeta_e = \infty$ $a.s.$ and $(S(t),I(t),R(t)) \in \mathbb{R}^3_+$ $a.s.$ If this statement is false, then there exists a pair of constant $T > 0$ and $\varepsilon \in (0,1)$ such that

$$\mathbb{P}(\zeta_\infty \leq T) \geq \varepsilon.$$

Thus, there is an integer $m_1 \geq m_0$, such that

$$\mathbb{P}(\zeta_\infty \leq T) \geq \varepsilon,\ \ for\ all\ \ m \geq m_1.$$

Define a C^2-function $U : \mathbb{R}^3_+ \longrightarrow \mathbb{R}_+$ by :

$$U(S,I,R,k) = S-1-\ln S+I-1-\ln I+R-1-\ln R.$$

From the Itô's formula, we have

$$\begin{aligned}dU &= \mathscr{L}Udt+\sigma_1(k)(S-1)dB_1(t)+\sigma_2(k)(I-1)dB_2(t)\\ &+\sigma_3(k)(R-1)dB_3(t)\\ &+\int_A [q_1(k,\alpha)S-\ln(1+q_1(k,\alpha))]\tilde{N}(dt,d\alpha)\\ &+\int_A [q_2(k,\alpha)I-\ln(1+q_2(k,\alpha))]\tilde{N}(dt,d\alpha)\\ &+\int_A [q_3(k,\alpha)R-\ln(1+q_3(k,\alpha))]\tilde{N}(dt,d\alpha),\end{aligned} \tag{5}$$

where

$$\begin{aligned}\mathscr{L}U &= \Lambda(k)-\mu(k)S-(\mu(k)+d(k))I-\mu(k)R-\frac{\Lambda(k)}{S}\\ &-\beta(k)S+\mu(k)+d(k)+\gamma(k)-\gamma(k)\frac{I}{R}\\ &+\mu(k)+\delta(k)+\frac{\sigma_1(k)+\sigma_2(k)+\sigma_3(k)}{2}\\ &+\int_A [q_1(k,\alpha)-\ln(1+q_1(k,\alpha))]\nu(d\alpha)\\ &+\int_A [q_2(k,\alpha)-\ln(1+q_2(k,\alpha))]\nu(d\alpha)\\ &+\int_A [q_3(k,\alpha)-\ln(1+q_3(k,\alpha))]\nu(d\alpha),\end{aligned}$$

which implies that

$$\begin{aligned}\mathscr{L}U \leq\ & \Lambda(k)+\mu(k)+d(k)+\gamma(k)+\mu(k)+\delta(k)\\ &+\frac{\sigma_1(k)+\sigma_2(k)+\sigma_3(k)}{2}\\ &+\int_A \left[q_1(k,\alpha)-\ln\left(1+q_1(k,\alpha)\right)\right]\nu(d\alpha)\\ &+\int_A \left[q_2(k,\alpha)-\ln\left(1+q_2(k,\alpha)\right)\right]\nu(d\alpha)\\ &+\int_A \left[q_3(k,\alpha)-\ln\left(1+q_3(k,\alpha)\right)\right]\nu(d\alpha).\end{aligned}$$

Since $z-\ln(1+z)\geq 0$ for all $z>-1$ and using the Assumption 2.2, we get

$$\begin{aligned}\mathscr{L}U \leq\ & \Lambda(k)+\mu(k)+d(k)+\gamma(k)+\mu(k)+\delta(k)\\ &+\frac{\sigma_1(k)+\sigma_2(k)+\sigma_3(k)}{2}+3K' := K,\end{aligned}$$

with

$$\begin{aligned}K' = \max\Big\{ & \int_A \left[q_1(k,\alpha)-\ln\left(1+q_1(k,\alpha)\right)\right]\nu(d\alpha),\\ & \int_A \left[q_2(k,\alpha)-\ln\left(1+q_2(k,\alpha)\right)\right]\nu(d\alpha),\\ & \int_A \left[q_3(k,\alpha)-\ln\left(1+q_3(k,\alpha)\right)\right]\nu(d\alpha)\Big\}.\end{aligned}$$

Integrating both sides of (5) between 0 and $\zeta_m\wedge T$ and taking expectation we obtain

$$\begin{aligned}0 &\leq \mathbb{E}U\left(S(\zeta_m\wedge T),I(\zeta_m\wedge T),R(\zeta_m\wedge T)\right)\\ &\leq U\left(S(0),I(0),R(0)\right)+KT.\end{aligned}$$

Define for each $s>0$, $H(s)=\inf\left\{U(u_1,u_2,u_3),\ u_i\geq s\ or\ u_i\leq\frac{1}{s},\ i=1,2,3\right\}$, with $u_1=S$, $u_2=I$ and $u_3=R$ then we have $\lim_{s\to\infty}H(s)=\infty$. Therefore

$$\begin{aligned}&U\left(S(0),I(0),R(0)\right)+KT\geq\\ &\mathbb{E}\left[1_{\{\zeta_m\leq T\}}U\left(S(\zeta_m\wedge T),I(\zeta_m\wedge T),R(\zeta_m\wedge T)\right)\right]\geq\varepsilon H(m).\end{aligned}$$

Letting $m\to\infty$ leads to $\infty>U\left(S(0),I(0),R(0)\right)+KT=\infty$ which is a contradiction, for consequent $\zeta_\infty=\infty$ $a.s.$ This completes the proof. □

3 Asymptotic behavior around the disease-free equilibrium

In this section, we investigate the behavior of the global positive solution $(S(t),I(t),R(t))$ around the disease-free equilibrium $E_0=(\frac{\Lambda}{\mu},0,0)$. Assume that $R_0\leq 1$ and let

$$\begin{aligned}M_1 &= 2\mu(k)-2\sigma_1^2(k)-4\int_A q_1^2(k,\alpha)\nu(d\alpha),\\ M_2 &= 2\mu(k)-\sigma_2^2(k)-4\int_A q_1^2(k,\alpha)\nu(d\alpha),\\ M_3 &= \frac{2\mu(k)\gamma(k)+2\mu(k)(\mu(k)+\delta(k))}{\gamma(k)}\\ &\quad-\left(4+\frac{2\mu(k)}{\gamma(k)}\right)\int_A q_3^2(k,\alpha)\nu(d\alpha),\\ &\quad-\sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right)\\ M_4 &= \frac{\Lambda^2(k)}{\mu^2(k)}\left[2\sigma_1^2(k)+4\int_A q_1^2(k,\alpha)\nu(d\alpha)\right],\\ M &= \min\{M_1,M_2,M_3\}.\end{aligned}$$

Theorem 3.1. Consider the stochastic system (4) with initial condition $(S(0),I(0),R(0))$ in $\mathbb{R}_+^3$. Assume that the following conditions hold

$$\begin{aligned}2\mu(k) &> 2\sigma_1^2(k)+4\int_A q_1^2(k,\alpha)\nu(d\alpha),\\ 2\mu(k) &> \sigma_2^2(k)+4\int_A q_2^2(k,\alpha)\nu(d\alpha),\\ \frac{2\mu(k)\gamma(k)+2\mu(k)(\mu(k)+\delta(k))}{\gamma(k)} &> \sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right)\\ &\quad+\left(4+\frac{2\mu(k)}{\gamma(k)}\right)\int_A q_3^2(k,\alpha)\nu(d\alpha).\end{aligned}$$

Then, the solution of model (4) has the property

$$\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[\left(S(u)-\frac{\Lambda}{\mu}\right)^2+I^2(u)+R^2(u)\right]du\leq\frac{M_4}{M}.$$

Proof. Let $X(t)=S(t)-\frac{\Lambda}{\mu}$; $Y(t)=I(t)$; $Z(t)=R(t)$. Then, model (4) becomes

$$\begin{aligned}dX &= \left[-\mu(k)X-\beta(k)XY-\beta(k)\frac{\Lambda(k)}{\mu(k)}Y+\delta(k)Z\right]dt\\ &\quad+\sigma_1(k)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)dB_1(t)\\ &\quad+\int_A q_1(k,\alpha)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)\tilde{N}(dt,d\alpha),\\ dY &= \left[\beta(k)XY-\left(\mu(k)+\gamma(k)-\beta(k)\frac{\Lambda(k)}{\mu(k)}\right)Y\right]dt\\ &\quad+\sigma_2(k)YdB_2(t)+\int_A q_2(k,\alpha)Y\tilde{N}(dt,d\alpha),\\ dZ &= \left[\gamma(k)Y-(\mu(k)+\delta(k))Z\right]dt+\sigma_3(k)ZdB_3(t)\\ &\quad+\int_A q_3(k,\alpha)Z\tilde{N}(dt,d\alpha).\end{aligned}$$

We consider the following function

$$U(X,Y,Z,k)=(X+Y+Z)^2+d_1(k)Y+d_2(k)Z^2,$$

where $d_1(k)$, $d_2(k)$ for all $k \in \mathbb{S}$, are two positive constants to be chosen later. Then, using Ito's formula, we have

$$
\begin{aligned}
dU =\ & \mathscr{L}U dt + 2(X+Y+Z)\left[\sigma_1(k)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)dB_1(t)\right. \\
& \left. +\sigma_2(k)Y dB_2(t) + \sigma_3(k)Z dB_3(t)\right] \\
& +d_1(k)\sigma_2(k)Y dB_2(t) + 2d_2(k)\sigma_3(k)Z^2 dB_3(t) \\
& +\int_A \left\{\left[q_1(k,\alpha)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)+q_2(k,\alpha)Y\right.\right. \\
& \left. +q_3(k,\alpha)Z\right]^2 + 2(X+Y+Z)\left[q_1(k,\alpha)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)\right. \\
& \left. +q_2(k,\alpha)Y + q_3(k,\alpha)Z\right] + d_1(k)q_2(k,\alpha)Y \\
& \left. +2d_2(k)q_3(k,\alpha)Z^2 + d_2(k)q_3^2(k,\alpha)Z^2\right\}\tilde{N}(dt,d\alpha),
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
\mathscr{L}U =\ & -2\mu(k)(X+Y+Z)X - 2\mu(k)(X+Y+Z)Y \\
& -2\mu(k)(X+Y+Z)Z + d_1(k)\beta(k)XY \\
& -d_1(k)\left(\mu(k)+\gamma(k)-\beta(k)\frac{\Lambda(k)}{\mu(k)}\right)Y \\
& +2d_2(k)\left[\gamma(k)Y - (\mu(k)+\delta(k))Z\right]Z \\
& +\sigma_1^2(k)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)^2 + \sigma_2^2(k)Y^2 \\
& +\sigma_3^2(k)(1+d_2(k))Z^2 \\
& +\int_A \left\{\left[q_1(k,\alpha)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)+q_2(k,\alpha)Y\right.\right. \\
& \left.\left. +q_3(k,\alpha)Z\right]^2 + d_2(k)q_3^2(k,\alpha)Z^2\right\}\nu(d\alpha) \\
& +\sum_{l=1}^{N}\phi_{kl}V(X,Y,Z,l) \\
=\ & -\left[2\mu(k)-\sigma_1^2(k)\right]X^2 - \left[2\mu(k)-\sigma_2^2(k)\right]Y^2 \\
& -\left[2\mu(k)+2d_2(k)(\mu(k)+\delta(k))\right. \\
& \left. -\sigma_3^2(k)(1+d_2(k))\right]Z^2 \\
& -d_1(k)(\mu(k)+\gamma(k))(1-R_0)Y \\
& +\left[d_1(k)\beta(k)-4\mu(k)\right]XY \\
& -4\mu(k)XZ + \left[2d_2(k)\gamma(k)-4\mu(k)\right]YZ \\
& +2\sigma_1^2(k)X\frac{\Lambda(k)}{\mu(k)} + \sigma_1^2(k)\frac{\Lambda^2(k)}{\mu^2(k)} \\
& +\int_A \left\{\left[q_1(k,\alpha)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)+q_2(k,\alpha)Y\right.\right. \\
& \left.\left. +q_3(k,\alpha)Z\right]^2 + d_2(k)q_3^2(k,\alpha)Z^2\right\}\nu(d\alpha) \\
& +\sum_{l=1}^{N}\phi_{kl}V(X,Y,Z,l),
\end{aligned}
$$

choosing

$$d_1(k) = \frac{4\mu(k)}{\beta(k)} \quad \text{and} \quad d_2(k) = \frac{2\mu(k)}{\gamma(k)},$$

such that

$$d_1(k)\beta(k) - 4\mu(k) = 0 \quad \text{and} \quad 2d_2(k)\gamma(k) - 4\mu(k) = 0,$$

which can be simplified to

$$
\begin{aligned}
\mathscr{L}U \le\ & -\left[2\mu(k)-\sigma_1^2(k)\right]X^2 - \left[2\mu(k)-\sigma_2^2(k)\right]Y^2 \\
& -\left[\frac{2\mu(k)\gamma(k)+4\mu(k)(\mu(k)+\delta(k))}{\gamma(k)}\right. \\
& \left. -\sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right)\right]Z^2 + 2\sigma_1^2(k)X\frac{\Lambda(k)}{\mu(k)} \\
& +\sigma_1^2(k)\frac{\Lambda^2(k)}{\mu^2(k)} + \int_A \left\{\left[q_1(k,\alpha)\left(X+\frac{\Lambda(k)}{\mu(k)}\right)\right.\right. \\
& \left. +q_2(k,\alpha)Y + q_3(k,\alpha)Z\right]^2 \\
& \left. +\frac{2\mu(k)}{\gamma(k)}q_3^2(k,\alpha)Z^2\right\}\nu(d\alpha) + \sum_{l=1}^{N}\phi_{kl}V(X,Y,Z,l).
\end{aligned}
$$

Next, let $\check{d}=\max\left\{\frac{d_i(l)}{d_i(k)} : 1 \le i \le 2,\ 1 \le l,k \le N\right\}$, then for any $l,k \in \mathbb{S}$, we obtain

$$
\begin{aligned}
U(X,Y,Z,l) &= (X+Y+Z)^2 + d_1(l)Y + d_2(l)Z^2 \\
&\le \check{d}\left[(X+Y+Z)^2 + d_1(k)Y + d_2(k)Z^2\right] \\
&= \check{d}U(X,Y,Z,k),
\end{aligned}
$$

then

$$
\begin{aligned}
\sum_{l=1}^{N}\phi_{kl}U(X,Y,Z,l) &\le \check{d}\left(\sum_{l=1}^{N}|\phi_{kl}|\right)U(X,Y,Z,k) \\
&:= C_2 U(X,Y,Z,k).
\end{aligned}
$$

Using the inequalities $2ab \le a^2 + b^2$ and $(a+b+c+d)^2 \le 4\left(a^2+b^2+c^2+d^2\right)$, we obtain

$$
\begin{aligned}
\mathscr{L}U \le\ & -2\left[\mu(k)-\sigma_1^2(k)\right]X^2 - \left[2\mu(k)-\sigma_2^2(k)\right]Y^2 \\
& -\left[\frac{2\mu(k)\gamma(k)+4\mu(k)(\mu(k)+\delta(k))}{\gamma(k)}\right. \\
& \left. -\sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right)\right]Z^2 + 2\sigma_1^2(k)\frac{\Lambda^2(k)}{\mu^2(k)} \\
& +4\int_A \left\{\left[q_1^2(k,\alpha)X^2 + q_1^2(k,\alpha)\frac{\Lambda^2(k)}{\mu^2(k)} + q_2^2(k,\alpha)Y^2\right.\right. \\
& \left.\left. +q_3^2(k,\alpha)Z^2\right] + \frac{2\mu(k)}{\gamma(k)}q_3^2(k,\alpha)Z^2\right\}\nu(d\alpha) \\
& +C_2 V(X,Y,Z,k).
\end{aligned}
$$

Integrating both sides of (6) from 0 to t and taking expectation, we get

$$
\begin{aligned}
0 &\le \mathbb{E}U(X(t),Y(t),Z(t),r(t)) \le U(X(0),Y(0),Z(0),r(0)) \\
&+ \mathbb{E}\int_0^t \Big\{ -\Big(2\left(\mu(k)-\sigma_1^2(k)\right) - 4\int_A q_1^2(k,\alpha)\nu(d\alpha)\Big) X^2(u) \\
&- \Big(\left(2\mu(k)-\sigma_2^2(k)\right) - 4\int_A q_2^2(k,\alpha)\nu(d\alpha)\Big) Y^2(u) \\
&- \Big[\frac{2\mu(k)\gamma(k)+4\mu(k)(\mu(k)+\delta(k))}{\gamma(k)} \\
&\quad -\sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right) \\
&- \left(4+\frac{2\mu(k)}{\gamma(k)}\right)\int_A q_3^2(k,\alpha)\nu(d\alpha)\Big] Z^2(u)\Big\} du \\
&+ 4t\int_A q_1^2(k,\alpha)\frac{\Lambda^2(k)}{\mu^2(k)}\nu(d\alpha) + 2\sigma_1^2(k)\frac{\Lambda(k)}{\mu(k)^2}^2 t \\
&+ C_2\int_0^t U(X(u),Y(u),Z(u),r(u)).
\end{aligned}
$$

using the Gronwall inequality, we obtain

$$
\begin{aligned}
0 &\le \mathbb{E}U(X(t),Y(t),Z(t),r(t)) \le \{U(X(0),Y(0),Z(0),r(0)) \\
&+ \mathbb{E}\int_0^t \Big[-\Big(2\left(\mu(k)-\sigma_1^2(k)\right) - 4\int_A q_1^2(k,\alpha)\nu(d\alpha)\Big) X^2(u) \\
&- \Big(\left(2\mu(k)-\sigma_2^2(k)\right) - 4\int_A q_2^2(k,\alpha)\nu(d\alpha)\Big) Y^2(u) \\
&- \Big(\frac{2\mu(k)\gamma(k)+4\mu(k)(\mu(k)+\delta(k))}{\gamma(k)} \\
&\quad -\sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right) \\
&- \left(4+\frac{2\mu(k)}{\gamma(k)}\right)\int_A q_3^2(k,\alpha)\nu(d\alpha)\Big) Z^2(u)\Big] du \\
&+ 2\sigma_1^2(k)\frac{\Lambda^2(k)}{\mu^2(k)}t + 4t\int_A q_1^2(k,\alpha)\frac{\Lambda^2(k)}{\mu^2(k)}\nu(d\alpha)\Big\} e^{C_2 t}.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&\mathbb{E}\int_0^t \Big[\Big(2\left(\mu(k)-\sigma_1^2(k)\right) - 4\int_A q_1^2(k,\alpha)\nu(d\alpha)\Big) X^2(u) \\
&+\Big(\left(2\mu(k)-\sigma_2^2(k)\right) - 4\int_A q_2^2(k,\alpha)\nu(d\alpha)\Big) Y^2(u) \\
&+\Big(\frac{2\mu(k)\gamma(k)+4\mu(k)(\mu(k)+\delta(k))}{\gamma(k)} \\
&-\sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right) \\
&-\left(4+\frac{2\mu(k)}{\gamma(k)}\right)\int_A q_3^2(k,\alpha)\nu(d\alpha)\Big) Z^2(u)\Big] du \\
&\le 2\sigma_1^2(k)\frac{\Lambda^2(k)}{\mu^2(k)}t + 4t\int_A q_1^2(k,\alpha)\frac{\Lambda^2(k)}{\mu^2(k)}\nu(d\alpha) \\
&+U(X(0),Y(0),Z(0),r(0)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \Big[\Big(2\left(\mu(k)-\sigma_1^2(k)\right) \\
&-4\int_A q_1^2(k,\alpha)\nu(d\alpha)\Big) X^2(u)\Big) \\
&+\Big(\left(2\mu(k)-\sigma_2^2(k)\right) - 4\int_A q_2^2(k,\alpha)\nu(d\alpha)\Big) Y^2(u) \\
&+\Big(\frac{2\mu(k)\gamma(k)+4\mu(k)(\mu(k)+\delta(k))}{\gamma(k)} \\
&-\sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right) \\
&-\left(4+\frac{2\mu(k)}{\gamma(k)}\right)\int_A q_3^2(k,\alpha)\nu(d\alpha)\Big) Z^2(u)\Big] du \\
&\le 2\sigma_1^2(k)\frac{\Lambda^2(k)}{\mu^2(k)} + 4\int_A q_1^2(k,\alpha)\frac{\Lambda^2(k)}{\mu^2(k)}\nu(d\alpha).
\end{aligned}
$$

Hence

$$
\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[\left(S(u)-\frac{\Lambda}{\mu}\right)^2 + I^2(u) + R^2(u)\right] du \le \frac{M_4}{M}.
$$

□.

Remark 3.1. If $R_0 \le 1$ and under the conditions of the theorem 3.1 we conclude that the solution of (4) fluctuates around the disease-free equilibrium.

4 Asymptotic behavior around the endemic equilibrium

In this section we handle the behavior of the global positive solution $(S(t),I(t),R(t))$ of the system (4) around the endemic equilibrium E^*. We assume that $R_0 > 1$. Let

$$
\begin{aligned}
T_1 &= \frac{\mu(k)+\delta(k)}{\mu(k)+\gamma(k)+\delta(k)} - \frac{1}{2}\sigma_1^2(k) - \frac{3}{2}\int_A q_1^2(k,\alpha)\nu d(\alpha), \\
T_2 &= \mu(k) - \frac{1}{2}\sigma_2^2(k) - \frac{3}{2}\int_A q_2^2(k,\alpha)\nu(d\alpha), \\
T_3 &= \frac{\mu(k)(\mu(k)+\delta(k))}{\gamma(k)} - \frac{2\mu(k)+\gamma(k)}{2\gamma(k)}\sigma_3^2(k) \\
&\quad -\left(\frac{2\mu(k)+3\gamma(k)}{2\gamma(k)}\right)\int_A q_3^2(k,\alpha)\nu(d\alpha), \\
T &= \frac{2\mu(k)}{\beta(k)} I^* \int_A \left[q_2(k,\alpha) - \log(1+q_2(k,\alpha))\right]\nu(d\alpha) \\
&\quad + \frac{\mu(k)}{\beta(k)}\sigma_2(k) I^* \\
&\quad + \frac{\mu(k)\left(\sigma_2^2(k) + 3\int_A q_2^2(k,\alpha)\nu(d\alpha)\right)}{2\mu(k) - \sigma_2^2(k) - 3\int_A q_2^2(k,\alpha)\nu(d\alpha)}(I^*)^2 \\
&\quad + \frac{\mu(k)(\mu(k)+\delta(k))[A]}{B}(S^*)^2
\end{aligned}
$$

$$+\frac{\mu(k)\left(\mu(k)+\delta(k)\right)C}{\gamma(k)D}\left(R^*\right)^2,$$

where
$A=\sigma_1^2(k)+3\int_A q_1^2(k,\alpha)\nu d(\alpha)$
$B = 2\mu(k)\left(\mu(k)+\delta(k)\right) - \sigma_1^2(k)\left(\mu(k)+\gamma(k)+\delta(k)\right) - 3\left(\mu(k)+\gamma(k)+\delta(k)\right)\int_A q_1^2(k,\alpha)\nu(d\alpha)$
$C=\sigma_3^2(k)\left(2\mu(k)+\gamma(k)\right) + \left(2\mu(k)+3\gamma(k)\right)\int_A q_3^2(k,\alpha)\nu(d\alpha)$
$D = 2\mu(k)\left(\mu(k)+\delta(k)\right) - \sigma_3^2(k)\left(2\mu(k)+\gamma(k)\right) - \left(2\mu(k)+3\gamma(k)\right)\int_A q_3^2(k,\alpha)\nu(d\alpha)$

Theorem 4.1. Under the following conditions

$$\begin{aligned}
\frac{\mu(k)+\delta(k)}{\mu(k)+\gamma(k)+\delta(k)} &> \frac{1}{2}\sigma_1^2(k)+\frac{3}{2}\int_A q_1^2(k,\alpha)\nu(d\alpha),\\
\mu(k) &> \frac{1}{2}\sigma_2^2(k)+\frac{3}{2}\int_A q_2^2(k,\alpha)\nu(d\alpha),\\
\frac{\mu(k)\left(\mu(k)+\delta(k)\right)}{\gamma(k)} &> \frac{2\mu(k)+\gamma(k)}{2\gamma(k)}\sigma_3^2(k)\\
&+\left(\frac{2\mu(k)+3\gamma(k)}{2\gamma(k)}\right)\int_A q_3^2(k,\alpha)\nu(d\alpha),
\end{aligned}$$

for any given initial condition $(S(0),I(0),R(0))\in\mathbb{R}_+^3$ the solution of system (4) satisfies

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[\left(S(s)-\frac{\mu(k)\left(\mu(k)+\delta(k)\right)}{\left(\mu(k)+\gamma(k)+\delta(k)\right)T_1}S^*\right)^2\right.\\
&+\left(I(s)-\frac{\mu(k)}{T_2}I^*\right)^2\\
&\left.+\left(R(s)-\frac{\mu(k)\left(\mu(k)+\delta(k)\right)}{\gamma(k)T_3}R^*\right)^2\right]ds\le\frac{T}{\hat{T}}
\end{aligned}$$

where $\hat{T}=\min\{T_1,T_2,T_3\}$.

Proof Consider the function U_2 expressed by

$$\begin{aligned}
U_2(S,I,R,k) &= \frac{1}{2}\left(S-S^*+I-I^*+R-R^*\right)^2\\
&+w_1(k)\left(I-I^*-I^*\ln\frac{I}{I^*}\right)\\
&+\frac{1}{2}w_2(k)\left(R-R^*\right)^2,
\end{aligned}$$

where $w_1(k),w_1(k)$ for each $k\in\mathbb{S}$ are two positive constants to be determined below. Then, using the Itô's formula we have

$$\begin{aligned}
dU_2 &= \mathscr{L}U_2dt+\left(S-S^*+I-I^*+R-R^*\right)\left[\sigma_1(k)SdB_1(t)\right.\\
&\left.+\sigma_2(k)IdB_2(t)+\sigma_3(k)RdB_3(t)\right]\\
&+w_1(k)\left(1-\frac{I^*}{I}\right)\sigma_2(k)IdB_2(t)\\
&+w_2(k)\sigma_3(k)\left(R-R^*\right)RdB_3(t)\\
&+\int_A\left\{\left(S-S^*+I-I^*+R-R^*\right)\left[q_1(k,\alpha)S\right.\right.\\
&\left.+q_2(k,\alpha)I+q_3(k,\alpha)R\right]\\
&+\frac{1}{2}\left[q_1(k,\alpha)S+q_2(k,\alpha)I+q_3(k,\alpha)R\right]^2\\
&+w_1(k)\left[q_2(k,\alpha)I-I^*\ln(1+q_2(k,\alpha))\right]\\
&+w_2(k)\left(R-R^*\right)q_3(k,\alpha)R\\
&\left.+\frac{1}{2}w_2(k)q_3^2(k,\alpha)R^2\right\}\tilde{N}(dt,d\alpha),
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{L}U_2 &= \left(S-S^*+I-I^*+R-R^*\right)\left[\Lambda(k)-\mu(k)S-\mu(k)I\right.\\
&\left.-\ \mu(k)R\right]+w_1(k)\left(1-\frac{I^*}{I}\right)\left[\beta(k)SI-\left(\mu(k)+\gamma(k)\right)I\right]\\
&+\frac{1}{2}\sigma_1(k)S^2+\frac{1}{2}\sigma_2(k)I^2+\frac{1}{2}w_1(k)\sigma_2(k)I^*\\
&+w_2(k)\left(R-R^*\right)\left[\gamma(k)I-\left(\mu(k)+\gamma(k)\right)R\right]\\
&+\frac{1}{2}\sigma_3(k)R^2+\frac{1}{2}w_2(k)\sigma_3(k)R^2\\
&+\int_A\left\{\frac{1}{2}\left[q_1(k,\alpha)S+q_2(k,\alpha)I+q_3(k,\alpha)R\right]^2\right.\\
&+w_1(k)I^*\left[q_2(k,\alpha)-\ln(1+q_2(k,\alpha))\right]\\
&\left.+\frac{1}{2}w_2(k)q_3^2(k,\alpha)R^2\right\}\nu(d\alpha)+\sum_{l=1}^N\phi_{kl}U_2(S,I,R,l).
\end{aligned}$$

Using the fact that the endemic equilibrium $E^*=(S^*,I^*,R^*)$ satisfies

$$\begin{aligned}
\Lambda(k)-\beta(k)S^*I^*-\mu(k)S^*+\delta(k)R^* &= 0,\\
\beta(k)S^*I^*-\left(\mu(k)+\gamma(k)\right)I^* &= 0,\\
\gamma(k)I^*-\left(\mu(k)+\delta(k)\right)R^* &= 0,
\end{aligned}$$

we obtain

$$\begin{aligned}\mathscr{L}U_2 &= (S-S^*+I-I^*+R-R^*)[-\mu(k)(S-S^*)\\&\quad -\mu(k)(I-I^*)-\mu(k)(R-R^*)]+\frac{1}{2}\sigma_1(k)S^2\\&\quad +w_1(k)\beta(k)(I-I^*)(S-S^*)+\frac{1}{2}\sigma_2(k)I^2\\&\quad +w_2(k)(R-R^*)[\gamma(k)(I-I^*)\\&\quad -(\mu(k)+\delta(k))(R-R^*)]\\&\quad +\frac{1}{2}w_1(k)\sigma_2(k)I^*+\frac{1}{2}\sigma_3(k)(1+w_2(k))R^2\\&\quad +\int_A\left\{\frac{1}{2}[q_1(k,\alpha)S+q_2(k,\alpha)I+q_3(k,\alpha)R]^2\right.\\&\quad +w_1(k)I^*[q_2(k,\alpha)-\ln(1+q_2(k,\alpha))]\\&\quad \left.+\frac{1}{2}w_2(k)q_3^2(k,\alpha)R^2\right\}\nu(d\alpha)+\sum_{l=1}^N\phi_{kl}U_2(S,I,R,l)\\&= -\mu(k)(S-S^*)^2-\mu(k)(I-I^*)^2\\&\quad -[\mu(k)+w_2(k)(\mu(k)+\delta(k))](R-R^*)^2\\&\quad +[w_1(k)\beta(k)-2\mu(k)](S-S^*)(I-I^*)\\&\quad +[w_2(k)\gamma(k)-2\mu(k)](I-I^*)(R-R^*)\\&\quad -2\mu(k)(S-S^*)(R-R^*)+\frac{1}{2}\sigma_1(k)S^2+\frac{1}{2}\sigma_2(k)I^2\\&\quad +\frac{1}{2}w_1(k)\sigma_2(k)I^*+\frac{1}{2}\sigma_3(k)(1+w_2(k))R^2\\&\quad +\int_A\left\{\frac{1}{2}[q_1(k,\alpha)S+q_2(k,\alpha)I+q_3(k,\alpha)R]^2\right.\\&\quad +w_1(k)I^*[q_2(k,\alpha)-\ln(1+q_2(k,\alpha))]\\&\quad \left.+\frac{1}{2}w_2(k)q_3^2(k,\alpha)R^2\right\}\nu(d\alpha)+\sum_{l=1}^N\phi_{kl}U_2(S,I,R,l).\end{aligned}$$

Choose $w_1(k)=\frac{2\mu(k)}{\beta(k)}$ and $w_2(k)=\frac{2\mu(k)}{\gamma(k)}$, such that

$$w_1(k)\beta(k)-2\mu(k)=0 \quad\text{and}\quad w_2(k)\gamma(k)-2\mu(k)=0.$$

Using the inequality $(a+b+c)^2\le 3\left(a^2+b^2+c^2\right)$ and $2ab\le\frac{a^2}{\varepsilon(k)}+\varepsilon(k)b^2$ where $\varepsilon(k)=\frac{\mu(k)+\gamma(k)+\delta(k)}{\gamma(k)}$, we get

$$\begin{aligned}\mathscr{L}U_2 &\le -\frac{\mu(k)(\mu(k)+\delta(k))}{\mu(k)+\gamma(k)+\delta(k)}(S-S^*)^2-\mu(k)(I-I^*)^2\\&\quad -\frac{\mu(k)(\mu(k)+\delta(k))}{\gamma(k)}(R-R^*)^2+\frac{1}{2}\sigma_1^2(k)S^2\\&\quad +\frac{1}{2}\sigma_3^2(k)\left(1+\frac{2\mu(k)}{\gamma(k)}\right)R^2+\frac{3}{2}S^2\int_A q_1^2(k,\alpha)\nu(d\alpha)\\&\quad +\frac{1}{2}\sigma_2^2(k)I^2+\frac{3}{2}I^2\int_A q_2^2(k,\alpha)\nu(d\alpha)\\&\quad +\frac{2\mu(k)}{\beta(k)}I^*\int_A[q_2(k,\alpha)-\ln(1+q_2(k,\alpha))]\nu(d\alpha)\\&\quad +\left(\frac{\mu(k)}{\gamma(k)}+\frac{3}{2}\right)R^2\int_A q_3^2(k,\alpha)\nu(d\alpha)\\&\quad +\frac{\mu(k)}{\beta(k)}\sigma_2(k)I^*+\sum_{l=1}^N\phi_{kl}U_2(S,I,R,l).\end{aligned}$$

There exists a constant C_3 such that

$$\sum_{l=1}^N\phi_{kl}U_2(S,I,R,l)\le C_3U_2(S,I,R,k).$$

Thus

$$\begin{aligned}\mathscr{L}U_2 &\le -T_1\left(S-\frac{\mu(k)(\mu(k)+\delta(k))}{(\mu(k)+\gamma(k)+\delta(k))T_1}S^*\right)^2\\&\quad -T_2\left(I-\frac{\mu(k)}{T_2}I^*\right)^2\\&\quad -T_3\left(R-\frac{\mu(k)(\mu(k)+\delta(k))}{\gamma(k)T_3}R^*\right)^2\\&\quad +T+C_3U_2(S,I,R,k).\end{aligned}$$

Taking expectation on both sides of (7) and using the Gronwall inequality, we get

$$\begin{aligned}0 &\le \mathbb{E}U_2(S(t),I(t),R(t),r(t))\\&\le U_2(S(0),I(0),R(0),r(0))\\&\quad -\mathbb{E}\int_0^t\left[T_1\left(S(s)-\frac{\mu(k)(\mu(k)+\delta(k))}{(\mu(k)+\gamma(k)+\delta(k))T_1}S^*\right)^2\right.\\&\quad +T_2\left(I(s)-\frac{\mu(k)}{T_2}I^*\right)^2\\&\quad \left.+T_3\left(R(s)-\frac{\mu(k)(\mu(k)+\delta(k))}{\gamma(k)T_3}R^*\right)^2\right]ds\\&\quad +C_3\int_0^t\mathbb{E}U_2(S(s),I(s),R(s),r(s))+Tt\end{aligned}$$

$$\begin{aligned}&\le \Big\{U_2(S(0),I(0),R(0),r(0))\\&\quad -\mathbb{E}\int_0^t\left[T_1\left(S(s)-\frac{\mu(k)(\mu(k)+\delta(k))}{(\mu(k)+\gamma(k)+\delta(k))T_1}S^*\right)^2\right.\\&\quad +T_2\left(I(s)-\frac{\mu(k)}{T_2}I^*\right)^2\\&\quad \left.+T_3\left(R(s)-\frac{\mu(k)(\mu(k)+\delta(k))}{\gamma(k)T_3}R^*\right)^2\right]ds+Tt\Big\}e^{C_3t}.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbb{E}\int_0^t\left[T_1\left(S(s)-\frac{\mu(k)(\mu(k)+\delta(k))}{(\mu(k)+\gamma(k)+\delta(k))T_1}S^*\right)^2\right.\\&\quad +T_2\left(I(s)-\frac{\mu(k)}{T_2}I^*\right)^2\\&\quad \left.+T_3\left(R(s)-\frac{\mu(k)(\mu(k)+\delta(k))}{\gamma(k)T_3}R^*\right)^2\right]ds\\&\le U_2(S(0),I(0),R(0),r(0))+Tt.\end{aligned}$$

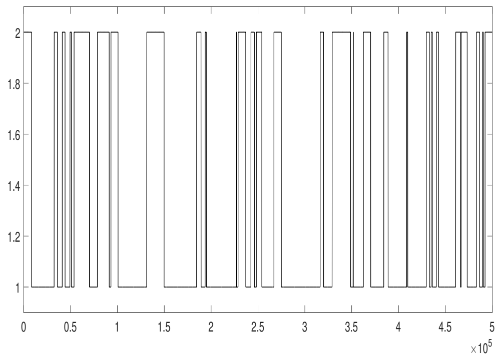

Fig. 1: The trajectory of the Markov chain r(t).

Hence

$$\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[\left(S(s)-\frac{\mu(k)\left(\mu(k)+\delta(k)\right)}{\left(\mu(k)+\gamma(k)+\delta(k)\right)T_1}S^*\right)^2+\left(I(s)-\frac{\mu(k)}{T_2}I^*\right)^2+\left(R(s)-\frac{\mu(k)\left(\mu(k)+\delta(k)\right)}{\gamma(k)T_3}R^*\right)^2\right]ds\le\frac{T}{\hat{T}}.$$

.□.

Remark 4.1. From theorem 4.1, one can conclude that if $R_0>1$ the solution will fluctuate around the endemic equilibrium.

5 Examples

In this section, we give simulations corresponding to the analytical results showed in the past sections using the Euler-Maruyama scheme [31].
We consider Markov chain $r(t)$ taking value in state space $\mathbb{S}=\{1,2\}$ with the generator

$$\Phi=\begin{pmatrix}-1 & 1\\ 2 & -2\end{pmatrix}.$$

Then, the Markov chain $r(t)$ has a unique stationary distribution,

$$\pi=(\pi_1,\pi_2)=\left(\frac{2}{3},\frac{1}{3}\right).$$

Given a step size $\Delta=0.0001$, the Markov chain $r(t)$ can be simulated by computing the one-step transition probability matrix $P=e^{\Delta\Phi}$ [32], the transition probability matrix is given by

$$P=\begin{pmatrix}0.9999 & 0.0001\\ 0.0002 & 0.9998\end{pmatrix}.$$

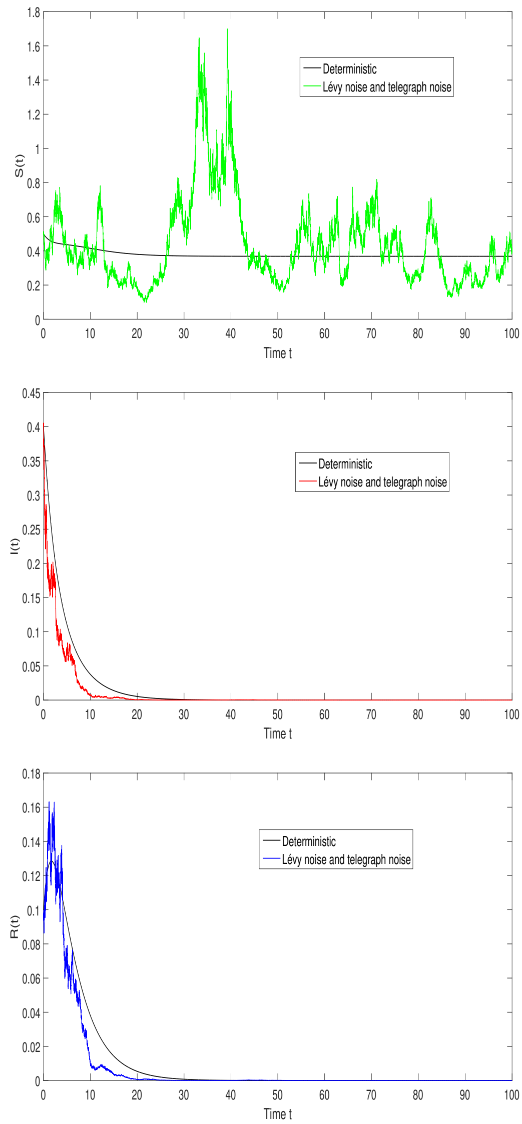

Fig. 2: Paths of stochastic and deterministic systems as given in Example 5.1.

Fig. 1 shows a result of one simulation run of the Markov chain $r(t)$.

Example 5.1. In this example, we set $\nu(\mathbb{A})=0.5$, $(S(0),I(0),R(0))=(0.5,0.4,0.1)$, $r(0)=2$, and the coefficients:
If $k=1$,

$$\Lambda(1)=0.25,\beta(1)=0.2,\mu(1)=0.19,\gamma(1)=0.2,$$
$$\delta(1)=0.2,\sigma_1(1)=0.3,\sigma_2(1)=0.4,\sigma_3(1)=0.26,$$
$$q_1(1,\alpha)=0.1,\ q_2(1,\alpha)=0.23,\ q_3(1,\alpha)=0.2$$

If $k = 2$,

$$\Lambda(2) = 0.24, \beta(2) = 0.22, \mu(2) = 0.18, \gamma(2) = 0.19,$$
$$\delta(2) = 0.2, \sigma_1(2) = 0.35, \sigma_2(2) = 0.34, \sigma_3(2) = 0.3,$$
$$q_1(2,\alpha) = 0.2, q_2(2,\alpha) = 0.2, q_3(2,\alpha) = 0.3.$$

This implies that

$$R_0 = 0.8421 < 1,$$
$$\mu(1) = 0.38 > 2\sigma_1^2(1) + 4\int_A q_1^2(1,\alpha)\nu(d\alpha) = 0.2,$$
$$2\mu(1) = 0.38 > \sigma_2^2(1) + 4\int_A q_2^2(1,\alpha)\nu(d\alpha) = 0.2658,$$
$$\frac{2\mu(1)\gamma(1) + 2\mu(1)(\mu(1)+\delta(1))}{\gamma(1)} = 1.121$$
$$> \sigma_3^2(1)\left(1 + \frac{2\mu(1)}{\gamma(1)}\right)$$
$$+ \left(4 + \frac{2\mu(1)}{\gamma(1)}\right)\int_A q_3^2(1,\alpha)\nu(d\alpha) = 0.3140.$$

And

$$R_0 = 0.7719 < 1,$$
$$2\mu(2) = 0.38 > 2\sigma_1^2(2) + 4\int_A q_1^2(2,\alpha)\nu(d\alpha) = 0.3250,$$
$$2\mu(2) = 0.38 > \sigma_2^2(2) + 4\int_A q_2^2(2,\alpha)\nu(d\alpha) = 0.2025,$$
$$\frac{2\mu(2)\gamma(2) + 2\mu(2)(\mu(2)+\delta(2))}{\gamma(2)} = 1.1210 >$$
$$\sigma_3^2(2)\left(1 + \frac{2\mu(2)}{\gamma(2)}\right)$$
$$+ \left(4 + \frac{2\mu(2)}{\gamma(2)}\right)\int_A q_3^2(2,\alpha)\nu(d\alpha) = 0.5265.$$

Which implies that the solution of (4) fluctuates around the disease-free equilibrium. Fig. 2 represents the trajectories of the solutions to (1) and (4).

Example 5.2. In this example we present a simulation of the trajectories of the solutions around the endemic equilibrium E^* with the following parameters:
If k=1,

$$\Lambda(1) = 0.25, \beta(1) = 0.4, \mu(1) = 0.19, \gamma(1) = 0.2,$$
$$\delta(1) = 0.2, \sigma_1(1) = 0.1, \sigma_2(1) = 0.12, \sigma_3(1) = 0.11,$$
$$q_1(1,\alpha) = 0.1,\ q_2(1,\alpha) = 0.23,\ q_3(1,\alpha) = 0.2.$$

If k=2,

$$\Lambda(2) = 0.24, \beta(2) = 0.38, \mu(2) = 0.18, \gamma(2) = 0.2,$$
$$\delta(2) = 0.21, \sigma_1(2) = 0.13, \sigma_2(2) = 0.1, \sigma_3(2) = 0.13,$$
$$q_1(2,\alpha) = 0.2, q_2(2,\alpha) = 0.2, q_3(2,\alpha) = 0.3.$$

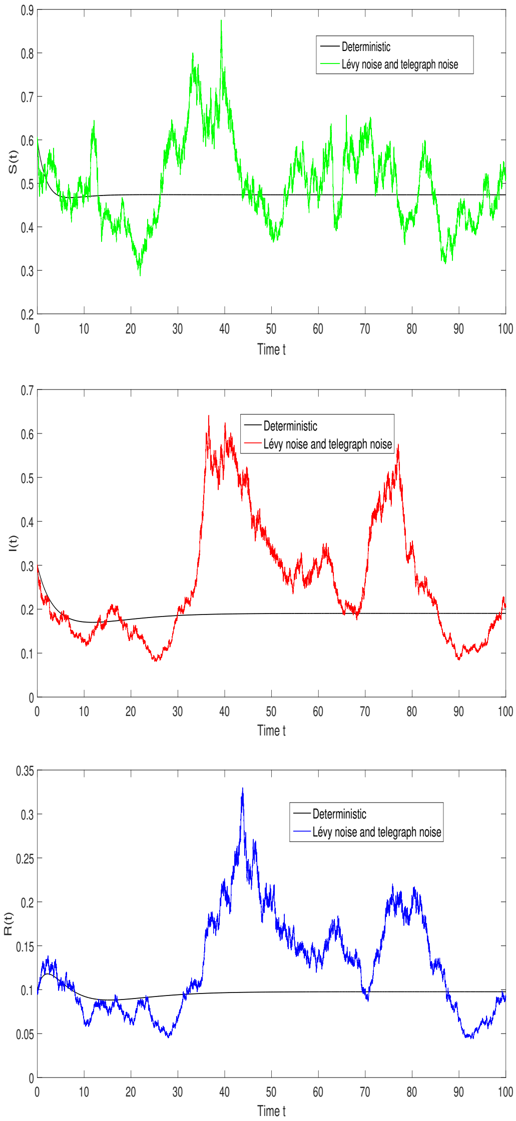

Fig. 3: Paths of stochastic and deterministic systems as given in Example 5.2.

A simple computation gives that

$$R_0 = 1.3495,$$
$$\frac{\mu(1)+\delta(1)}{\mu(1)+\gamma(1)+\delta(1)} = 0.6610$$
$$> \frac{1}{2}\sigma_1^2(1) + \frac{3}{2}\int_A q_1^2(1,\alpha)\nu(d\alpha) = 0.0125,$$
$$\mu(1) = 0.19 > \frac{1}{2}\sigma_2^2(1) + \frac{3}{2}\int_A q_2^2(1,\alpha)\nu(d\alpha) = 0.0469,$$
$$\frac{\mu(1)(\mu(1)+\delta(1))}{\gamma(1)} = 0.3705 > \frac{2\mu(1)+\gamma(1)}{2\gamma(1)}\sigma_3^2(1)$$
$$+ \left(\frac{2\mu(1)+3\gamma(1)}{2\gamma(1)}\right)\int_A q_3^2(1,\alpha)\nu(d\alpha) = 0.0699.$$

And

$$\begin{aligned}
&R_0 = 1.2982,\\
&\frac{\mu(2)+\delta(2)}{\mu(2)+\gamma(2)+\delta(2)} = 0.6610\\
&> \frac{1}{2}\sigma_1^2(2) + \frac{3}{2}\int_A q_1^2(2,\alpha)\nu(d\alpha) = 0.0385,\\
&\mu(2) = 0.18 > \frac{1}{2}\sigma_2^2(2) + \frac{3}{2}\int_A q_2^2(2,\alpha)\nu(d\alpha) = 0.0385,\\
&\frac{\mu(2)(\mu(2)+\delta(2))}{\gamma(2)} = 0.3510 >\\
&\frac{2\mu(2)+\gamma(2)}{2\gamma(2)}\sigma_3^2(2)\\
&+\left(\frac{2\mu(2)+3\gamma(2)}{2\gamma(2)}\right)\int_A q_3^2(2,\alpha)\nu(d\alpha) = 0.1317.
\end{aligned}$$

Then, the condition of theorem 4.1 is satisfied. Hence, the solution of (4) fluctuates around the endemic equilibrium. The computer simulations in Fig 3, illustrate these results.

6 Conclusion

This paper addressed a stochastic SIRS epidemic model (4) driven by Brownian motion, regime switching and Lévy noise, which accurately represent the natural effects. We first showed the existence and the uniqueness of the global positive solution for the stochastic system (4), Then, we proved that the solution of the system (4) fluctuates around the equilibria under sufficient conditions using Lyapunov method. We illustrated our theoretical results by numerical simulations. In future studies, we will study the effect of white, color and Lévy noise on other epidemic models such as SVIS model and SEIR model.

Competing interests

The authors declare that they have no competing interests.

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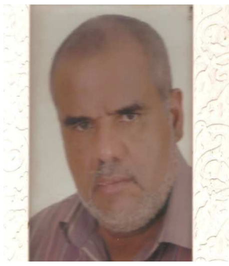

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