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Geometric Phase of a Moving Four-Level Lambda-Type Atom in a Dissipative Cavity

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Abstract: In this paper, we investigate some properties through four-level λ-type atom interacting with a single-mode quantized field with multi-photon transitions. We study this system in the presence of detuning parameter, Kerr nonlinearity, and intensity-dependent atom-field coupling in a dissipative cavity (i.e. the field is suffering decay rate). Also, the coupling parameter modulated to be time dependent. The exact solution of this model is given using the Schrődinger equation when the atom and the field are initially prepared in superposition state and coherent state, respectively. We employed the results to calculate some aspects such as atomic population inversion, geometric phase and Husimi Q-function. It is emphasized that the system can be used as a quantum memory.

Keywords: Atomic population inversion, Four-level atom, Geometric phase, Husimi Q-function, Kerr-medium

1 Introduction

Jaynes-Cummings model (JCM) [1] is a well-known and important model in quantum optics. It is an exactly solvable model which clearly describes the interaction between a two-level atom and a single-mode of a quantized radiation field, when the Rotating-Wave Approximation (RWA) is considered. Much attention has been paid to generalize the JCM in many different directions such as multi-photon transition, multi-level atoms, intensity-dependent coupling, multi-atoms interaction, multi-mode fields, Stark shift and Kerr nonlinearity, which have been recently investigated [2–15].

Numerous efforts have been devoted to analytical solutions of multi-level atoms interacting with the cavity field problems. One of the interesting example is the system of three-level atom different configurations (Λ, V, and Ξ) and one- or two-mode field [2, 7, 16–21]. Many studies have been conducted on the atom-field entanglement and geometric phase in such systems [2, 7, 16, 18–22]. Several studies of a three-level atom in motion which interacts with a single-mode field in an optical cavity in an intensity-dependent coupling regime have been conducted [23]. On the other hand, when the quantum systems interact with their surrounding environments, this results in the dissipation and then the decoherence which leads to reducing the quantum system entanglement. Several studies have been conducted in this domain. For instance, information dynamics of a three-level atom interacting with a damped cavity field has been recently investigated taking into consideration that the optical cavity is coupled to the environment [24]. Also, the case of a dissipative cavity is studied for a three-level atomic system through master equation methods [25, 26] and postulating a non-Hermitian model Hamiltonian [27–29].

Another example of multi-level atoms interacting with the cavity field problems is the system of four-level atom. Several systems of the four-level atom configurations such as Ξ, N, X, Y, Λ, λ, double-Λ, etc. have been introduced and some of their aspects have been visualized [21, 30–36]. For instance, the author in [32] has considered the quantum mutual entropy of a single four-level atom strongly coupled to a cavity field and driven by a laser field. Another study has been done by Abdel-Aty et al. in [33]. They have studied an intensity-dependent coupling regime that consists of a λ-type Four-level atom interacting with a single-mode quantized field and some physical properties of the atom-field entangled have been investigated.

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In the recent years, much attention has been focused on the properties of the four-level atomic systems when time-dependent coupling with the field is considered [37–42]. Theoretical efforts have been stimulated by experimental progress in cavity QED. In addition to the experimental drive, there also exists a theoretical motivation to include the atomic motion effect in the four-level atomic systems. More recently, dynamics of entanglement of time-dependent three- and four-Level atomic system under Stark effect and Kerr-medium has been investigated [21].

In this work, we extend the models in [33, 41, 42] by considering the coupling parameter to be intensity-dependent. Also, the four-level atom is moving in a dissipative cavity. The atomic inversion, the geometric phase and the Husimi Q-function expressions are calculated. The present paper is organized as follows: In Sec. 2, we introduce the model and its solution under certain approximation similar to that of the Rotating-Wave Approximation (RWA) at any time $t > 0$, In Sec. 3, we investigate the atomic inversion and the dynamical properties for different regimes. Numerical results for the atomic inversion are discussed in this section, as well. In Sec. 4, the geometric phase is investigated. We devote Sec. 5 to study one of the quasi-probability distribution functions. In particular, the expression of the Husimi Q-function is presented. The main results and conclusion are presented in Sec. 6.

2 Physical Model

The considered model is an intensity-dependent regime that consists of a moving four-level λ-type atom with the energy levels $\omega_4 > \omega_3 > \omega_2 > \omega_1$, interacting with a single-mode quantized field of frequency Ω in an optical cavity surrounded by Kerr nonlinearity in the presence of detuning parameters. The transitions $|4\rangle \longleftrightarrow |3\rangle$, $|3\rangle \longleftrightarrow |1\rangle$, and $|3\rangle \longleftrightarrow |2\rangle$ are allowed while the transition $|2\rangle \longleftrightarrow |1\rangle$ is forbidden as shown in Fig. 1. This means that the number of photons of the atomic states $|2\rangle$ and $|1\rangle$ is the same. To include damping effects, we propose the following non-Hermitian Hamiltonian in the Rotating Wave Approximation (RWA) of the introduced physical system ($\hbar = 1$) [33, 41, 42]:

$$\hat{H} = \sum_{i=1}^{4} \omega_i \hat{\sigma}_{ii} + \Omega \hat{a}^{\dagger}\hat{a} + \gamma_1(t)(\hat{R}^k \hat{\sigma}_{31} + \hat{R}^{\dagger k}\hat{\sigma}_{13})$$
$$+\gamma_2(t)(\hat{R}^k \hat{\sigma}_{32} + \hat{R}^{\dagger k}\hat{\sigma}_{23}) + \gamma_3(t)(\hat{R}^k \hat{\sigma}_{43} + \hat{R}^{\dagger k}\hat{\sigma}_{34}) \quad (1)$$
$$+\chi \hat{a}^{\dagger 2}\hat{a}^2 - \frac{i}{2}\Gamma \hat{a}^{\dagger}\hat{a}. \quad (2)$$

In which, the operators $\hat{\sigma}_{ij} = |i\rangle\langle j|$ are the atomic raising or lowering operator, the bosonic operators $\hat{a}^{\dagger}$ and $\hat{a}$ are the field creation and annihilation operators, respectively, Ω is the frequency of the single-mode field,

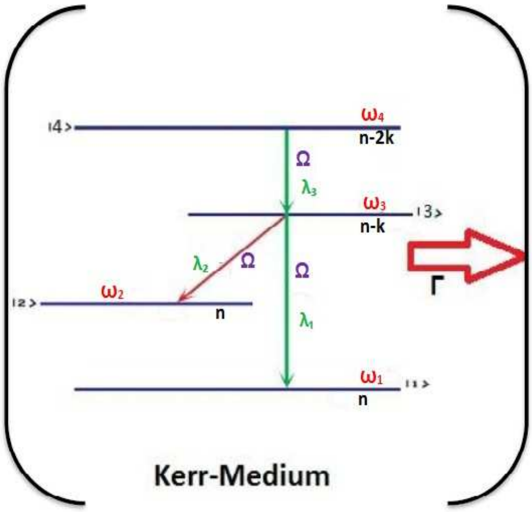

Fig. 1: Schematic diagram of a four-level λ-type atom with frequencies ω_i (i=1, 2, 3, 4) interacting with single-mode electromagnetic field with frequency Ω in a cavity suffers a decay rate Γ for field.

$\gamma_i(t)$, $i = 1, 2, 3$ are the atom-field coupling parameters, χ is the third-order nonlinearity of the Kerr-medium and Γ is the decay (dissipation) coefficient of the field. In addition, the operators $\hat{R}^{\dagger} = f(\hat{n})\hat{a}^{\dagger}$ and $\hat{R} = \hat{a}f(\hat{n})$ with $\hat{n} = \hat{a}^{\dagger}\hat{a}$ are, respectively, the nonlinear ($f-$deformed) creation and annihilation operators, which satisfy the following communication relations:

$$[\hat{R}, \hat{R}^{\dagger}] = (\hat{n}+1)f^2(\hat{n}+1) - \hat{n}f^2(\hat{n})$$

$$[\hat{R}^{\dagger}, \hat{n}] = -\hat{R}^{\dagger}, \qquad [\hat{R}, \hat{n}] = \hat{R}, \quad (3)$$

where $f(\hat{n})$ is a function of the number operator (intensity of light). To observe what we have really done, we put

$$\hat{H} = \sum_{j=1}^{4} \omega_j \hat{\sigma}_{jj} + \Omega \hat{a}^{\dagger}\hat{a} + +\chi \hat{a}^{\dagger 2}\hat{a}^2 - \frac{i}{2}\Gamma \hat{a}^{\dagger}\hat{a}$$
$$+\gamma_1(t)(\hat{a}^k \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{\sigma}_{31} + \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{a}^{\dagger k}\hat{\sigma}_{13})$$
$$+\gamma_2(t)(\hat{a}^k \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{\sigma}_{32} + \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{a}^{\dagger k}\hat{\sigma}_{23})$$
$$+\gamma_3(t)(\hat{a}^k \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{\sigma}_{43} + \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{a}^{\dagger k}\hat{\sigma}_{34}). \quad (4)$$

The interaction Hamiltonian can be rewritten in the following manner

$$\hat{H}_I = \gamma_1(t)(\hat{a}^k \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!} e^{-i\Delta_1 t}\hat{\sigma}_{31} + \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{a}^{\dagger k} e^{i\Delta_1 t}\hat{\sigma}_{13}) + \gamma_2(t)(\hat{a}^k \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!} e^{-i\Delta_2 t}\hat{\sigma}_{32} + \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{a}^{\dagger k} e^{i\Delta_2 t}\hat{\sigma}_{23}) + \gamma_3(t)(\hat{a}^k \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!} e^{-i\Delta_3 t}\hat{\sigma}_{43} + \frac{[f(\hat{n})]!}{[f(\hat{n}-k)]!}\hat{a}^{\dagger k} e^{i\Delta_3 t}\hat{\sigma}_{34}) + \chi \hat{a}^{\dagger 2}\hat{a}^2 - \frac{i}{2}\Gamma \hat{a}^\dagger \hat{a}. \quad (5)$$

Where, the detuning parameters Δ_1, Δ_2 and Δ_3 are given by

$$\Delta_1 = \omega_1 - \omega_3 + k\Omega,$$
$$\Delta_2 = \omega_2 - \omega_3 + k\Omega,$$
$$\Delta_3 = \omega_3 - \omega_4 + k\Omega.$$

For simplicity, we consider $\gamma_1(t) = \gamma_2(t) = \gamma_3(t) = \gamma(t) = \lambda_i \sin(\delta t)$, where λ_i, $i = 1,2,3$ are arbitrary constants. We assume that the wave function of the atom-field at any time $t > 0$ can be expressed as

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} [A_1(n,t)|1,n\rangle + A_2(n,t)|2,n\rangle + A_3(n-2,t)|3,n-2\rangle + A_4(n-4,t)|4,n-4\rangle]. \quad (6)$$

To reach this goal, suppose that the atoms enter the cavity in the coherent superposition of the states $|1\rangle$, $|2\rangle$, $|3\rangle$ and $|4\rangle$, i.e.,

$$|\Psi(0)\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle + \alpha_4|4\rangle, \quad (7)$$

where α_1, α_2, α_3 and α_4 are arbitrary constants and obey the normalization relation

$$\sum_{i=1}^{4} |\alpha_i|^2 = 1, \quad (8)$$

and the field is assumed to be initially in the coherent state, i.e.,

$$|\Psi(0)\rangle_F = \sum_{n=0}^{\infty} q_n |n\rangle, \quad (9)$$

where $q_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}$, $|\alpha|^2$ is the initial mean photon number for the mode. Now, substituting $|\Psi(t)\rangle$ from Eq. (5) and $\hat{H}_I$ from Eq. (4) in the time-dependent Schrődinger equation $i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}_I\,|\Psi(t)\rangle$, one may arrive at the following coupled differential equations for the atomic probability amplitudes

$$i\dot{A}_1(n,t) = v_1 A_1(n,t) + g_1 e^{i\Delta_1 t}\sin(\delta t) A_3(n-k,t),$$
$$i\dot{A}_2(n,t) = v_1 A_2(n,t) + g_2 e^{i\Delta_2 t}\sin(\delta t) A_3(n-k,t),$$
$$i\dot{A}_3(n-k,t) = v_2 A_3(n-k,t) + g_1 e^{-i\Delta_1 t}\sin(\delta t) A_1(n,t) + g_2 e^{-i\Delta_2 t}\sin(\delta t) A_2(n,t) + g_3 e^{i\Delta_3 t}\sin(\delta t) A_4(n-2k,t)$$
$$i\dot{A}_4(n-2k,t) = v_3 A_4(n-2k,t) + g_3 e^{-i\Delta_3 t}\sin(\delta t) A_3(n-k,t). \quad (10)$$

Where,

$$v_1 = \chi n(n-1) - \frac{i}{2}\Gamma n, \quad v_2 = \chi(n-k)(n-k-1) - \frac{i}{2}\Gamma(n-k),$$
$$v_3 = \chi(n-2k)(n-2k-1) - \frac{i}{2}\Gamma(n-2k),$$
$$g_1 = \lambda_1 \sqrt{\frac{n!}{(n-k)!}\frac{[f(n)]!}{[f(n-k)]!}}, \quad g_2 = \lambda_2 \sqrt{\frac{n!}{(n-k)!}\frac{[f(n)]!}{[f(n-k)]!}},$$
$$g_3 = \lambda_3 \sqrt{\frac{(n-k)!}{(n-2k)!}\frac{[f(n-k)]!}{[f(n-2k)]!}},$$
$$[f(n)]! = f(n)f(n-1).....f(1), \; [f(0)]! = 1.$$

As one can see there are two exponential terms in each equation: one is rapidly oscillating terms $e^{\pm i(\delta+\Delta_i)t}$ and the other is slowly varying terms $e^{\pm i(\delta-\Delta_i)t}$. In this case if we neglect the rapidly varying terms compared with the slowly varying terms, then Eq. (9) reduces to

$$i\frac{d}{dt}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} v_1 & 0 & -\bar{g}_1 e^{i\bar{\Delta}_1 t} & 0 \\ 0 & v_1 & -\bar{g}_2 e^{i\bar{\Delta}_2 t} & 0 \\ \bar{g}_1 e^{-i\bar{\Delta}_1 t} & \bar{g}_2 e^{-i\bar{\Delta}_2 t} & v_2 & -\bar{g}_3 e^{i\bar{\Delta}_3 t} \\ 0 & 0 & \bar{g}_3 e^{-i\bar{\Delta}_3 t} & v_3 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \quad (11)$$

where $\bar{\Delta}_j = \Delta_j - \delta$, $\bar{g}_j = g_1/2i$, $j = 1,2,3$. It is obvious that the coefficients of this coupled system of differential equations are time-dependent ones. We can avoid this problem using the transforms

$$A_1(n,t) = \bar{A}_1(n,t)e^{\frac{i\bar{\Delta}_1}{2}t},$$
$$A_2(n,t) = \bar{A}_2(n,t)e^{i(\bar{\Delta}_2-\frac{\bar{\Delta}_1}{2})t},$$
$$A_3(n-k,t) = \bar{A}_3(n-k,t)e^{-\frac{i\bar{\Delta}_1}{2}t},$$
$$A_4(n-2k,t) = \bar{A}_4(n-2k,t)e^{-i(\bar{\Delta}_3+\frac{\bar{\Delta}_1}{2})t}, \quad (12)$$

and the Laplace transform to arrive at

$$\begin{pmatrix} s+i\bar{v}_1 & 0 & -i\bar{g}_1 & 0 \\ 0 & s+i\bar{v}_2 & -i\bar{g}_2 & 0 \\ i\bar{g}_1 & i\bar{g}_2 & s+i\bar{v}_3 & -i\bar{g}_3 \\ 0 & 0 & i\bar{g}_3 & s+i\bar{v}_4 \end{pmatrix}\begin{pmatrix} L_1(s) \\ L_2(s) \\ L_3(s) \\ L_4(s) \end{pmatrix} = \begin{pmatrix} \bar{A}_1(0) \\ \bar{A}_2(0) \\ \bar{A}_3(0) \\ \bar{A}_4(0) \end{pmatrix}. \quad (13)$$

Where

$$\bar{v}_1 = v_1 + \frac{\bar{\Delta}_1}{2}, \quad \bar{v}_2 = v_1 + (\bar{\Delta}_2 - \frac{\bar{\Delta}_1}{2}),$$
$$\bar{v}_3 = v_2 - \frac{\bar{\Delta}_1}{2}, \quad \bar{v}_4 = v_3 - (\bar{\Delta}_3 + \frac{\bar{\Delta}_1}{2}).$$

We used Cramer's rule [43] to solve the set of algebraic equations (12) and obtained

$$L_1(s) = \frac{f_1(s)}{F(s)}, \; L_2(s) = \frac{f_2(s)}{F(s)}, \; L_3(s) = \frac{f_3(s)}{F(s)}, \; L_4(s) = \frac{f_4(s)}{F(s)}, \quad (14)$$

where

$$f_1(s)=\begin{vmatrix}\bar{A}_1(0) & 0 & -i\bar{g}_1 & 0\\ \bar{A}_2(0) & s+i\bar{v}_2 & -i\bar{g}_2 & 0\\ \bar{A}_3(0) & i\bar{g}_2 & s+i\bar{v}_3 & -i\bar{g}_3\\ \bar{A}_4(0) & 0 & i\bar{g}_3 & s+i\bar{v}_4\end{vmatrix}, \quad (15)$$

$$f_2(s)=\begin{vmatrix}s+i\bar{v}_1 & \bar{A}_1(0) & -i\bar{g}_1 & 0\\ 0 & \bar{A}_2(0) & -i\bar{g}_2 & 0\\ i\bar{g}_1 & \bar{A}_3(0) & s+i\bar{v}_3 & -i\bar{g}_3\\ 0 & \bar{A}_4(0) & i\bar{g}_3 & s+i\bar{v}_4\end{vmatrix}, \quad (16)$$

$$f_3(s)=\begin{vmatrix}s+i\bar{v}_1 & 0 & \bar{A}_1(0) & 0\\ 0 & s+i\bar{v}_2 & \bar{A}_2(0) & 0\\ i\bar{g}_1 & i\bar{g}_2 & \bar{A}_3(0) & -i\bar{g}_3\\ 0 & 0 & \bar{A}_4(0) & s+i\bar{v}_4\end{vmatrix}, \quad (17)$$

$$f_4(s)=\begin{vmatrix}s+i\bar{v}_1 & 0 & -i\bar{g}_1 & \bar{A}_1(0)\\ 0 & s+i\bar{v}_2 & -i\bar{g}_2 & \bar{A}_2(0)\\ i\bar{g}_1 & i\bar{g}_2 & s+i\bar{v}_3 & \bar{A}_3(0)\\ 0 & 0 & i\bar{g}_3 & \bar{A}_4(0)\end{vmatrix}, \quad (18)$$

$$F(s)=\prod_{j=1}^{4}(s-\mu_j), \quad (19)$$

and $\mu_j,\ j=1,2,3,4$ are the roots of the equation

$$s^4+a_3s^3+a_2s^2+a_1s+a_0=0, \quad (20)$$

which can be given using MATHEMATICA, where

$$\begin{aligned}a_0 &= \bar{v}_1\bar{v}_2\bar{v}_3\bar{v}_4+\bar{g}_1^2\bar{v}_2\bar{v}_4+\bar{g}_2^2\bar{v}_1\bar{v}_4+\bar{g}_3^2\bar{v}_1\bar{v}_2,\\ a_1 &= -i[\bar{v}_3\bar{v}_4(\bar{v}_1+\bar{v}_2)+\bar{v}_1\bar{v}_2(\bar{v}_3+\bar{v}_4)+\bar{g}_1^2(\bar{v}_2+\bar{v}_4)\\ &\quad+\bar{g}_2^2(\bar{v}_1+\bar{v}_4)+\bar{g}_3^2(\bar{v}_1+\bar{v}_2)],\\ a_2 &= -[\bar{v}_1(\bar{v}_2+\bar{v}_3+\bar{v}_4)+\bar{v}_2(\bar{v}_3+\bar{v}_4)+\bar{v}_3\bar{v}_4+\bar{g}_1^2+\bar{g}_2^2+\bar{g}_3^2],\\ a_3 &= i(\bar{v}_1+\bar{v}_2+\bar{v}_3+\bar{v}_4). \end{aligned} \quad (21)$$

Taking the inverse Laplace transform for $L_1(s)$, $L_2(s)$, $L_3(s)$, and $L_4(s)$, we get the expressions for $\bar{A}_1(n,t)$, $\bar{A}_2(n,t)$, $\bar{A}_3(n-k,t)$, and $\bar{A}_4(n-2k,t)$ and then the expressions for $A_1(n,t)$, $A_2(n,t)$, $A_3(n-k,t)$, and $A_4(n-2k,t)$. At any time $t>0$ the reduced density matrix of the atom is given by:

$$\hat{\rho}_A(t)=Tr_F[|\Psi(t)\rangle\langle\psi(t)|]=\begin{pmatrix}\rho_{44}(t) & \rho_{43}(t) & \rho_{42}(t) & \rho_{41}(t)\\ \rho_{34}(t) & \rho_{33}(t) & \rho_{32}(t) & \rho_{31}(t)\\ \rho_{24}(t) & \rho_{23}(t) & \rho_{22}(t) & \rho_{21}(t)\\ \rho_{14}(t) & \rho_{13}(t) & \rho_{12}(t) & \rho_{11}(t)\end{pmatrix}, \quad (22)$$

$$\begin{aligned}\rho_{11}(t) &= \sum_{n=0}^{\infty}|A_1(n,t)|^2,\\ \rho_{22}(t) &= \sum_{n=0}^{\infty}|A_2(n,t)|^2,\\ \rho_{33}(t) &= \sum_{n=0}^{\infty}|A_3(n-2,t)|^2,\\ \rho_{44}(t) &= \sum_{n=0}^{\infty}|A_4(n-4,t)|^2. \end{aligned} \quad (23)$$

The reduced density operator of the field $\hat{\rho}_F(t)$ is given by

$$\hat{\rho}_F(t)=Tr_A(|\Psi(t)\rangle\langle\Psi(t)|). \quad (24)$$

In the next sections, the initial mean photon numbers are fixed at $|\alpha|^2=25$, and the atom is prepared initially in the superposition state ($\alpha_1=\alpha_2=\alpha_3=\alpha_4=1/2$). For simplicity, we consider the constants $\lambda_i=\lambda$ have been taken to be real and the interaction time is the scaled time $\tau=\lambda t$. Also, all plots correspond to the intensity-dependent coupling with the nonlinearity function $f(n)=1/\sqrt{n}$, and 2-photon transition ($k=2$).

3 Atomic Population Inversion

Through the collapse and revival phenomenon, we can get information about the behavior of the atom-field interaction. Thus, we shall study the dynamics of an important quantity, i.e. atomic population inversion. The atomic inversion is defined as the difference between the population of the exited state $|4\rangle$ and the ground state $|1\rangle$ which is written, as follows [2,44]:

$$W(t)=\rho_{44}(t)-\rho_{11}(t). \quad (25)$$

No, we discuss the behavior of the atomic population inversion that corresponds to the intensity-dependent coupling with nonlinearity function ($f(n)=1/\sqrt{n}$). This will be done on the basis of the previous calculations. We examine the influence of the time-dependent coupling parameter, detuning parameter, Kerr-medium and the field damping factor on the behavior of the atomic population inversion. The temporal evolution atomic population inversion is presented in Figs. 2-4. In Fig. 2, we have considered the cases in which the values of the time-dependent coupling parameter $\delta/\lambda=0.01$, Pi and $5Pi$ in the absence (left plots) or presence (right plots) of the field damping factor Γ. Moreover, we have considered $\Delta_1/\lambda=\Delta_2/\lambda=\Delta_3/\lambda=\chi/\lambda=0$. When $\delta/\lambda=0.01$, $\Gamma=0$, the periodic behavior of the atomic population inversion function appears (see Fig. 2(a)). However, when $\Gamma=0.01$, the periodicity that appeared in Fig. 2(a) disappears and the atomic population inversion function equals zero in time evolution process in the considered time interval (see Fig. 2(b)). When $\delta/\lambda=Pi$, $\Gamma=0$, the time interval of the period and the maximum value of oscillations reduces compared to the previous case.

Furthermore the mean value of oscillations is shifted downward (see Fig. 2(c)). The time interval of the period of oscillations increases again and the mean value of oscillations is zero in Fig. 2(e) when $\delta/\lambda=5Pi$, $\Gamma=0$, and the maximum values of oscillations decrease compared to the previous cases. The atomic population inversion function equals zero as the time develops when $\delta/\lambda=5Pi$, $\Gamma=0.01$, but less than the previous cases (see Fig. 2(f) compared to Fig. 2(b) and Fig. 2(d)). To

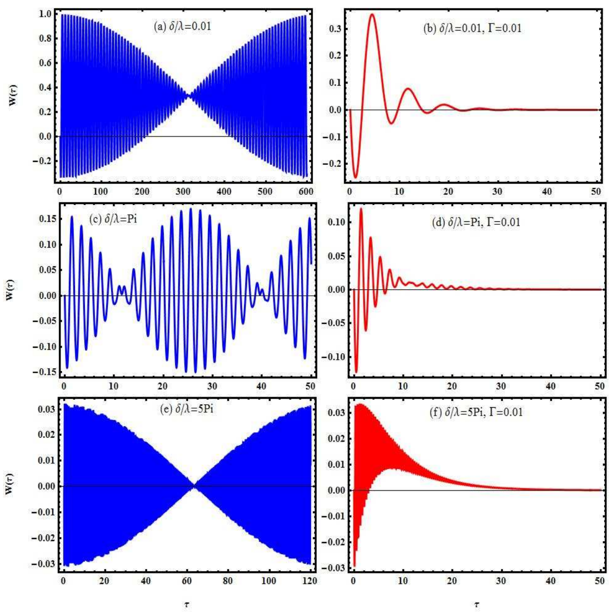

Fig. 2: Evolution of the atomic population inversion $W(t)$ for $f(n) = 1/\sqrt{n}$ of a four-level atom interacting with a single-mode coherent field for the parameters $\overline{n} = 25$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$, $\chi/\lambda = 0$ and for: (a) $\delta/\lambda = 0.01$, $\Gamma = 0$, (b) $\delta/\lambda = 0.01$, $\Gamma = 0.01$ (c) $\delta/\lambda = Pi$, $\Gamma = 0$, (d) $\delta/\lambda = Pi$, $\Gamma = 0.01$, (e) $\delta/\lambda = 5Pi$, $\Gamma = 0$, (f) $\delta/\lambda = 5Pi$, $\Gamma = 0.01$.

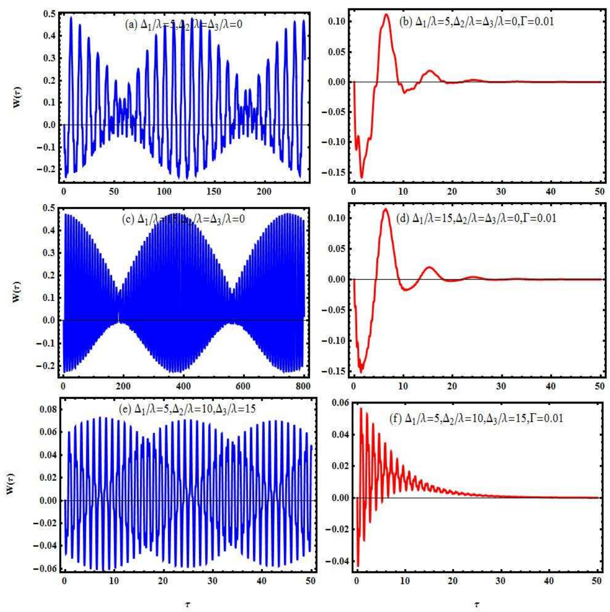

Fig. 3: Dynamics of the atomic population inversion with the same conditions as stated in Fig. 2 but for $\delta/\lambda = \chi/\lambda = 0$ and for: (a, b) $\Delta_1 = 5\lambda$, $\Delta_2 = \Delta_3 = 0$, (c, d) $\Delta_1 = 15\lambda$, $\Delta_2 = \Delta_3 = 0$, (e, f) $\Delta_1 = 5\lambda$, $\Delta_2 = 10\lambda$, $\Delta_3 = 15\lambda$.

analyze the behavior of the atomic population inversion function in the presence of the detuning parameter and in the presence or absence of the field damping factor, we have plotted Fig. 3. In Fig. 3(a), when $\Delta_1/\lambda = 5$, $\Delta_2/\lambda = \Delta_3/\lambda = 0$ and $\Gamma = 0$, the periodicity appears, but when $\Gamma = 0.01$, it disappears as time develops (see Fig. 3(b)). The time interval of the period of oscillations elongates in Fig. 3(c) compared with Fig. 3(a). A new kind of periodicity appears when $\Delta_1/\lambda = 5$, $\Delta_2/\lambda = 10$, $\Delta_3/\lambda = 15$, $\Gamma = 0$. Also, the range of oscillations is less than the previous cases (see Fig. 3(e)). The effect of Kerr-medium in the absence or presence of the field damping factor has plotted in Fig. 4. We observe that the number of oscillations decreases compared with that in Fig. 2 and Fig. 3. Furthermore, the range of oscillations decreases as the parameter χ/λ increases. In Fig. 4(a), when $\chi/\lambda = 0.7$, $\Gamma = 0$, the mean value of oscillations is shifted upward in the time evolution process. The field damping factor leads to

destruction of the behavior of the atomic inversion function (see Fig. 4(b, d, f)).

4 Geometric Phase

Through a cyclic evolution governed by a slow change of parameters, the quantum system acquires a phase factor , i.e. the geometric phase. The most common formulations are known as the Pancharatnam [45], Berry phase [46] and the Aharonov-Bohm phase [47]. The phase prescription is not trivial when the evolution is not cyclic because the initial and final states are different. On subtracting the dynamical phase $\phi_d(t)$ from the Pancharatnam phase [45] (the total phase $\phi_t(t)$) we obtain the geometric phase $\phi_g(t)$. Pancharatnam prescribes the phase between the vectors $|\Psi(0)\rangle$ and $|\Psi(t)\rangle$ as

$$\phi_t(t) = \arg\langle\Psi(0)|\Psi(t)\rangle. \tag{26}$$

In our model, the interaction is time-dependence ,so the geometric phase is just the total phase, i.e.

$$\phi_g(t) = \arg\langle\Psi(0)|\Psi(t)\rangle. \tag{27}$$

Now, we address the evolution of geometric phase $\phi_g(t)$ versus the scaled time λt under the influence of the system parameters. In Figs. 5-7, we have plotted the evolution of geometric phase $\phi_g(t)$ versus the scaled time λt when the field is initially prepared in the coherent state and the atom is prepared in superposition between the levels $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$, for many values of the system parameters δ/λ, Δ_1/λ, Δ_2/λ, Δ_3/λ, χ/λ and Γ.

In Fig. 5, we have examined the behavior of the geometric phase in the presence of the time-dependent coupling parameter ($\delta/\lambda = 0.01$, Pi, $5Pi$) and in the absence (left plots) or presence (right plots) of the field damping factor Γ. According to Fig. 5(a), the oscillations

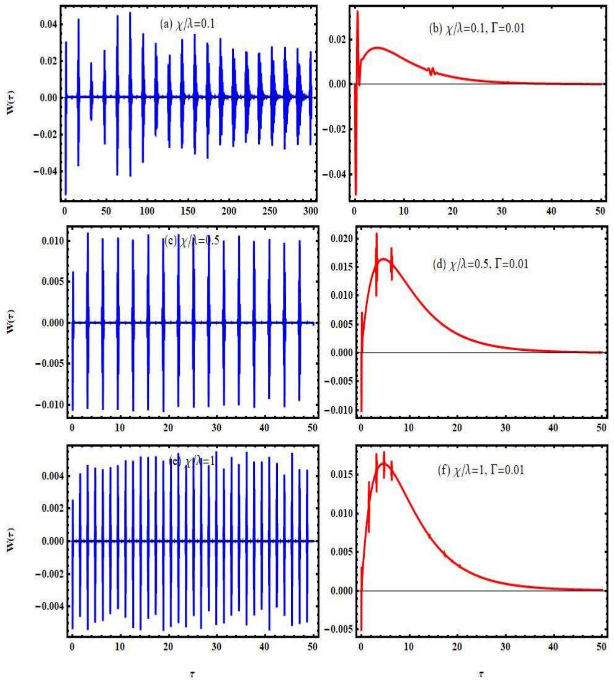

Fig. 4: Dynamics of the atomic population inversion with the same conditions as stated in Fig. 2 but for $\delta/\lambda = 0$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$ and for: (a, b) $\chi/\lambda = 0.1$, (c, d) $\chi/\lambda = 0.5$, (e, f) $\chi/\lambda = 1$.

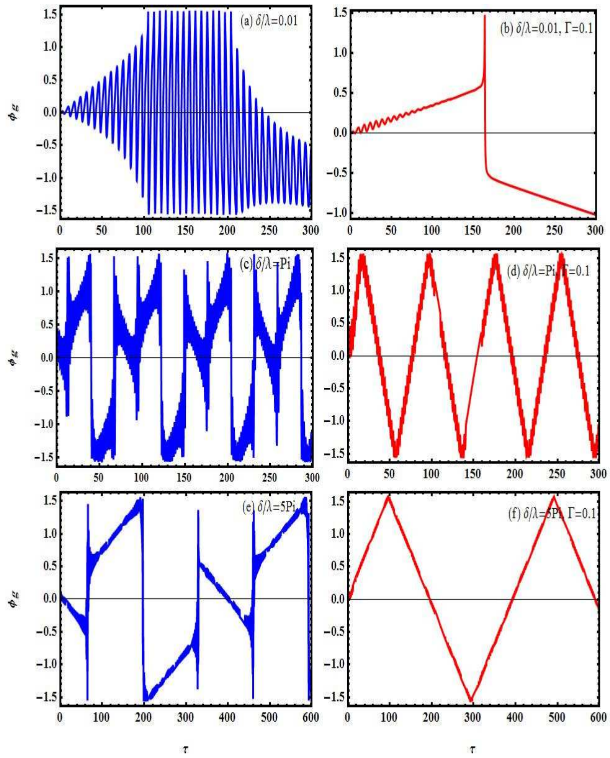

Fig. 5: Dynamics of the geometric phase with the same conditions as stated in Fig. 2 but for $\Gamma = 0.1$ in (b, d, f).

of the geometric phase function started from zero and oscillated approximately between -1.5 and 1.5. The addition of the field damping factor ($\Gamma = 0.1$) decreases the number of oscillations (see Fig. 5(b)). In Fig. 5 (c),when $\delta/\lambda = Pi$, $\Gamma = 0$ and in Fig. 5 (e), when $\delta/\lambda = 5Pi$, $\Gamma = 0$, new types of periodicity appear. It is clear that in Fig. 5 (d) and Fig. 5 (f), the field damping factor ($\Gamma = 0.1$) has changed the behavior of the geometric phase function to another type of periodicity.

On the other hand, to visualize the influence of the detuning parameters, we have plotted Fig. 6. In Fig. 6(a), when $\Delta_1/\lambda = 5$ and all the other parameters are zero, the behavior of the geometric phase function changes, i.e. a new type of periodicity appears. The effect of the field damping factor ($\Gamma = 0.1$), results in changing this type of periodicity to another type. Also, the mean value of oscillations is shifted upward as the time develops in the considered time interval (see Fig. 6(b)). When $\Delta_1/\lambda = 15$, the time interval of the period is elongated compared with the previous case (see Fig. 6(c) and Fig. 6(d)). When $\Delta_1/\lambda = 5$, $\Delta_2/\lambda = 10$, $\Delta_3/\lambda = 15$, and all the other parameters are zero, the periodicity that appears in Fig. 6(a) and Fig. 6(c) disappears. However, it appears again when $\Gamma = 0.1$ (see Fig. 6(e) and Fig. 6(f)). By entering the effect of Kerr-medium, the behavior of the geometric phase function is unpronounced in the time evolution process (see Fig.7(a) and Fig. 7(b)).

5 Husimi Q-Function

The quasi-probability distribution functions such as the Wigner W-function, Glauber-Sudarshan P-representation and Husimi Q-function [48–51] are important tools for detecting quantum states of the systems. In homodyne experiments, these functions can be detected [52]. The quasi-probability distribution functions are c-number functions which can take negative values that allow us to study the non-classical features of the radiation fields.

Therefore, we devote the present section to one of these functions, i.e. the Husimi Q-function which has the nice properties of being always positive and no singularity problems arise at all. Also, it exists for any density matrix because it is simply expressed as the coherent expectation value of the reduced field density matrix. The width of the Q-function gives a measure for the light squeezing. Therefore, it is interesting to study the behavior of the Husimi Q-function which can be given in the following form [50, 53, 54]

$$Q(\beta,t) = \frac{1}{\pi}\langle\beta|\hat{\rho}_F|\beta\rangle = \frac{1}{\pi}\sum_{j=1}^{4}|\psi_j(\beta,t)|^2, \qquad (28)$$

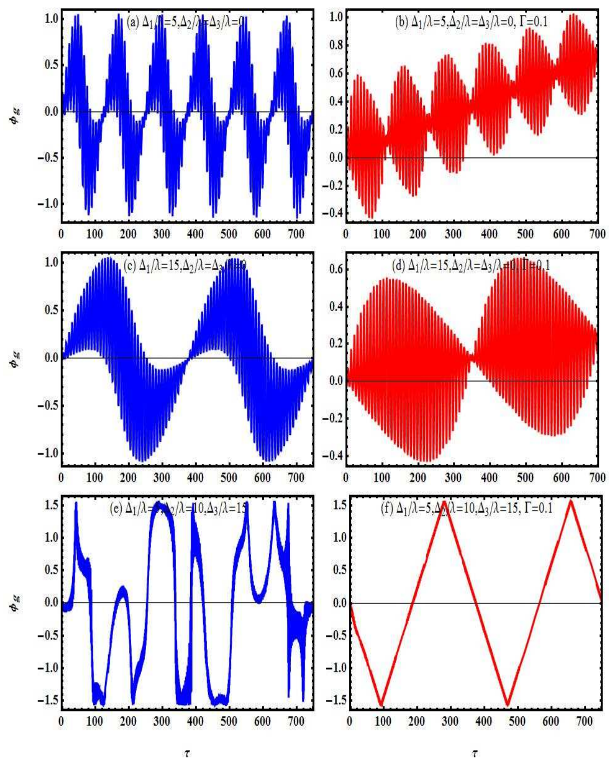

Fig. 6: Dynamics of the geometric phase with the same conditions as stated in Fig. 3 but for $\Gamma = 0.1$ in (b, d, f).

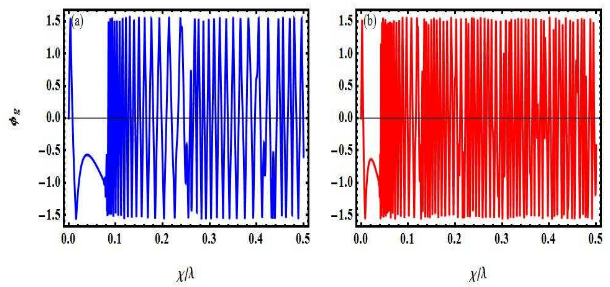

Fig. 7: Dynamics of the geometric phase versus χ/λ in the absence of all other parameters at (a) $\tau = Pi/4$, (b) $\tau = Pi/2$.

where $\hat{\rho}_F$ is the reduced density operator of the cavity field and $|\beta\rangle$ is the coherent state which is defined by

$$|\beta\rangle = e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle, \ \beta = x + iy, \tag{29}$$

and $\psi_j(\beta,t)$ are given by

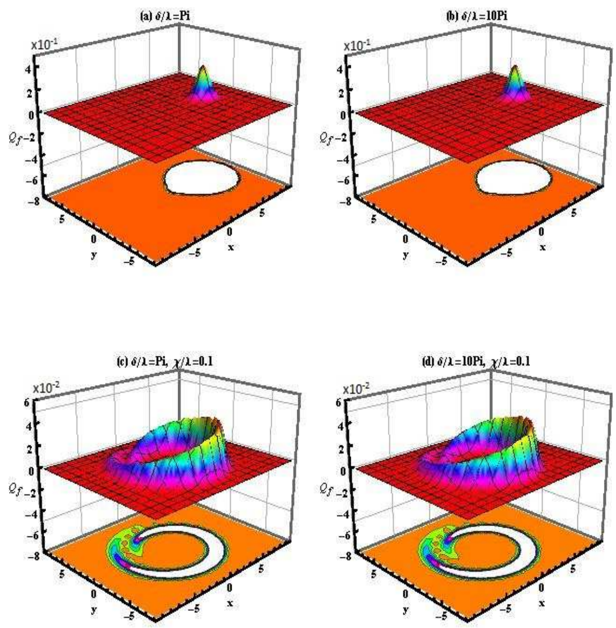

Fig. 8: The 3D sketch (upper) and the contour plot (lower) of Husimi Q-function in the subspace β in the complex β-plane when $\tau = Pi/2$, $\overline{n} = 25$ with $f(n) = 1/\sqrt{n}$ of a four-level atom interacting with a single-mode coherent field for the parameters $\Delta_1 = \Delta_2 = \Delta_3 = 0$, $\Gamma = 0$ and for: (a) $\delta/\lambda = Pi$, $\chi/\lambda = 0$, (b) $\delta/\lambda = 10Pi$, $\chi/\lambda = 0$ (c) $\delta/\lambda = Pi$, $\chi/\lambda = 0.1$, (d) $\delta/\lambda = 10Pi$, $\chi/\lambda = 0.1$.

$$\begin{aligned}
\psi_1(\beta,t) &= e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^{*^n}}{\sqrt{n!}} A_1(n,t), \\
\psi_2(\beta,t) &= e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^{*^n}}{\sqrt{n!}} A_2(n,t), \\
\psi_3(\beta,t) &= e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^{*^n}}{\sqrt{(n-k)!}} A_3(n-k,t), \\
\psi_4(\beta,t) &= e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^{*^n}}{\sqrt{(n-2k)!}} A_4(n-2k,t).
\end{aligned} \tag{30}$$

In Figs. 8-11, we sketch mesh plots (upper) and the corresponding contour plots (lower) of the Husimi Q-function in the subspace β . First, in Fig. 8, we examine the effect of the time-dependent coupling parameter in the absence or presence of Kerr-medium on the Q-function behavior by plotting the mesh and contour plot of the Q-function for a fixed scaled time $\tau = \frac{Pi}{2}$. We notice that the distribution of Q-function is represented by only one peak for the considered values of the time-dependent coupling parameter ($\delta/\lambda = Pi, \ 10Pi$) (see Fig. 8(a) and Fig. 8(b)). The behavior of the

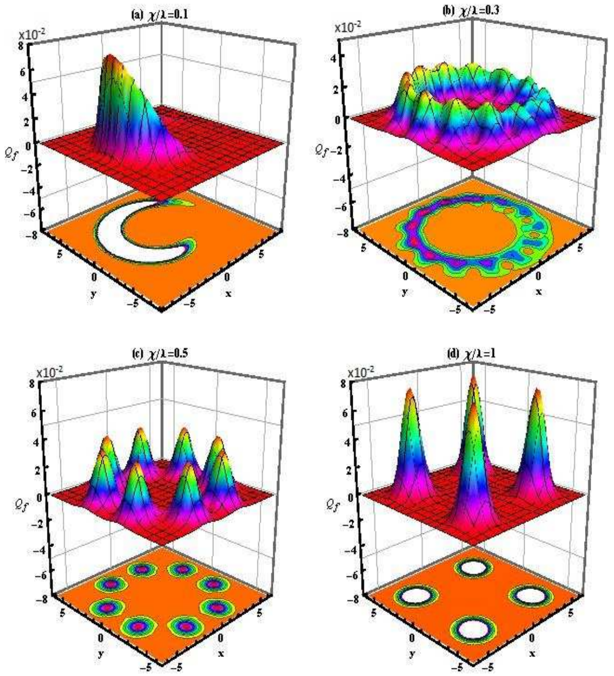

Fig. 9: The 3D sketch (upper) and the contour plot (lower) of Husimi Q-function in the subspace β in the complex β-plane when $\tau = Pi/4$, $\overline{n} = 25$ with $f(n) = 1/\sqrt{n}$ of a four-level atom interacting with a single-mode coherent field for the parameters $\delta/\lambda = 0$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$, $\Gamma = 0$ and for: (a) $\chi/\lambda = 0.1$, (b) $\chi/\lambda = 0.3$ (c) $\chi/\lambda = 0.5$, (d) $\chi/\lambda = 1$.

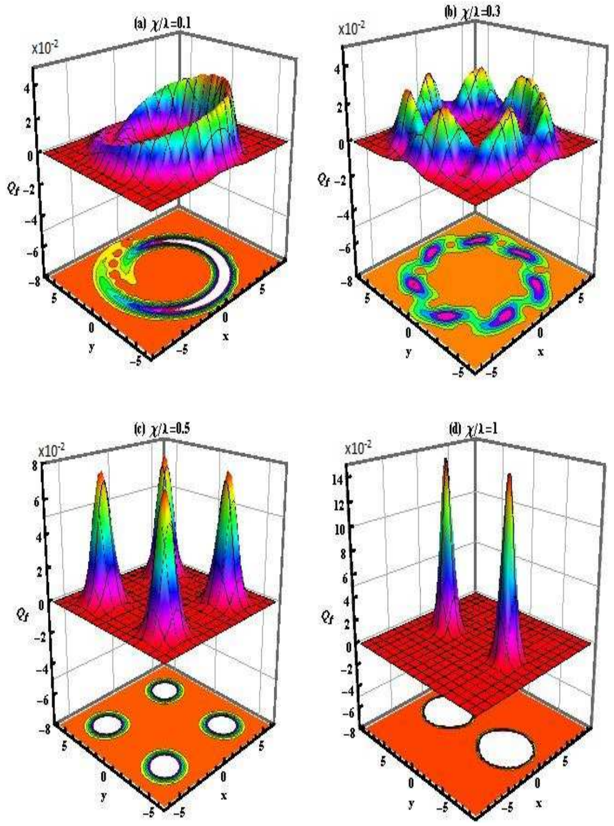

Fig. 10: The same as in Fig. 9 but for $\tau = Pi/2$.

Q-function distribution in Fig. 8(a) and Fig. 8(b) has changed by considering the Kerr-medium parameter ($\chi/\lambda = 0.1$). We can see that the time-dependent coupling parameter has no effect on the Q-function distribution in the presence of Kerr-medium effect (see Fig. 8(c) and Fig. 8(d)).

To investigate the effect of Kerr-medium on the evolution of the Q-function in the absence of all the other parameters at $\tau = \frac{Pi}{4}$, we sketched the numerical results in Fig. 9 for mesh plots and contour plots for different values of the Kerr-medium parameter χ/λ. We have started with $\chi/\lambda = 0.1$. We noted that the shape and height of Husimi Q-function are changed. The height decreased and the contour had a crescent-like shape (see Fig. 9(a)). When $\chi/\lambda = 0.3$, the peak height is compressed and splitted into many contentious small peaks and the contour plot takes a semi-ring shape (see Fig. 9(b)). When $\chi/\lambda = 0.5$, the height of Husimi Q-function increases again. Also, we note that the contour is separated into 8 squeezed circles. Hence, we can say that in Fig. 9(c), the system is in multi Schrődinger cat states. It is noted that by applying a strong Kerr-medium ($\chi/\lambda = 1$) to the system, Q-function is splitted into four fully-separated peaks with a height greater than the previous case. Also, the contour changes as χ/λ increases. In Fig. 10, we have investigated the influence of Kerr-medium on the evolution of the Husimi Q-function in the absence of all the other parameters, but we have fixed the scaled time at $\tau = \frac{Pi}{2}$. In Fig. 11, we have examined the influence of Kerr-medium ($\chi/\lambda = 0.5$) on the evolution of the Husimi Q-function for different values of the field damping factor at the scaled time $\tau = \frac{Pi}{2}$. We observed that the shape and the peaks height of Husimi Q function are changed. Also, the contour shape slightly changes as the parameter Γ increases. By increasing the value of Γ, the peak height decreases and the number of peaks decreases. Consequently, if the system suffers a field decay rate, its stability decreases.

6 Conclusion

We have considered a nonlinear system of four-level λ-type atom interacting with a single-mode quantized field through two-photon transition in a dissipative cavity. The Kerr-medium, detuning parameter, the field damping factor and intensity-dependent coupling were considered. Also, the coupling parameter modulated to be time-dependent. Under an approximation similar to that of the Rotating-Wave Approximation (RWA), the exact

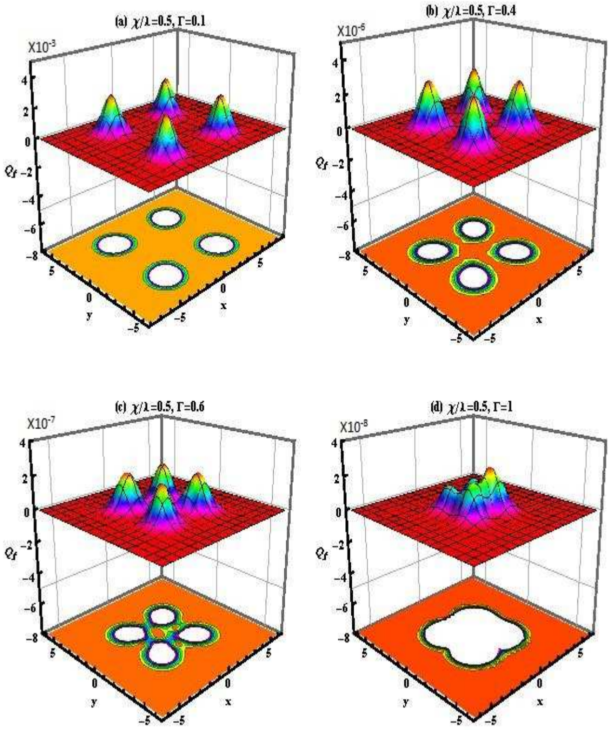

Fig. 11: The 3D sketch (upper) and the contour plot (lower) of Husimi Q-function in the subspace β in the complex β-plane when $\tau = Pi/2$, $\overline{n} = 25$ with $f(n) = 1/\sqrt{n}$ of a four-level atom interacting with a single-mode coherent field for the parameters $\delta/\lambda = 0$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$ and for: (a) $\chi/\lambda = 0.5$, $\Gamma = 0.1$, (b) $\chi/\lambda = 0.5$, $\Gamma = 0.4$, (c) $\chi/\lambda = 0.5$, $\Gamma = 0.6$, (d) $\chi/\lambda = 0.5$, $\Gamma = 1$.

expression of atom-field wave function was obtained. After obtaining the exact analytical form of the state vector of the whole system, the influence of the time-dependent coupling parameter, detuning parameter, Kerr nonlinearity, and the field damping factor (when the nonlinearity function $f(n) = 1/\sqrt{n}$) on the atomic population inversion, the geometric phase and Husimi Q-function were examined. One can study this system when both the field and the atom are initially prepared in other states. Moreover, an exciting extension of this model can be investigated by considering other nonlinearity functions.

Conflict of Interest

The authors declare that they have no conflict of interest.

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