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Geodetic Number Of Circulant Graphs $C_m(\{2,4,...,\lfloor \frac{m}{2} \rfloor-1,\lfloor \frac{m}{2} \rfloor\})$

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Abstract: In this paper, we compute the geodetic set and geodetic number of circulant graphs $C_m(\{2,4,\cdots,\lfloor \frac{m}{2} \rfloor-1,\lfloor \frac{m}{2} \rfloor\})$ where $m=2p$ or $m=p$.

Keywords: Circulant graph, Geodetic set; Geodetic number; Power of graph

1 Introduction

Through out this article, a graph G is a finite simple connected graph without loops and multiple edges. The distance between any two vertices v_1 and v_2 is the length of the shortest path between v_1 and v_2. We represent it by $d(v_1,v_2)$. The shortest path between two vertices (points) v_1 and v_2 is called a v_1-v_2 geodesic. The set of all vertices in G that lie on v_1-v_2 geodesic is denoted by $I_G[v_1,\ v_2]$. For any subset S of G, let $I[S]=\bigcup_{u,v\in S} I_G[v_1,\ v_2]$. If $I_G[v_1,\ v_2]=V(G)$, then S is called a geodetic set of G. The minimum number of a geodetic set of G is called the geodetic number and this number is denoted by $g(G)$. The geodetic number of a graph was intensively investigated in [1,2,3,4]. The maximum of the distance between any two difference vertices x and y in G is called diameter and is denoted by $diam(G)$. Two vertices $v_1,\ v_2$ of a graph G are called antipodal in G if $d(v_1,v_2)=diam(G)$.

Moreover, the geodetic, hull, and Steiner numbers of powers of paths were investigated by AbuGhneim et al. [5]. Moreover, for connected graphs, the closed intervals $I_G[u,v]$ were studied extensively by Nebesky [6]. In addition, power graphs also have applications in quantum random walks in physics and routing in networks and so have generated interest in past and current paper [7,8,9].

Let $1\leq a_1<a_2<\cdots<a_m\leq\lfloor \frac{n}{2} \rfloor$, where m,n,a_i are integers, $1\leq i\leq m$, and $n\geq 3$. Set $S=\{a_1,a_2,\cdots,a_m\}$. A graph G with the vertex set $\{1,2,\cdots,n\}$ and with the edge set $\{\{i,j\}:|i-j|\equiv a_t(mod\ n)$ for some $1\leq t\leq m\}$ is called a circulant graph with respect to set S (or with connection set S), and denotes by $C_n(S)$ or $C_n(a_1,a_2,\cdots,a_m)$. For example, every power of cycle is Circulant graph, and Circulant graphs play a very important and crucial role in Networks design [10,11,12] and have an applications to telecommunication network, VLSI design as well as parallel and distributed computing.

The present paper aims to compute the geodetic number of the circulant graphs $C_m(\{2,4,\cdots,\lfloor \frac{m}{2} \rfloor-1,\lfloor \frac{m}{2} \rfloor\})$.

2 Preliminary Lemmas

Let $C_m(\{2,4,...,\lfloor \frac{m}{2} \rfloor-1,\lfloor \frac{m}{2} \rfloor\})$ be the circulant graphs with $m=p$ or $m=2p$.
In this section, we present the necessary lemmas to compute the geodetic number of the circulant graphs $C_m(\{2,4,...,\lfloor \frac{m}{2} \rfloor-1,\lfloor \frac{m}{2} \rfloor\})$. We will also present some crucial lemmas.

To illustrate the idea of the following lemma, a vertex v in a graph G is called a extreme point if the subgraph induced by its neighbors is complete. If S is a geodetic, then S contains the set of extreme points. Now, we give some lemmas of properties $C_m(\{2,4,...,\lfloor \frac{m}{2} \rfloor-1,\lfloor \frac{m}{2} \rfloor\})$.

Lemma 1.*The circulant graphs $C_m(\{2,4,...,\lfloor \frac{m}{2} \rfloor-1,\lfloor \frac{m}{2} \rfloor\})$ have no extreme point.*

*Proof.*Let $L_1=\{v_0,\ v_1,\ ...,\ v_{\frac{p-1}{2}}\}$ and $L_2=\{v_{\frac{p-1}{2}+1},\ v_{\frac{p-1}{2}+2},,,,\ v_{p-1}\}$. We want to prove that

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the circulant graph $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$ has no extreme point. We can use the following cases:

–**Case 1**: If $m=2p$ where p is a prime number, then let the three vertices v_a, $v_{b=a+2}$ and $v_{c=p+a}$ exist in the circulant graph $C_{2p}(\{2,4,...,p-3,p-1,p\})$.
Now, v_a is adjacent to v_b and v_c where the two vertices v_b and v_c are not adjacent in $C_{2p}(\{2,4,...,p-3,p-1,p\})$, since $c-b=p-2\notin\{2,4,...,p-3,p-1,p\}$.
–**Case 2**: If $m=p$ where p is a prime number, then we have two subcases:
–**Subcase 2.1**: If $\frac{p-1}{2}$ is even, then there exist three vertices v_a, $v_{b=a+2}$ and $v_{c=\frac{p-1}{2}-1+a}$ in the circulant graphs $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$.
Now, the vertex v_a is adjacent to v_b and v_c where the two vertices v_b and v_c are not adjacent in $C_m(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$, since $c-b=\frac{p-1}{2}-3\notin\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}$.
–**Subcase 2.2**: If $\frac{p-1}{2}$ is odd, then there exist three vertices v_a, $v_{b=a+2}$ and $v_{c=\frac{p-1}{2}+a}$ in the circulant graphs $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$.
Now,the vertex v_a is adjacent to v_b and v_c where the two vertices v_b and v_c are not adjacent in $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$, since $c-b=\frac{p-1}{2}-2\notin\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}$.

Thus, v_a is not an extreme point in the circulant graph $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$ for any vertex v_a.

The following Lemma is well known, see [13].

Lemma 2. *The circulant graphs $C_n(\{S\})$, where $S=\{a_1,\cdots,a_k\}$, is connected if and only if $gcd(a_1,\ ...,\ a_k)=1$.*

We have the following lemma to determine the diameter of the circulsnt grsph $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$.

Lemma 3. *The diameter of the circulant graphs $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$ is 2.*

Proof. We have the following two cases:

–**Case 1**: If $m=2p$ where p is a prime number, then we have $S_1=\{v_1,v_3,...,v_p,v_{p+2},...,v_{2p-1}\}$ and $S_2=\{v_0,v_2,...,v_{p-1},v_{p+1},...,v_{2p-2}\}$ which are a complete subgraph. Each vertex in S_1 is adjacent to only one vertex in S_2 such that $\{v_i : v_i\in S_1\}$ is adjacent to $\{v_{i+p} : v_{i+p}\in S_2\}$, then $diam(C_{2p}(\{2,4,...,p-3,p-1,p\})=2$.

–**Case 2**: If $m=p$ where p is a prime number, then we have two subcases
–**Subcase 2.1**: If $\frac{p-1}{2}$ is even, then we have $S_1=\{v_0,v_2,...,v_{\frac{p-1}{2}-2},v_{\frac{p-1}{2}}\}$, $S_2=\{v_1,v_3,...,v_{\frac{p-1}{2}-3},v_{\frac{p-1}{2}-1}\}$, $S_3=\{v_{\frac{p-1}{2}+1},v_{\frac{p-1}{2}+3},...,v_{p-4},v_{p-2}\}$ and $S_4=\{v_{\frac{p-1}{2}+2},v_{\frac{p-1}{2}+4},...,v_{p-3},v_{p-1}\}$. These sets have the distances between their vertices as follows:

–**Sub-subcase 2.1.1**: If two distinct vertices (points) in S_1 are adjacent, S_1 is a complete subgraph.
–**Sub-subcase 2.1.2**: If two distinct vertices (points) in S_2 are adjacent, S_2 is a complete subgraph.
–**Sub-subcase 2.1.3**: If two distinct vertices (points) in S_3 are adjacent, S_3 is a complete subgraph.
–**Sub-subcase 2.1.4**: If two distinct vertices (points) in S_4 are adjacent, S_4 is a complete subgraph.
–**Sub-subcase 2.1.5**: Let v_i be a vertex in S_1 and v_j be a vertex in S_2. Then
If $i>j$, then v_i is adjacent to the vertex $v_{\frac{p-1}{2}+i}$ and $v_{\frac{p-1}{2}+i}$ is adjacent to v_j so $d(v_i,\ v_j)\leq 2$.
If $i<j$, then v_i is adjacent to the vertex $v_{\frac{p-1}{2}+j}$ and $v_{\frac{p-1}{2}+j}$ is adjacent to v_j so $d(v_i,\ v_j)\leq 2$.
–**Sub-subcase 2.1.6**: If vertex $v_i\in S_1$ is adjacent to the vertex $v_{\frac{p-1}{2}+i+1}$ and $v_{\frac{p-1}{2}}$ is adjacent to each vertex $v_j\in S_3$, $d(v_i,\ v_j)\leq 2$.
–**Sub-subcase 2.1.7**: If $v_i\in S_1$ is adjacent to the vertex $v_{\frac{p-1}{2}}$ and $v_{\frac{p-1}{2}}$ is adjacent to each vertex $v_j\in S_4$, $d(v_i,\ v_j)\leq 2$.
–**Sub-subcase 2.1.8**: If $v_i\in S_2$ is adjacent to the vertex $v_{\frac{p-1}{2}-1}$ and $v_{\frac{p-1}{2}-1}$ is adjacent to each vertex $v_j\in S_3$, $d(v_i,\ v_j)\leq 2$.
–**Sub-subcase 2.1.9**: If $v_i\in S_2$ is adjacent to the vertex v_1 and v_1 is adjacent to each vertex $v_j\in S_4$, $d(v_i,\ v_j)\leq 2$.
–**Sub-subcase 2.1.10**: For any two distinct vertices in S_3 and S_4, let $i=\{1,3,...,\frac{p-1}{2}-3,\frac{p-1}{2}-1\}$, $j=\{2,4,...,\frac{p-1}{2}-2,\frac{p-1}{2}\}$. Then for any vertex $v_{\frac{p-1}{2}+i}\in S_3$ is adjacent to the vertex v_i and the vertex v_i is adjacent to each vertex $v_{j+\frac{p-1}{2}}\in S_4$ for each $i<j$. On other hand for each $i>j$ every element $v_{\frac{p-1}{2}+i}\in S_3$ is adjacent to the vertex v_j and v_j is adjacent to each vertex $v_j\in S_4$, so $d(v_i,\ v_j)\leq 2$.
–**Subcase 2.2**: If $\frac{p-1}{2}$ is odd, then we have $S_1=\{v_0,v_2,...,v_{\frac{p-1}{2}-3},v_{\frac{p-1}{2}-1}\}$, $S_2=\{v_1,v_3,...,v_{\frac{p-1}{2}-2},v_{\frac{p-1}{2}}\}$, $S_3=\{v_{\frac{p-1}{2}+1},v_{\frac{p-1}{2}+3},...,v_{p-3},v_{p-1}\}$ and $S_4=\{v_{\frac{p-1}{2}+2},v_{\frac{p-1}{2}+4},...,v_{p-4},\ v_{p-2}\}$. Now we have the following sub-subcases.

–**Sub-subcase 2.2.1**: If two distinct vertices in S_1 are adjacent, S_1 is a complete subgraph.
The rest Sub-subcases follow analogously.

Now, before we start to compute the geodetic number of the circulant graphs $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$, consider this illustrative example to compute the geodetic number of the circulant graphs $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$, see

Figure 1.

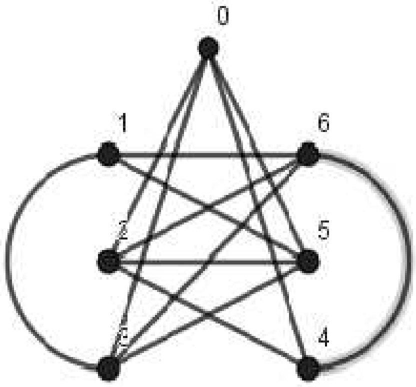

Fig. 1: $C_7(2,3)$.

3 The geodetic number of the circulant graphs $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$

In this section we determine the geodetic number of the circulant graphs $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$.
We also assume the vertex set of $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$ is $\{v_0,v_1,..., v_{2p-1}\}$ if $m=2p$, and the vertex set is $\{v_0,v_1,...,v_{p-1}\}$ if $m=p$.

Lemma 4. *If $m=2p$ where p is a prime number, then $g(C_{2p}(\{2,4,...,p-1,p\}))=p$.*

Proof. If $m=2p$, then let $S_1=\{v_1,v_3,...,v_p,v_{p+2},...,v_{2p-1}\}$ is a complete subgraph.
Let $S_2=\{v_0,v_2,...,v_{p-1},v_{p+1},...,v_{2p-2}\}$ be a complete subgraph and each vertex in S_1 is adjacent to only one vertex in S_2 such that $\{v_i : v_i\in S_1\}$ is adjacent to $\{v_{i+p} : v_{i+p}\in S_2\}$, then consider $S=\{v_3,v_5,...,v_p,v_{p+2},...,v_{2p-1}, v_{p+1}\}$ geodesics cover all the vertices of $C_{2p}(\{2,4,...,p-3,p-1,p\})$.

Now, we discuss the cases for the geodetic number when $m=p$.

Lemma 5. *If $m=p$ where p is a prime number, then $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=3$ if and only if $p=5$ or $p=7$.*

Proof. It is clear that when $p=5$, $C_5(2)\cong C_5$, $g(C_5(2)))=3$. When $p=7$ with $S=\{v_0,\ v_1,\ v_6\}$, $g(C_7(\{2,\ 3\})))=3$.
On the other hand if $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=3$. Then we have the following cases:

– **Case 1**: If $\frac{p-1}{2}$ is even, then consider $S=\{v_1,\ v_a,\ v_b\}$ where v_1 is not adjacent to v_a and v_b so $v_a,\ v_b\in\{S_1,\ S_3\}\setminus\{v_{\frac{p-1}{2}},\ v_{\frac{p-1}{2}+1}\}$.
Now, without loss of generality let $b\in S_1$ and $a\in S_3$. Thus, $v_1-S_2-v_a$, $a\in S_3\setminus\{v_{\frac{p-1}{2}+1}\}$, then $v_a=v_{\frac{p-1}{2}+5}$. Otherwise, not all vertices in S_2 are adjacent to v_a. Also, $v_1-S_3-v_b$, $b\in S_1\setminus\{v_{\frac{p-1}{2}}\}$, then $b=v_{\frac{p-1}{2}-2}$. Otherwise, not all vertices in S_3 are adjacent to v_b. This leads to the vertex $v_0=v_{\frac{p-1}{2}-2}$ i.e $p=5$. Otherwise, v_0 does not lie on any $v_1-v_{\frac{p-1}{2}+5}$, $v_1-v_{\frac{p-1}{2}-2}$ geodetic.

– **Case 2**: If $\frac{p-1}{2}$ is odd, then consider $S=\{v_0,\ v_a,\ v_b\}$ with v_0 is not adjacent to v_a and v_b where $v_a,\ v_b\in\{S_2,\ S_3\}\setminus\{v_{\frac{p-1}{2}+1},\ v_{\frac{p-1}{2}}\}$.
Now, without loss of generality let $v_b\in S_2$ and $v_a\in S_3$. So, $v_0-S_4-v_b$, $v_b\in S_2\setminus\{v_{\frac{p-1}{2}}\}$, then $v_b=v_{\frac{p-1}{2}-2}$. Otherwise, not all vertices in S_4 are adjacent to v_b. Also, $v_0-S_1-v_a$, $v_a\in S_3\setminus\{v_{\frac{p-1}{2}+1}\}$, then $v_a=v_{\frac{p-1}{2}+3}$. Otherwise, not all vertex in S_3 adjacent to v_a. This leads to the vertex $v_1=v_{\frac{p-1}{2}-2}$ i.e $p=7$. Otherwise, v_1 does not lie on any $v_0-v_{\frac{p-1}{2}-2}$, $v_0-v_{\frac{p-1}{2}+3}$ geodetic.

Observation 1.
Based on the proof of Lemma 5, we conclude that if $n=p$ where $p\geq 5$ is a prime number, then we have the following cases:

– **Case 1**: If $\frac{p-1}{2}$ is even, then every geodetic set of $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$ contains the set of $\{v_1,v_{\frac{p-1}{2}+5},\ v_{\frac{p-1}{2}-2}\}$.

– **Case 2**: If $\frac{p-1}{2}$ is odd, then every geodetic set of $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$ contains the set of $\{v_0,v_{\frac{p-1}{2}+3},\ v_{\frac{p-1}{2}-2}\}$.

The following lemma states that for any prime number $p\geq 11$, the geodetic number of $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$ is greater than or equal to 5.

Lemma 6. *If $m=p$ where $p\geq 11$ is a prime number, then $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))\geq 5$.*

Proof. Using similar argument to the proof of Lemma 5 , we get $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))\geq 5$. We have two cases to consider.

– **Case 1**: If $\frac{p-1}{2}$ is even, then the set $\{v_1,v_{\frac{p-1}{2}+5},v_{\frac{p-1}{2}-2}\}$ is a subset of any geodetic set of $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$. Let

$S=\{v_1,v_{\frac{p-1}{2}+5},v_{\frac{p-1}{2}-2},v_e\}$. We want to show that S cannot be a geodetic set of $C_p(\{2,4,..., \frac{p-1}{2}-1, \frac{p-1}{2}\})$. Now, v_e is different from the vertices $\{v_0,v_{\frac{p-1}{2}+3},v_{\frac{p-1}{2}-2}\}$. Since $diam(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=2$, v_e does not lie on any $v_1-v_{\frac{p-1}{2}+5}, v_1-v_{\frac{p-1}{2}-2}$ geodetic, i.e. $v_e\notin\{S_2,S_4\}$ by the symmetric of $C_P(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$ we can let $v_e=v_0$. This means $I_{C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})}[v_1,v_{\frac{p-1}{2}+5}]=\{v_1,S_2,v_{\frac{p-1}{2}+1},v_{\frac{p-1}{2}+3},v_{\frac{p-1}{2}+5}\}$, $I_{C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})}[v_1, v_{\frac{p-1}{2}-2}]=\{v_1,S_4,v_{\frac{p-1}{2}},v_{\frac{p-1}{2}+2},v_{\frac{p-1}{2}-2}\}$ and $I_{C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})}[v_0,v_1]=\{v_0, v_{\frac{p-1}{2}},v_{\frac{p-1}{2}-1},v_{\frac{p-1}{2}+1},v_1\}$. Therefore, S is not geodetic set of $C_p(\{2,4,..., \frac{p-1}{2}-1, \frac{p-1}{2}\})$.

–Case 2: If $\frac{p-1}{2}$ is odd, then the set $\{v_0,v_{\frac{p-1}{2}+3},v_{\frac{p-1}{2}-2}\}$ is a subset of any geodetic set of $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$. Let $S=\{v_0,v_{\frac{p-1}{2}+3}, v_{\frac{p-1}{2}-2}, v_e\}$. We want to show that S cannot be a geodetic set of $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$. Now, v_e is different from the vertices $\{v_0,v_{\frac{p-1}{2}+3}, v_{\frac{p-1}{2}-2}\}$. Since $diam(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=2$, v_e does not lie on any $v_0-v_{\frac{p-1}{2}+3}, v_0-v_{\frac{p-1}{2}-2}$ geodetic, i.e. $v_e\notin\{S_1, S_4\}$ by the symmetric of $C_P(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$. We can let $v_e=v_1$. This means $I_{C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})}[v_0,v_{\frac{p-1}{2}+3}]=\{v_0, S_1, v_{\frac{p-1}{2}+1},v_{\frac{p-1}{2}+3}\}$, $I_{C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})}[v_0,v_{\frac{p-1}{2}-2}]=\{v_0,S_4,v_{\frac{p-1}{2}},v_{\frac{p-1}{2}-2}\}$ and $I_{C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})}[v_0,v_1]=\{v_0,v_{\frac{p-1}{2}+1},v_{\frac{p-1}{2}},v_{\frac{p-1}{2}+2},v_1\}$. Therefore, S is not geodetic set of $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$.

Now, we turn our discussion to study the cases for when the geodetic number is 5 of the circulant graphs $C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$.

Lemma 7. *If $m=p$ where p is a prime number, then $g(C_m(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=5$ if and only if $p=11$ or $p=13$.*

Proof. It is clear that when $p=11$, consider $S=\{v_0,v_1,v_2,v_3,v_9\}$ is a geodetic set and $g(C_{11}(\{2,4, 5\})))=5$. Also when $p=13$ consider $S=\{v_0,v_1,v_{11},v_4,v_{10}\}$ is a geodetic set and $g(C_{11}(\{2,4,5\})))=5$.
On the other hand if $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=5$. Then we have two cases:

–Case 1: If $\frac{p-1}{2}$ is even, then consider $S=\{v_1,v_{\frac{p-1}{2}+5},v_{\frac{p-1}{2}-2},v_0,v_e\}$ with v_0 is not adjacent to v_e where $v_e\in\{S_2,S_4\}\setminus\{v_{\frac{p-1}{2}-1}, v_{\frac{p-1}{2}+2}\}$.
Now, if the vertex $v_e\in S_4$. So, $v_0-S_1-v_e$, then $v_e=v_{\frac{p-1}{2}+4}$. Otherwise, not all vertices in S_1 are adjacent to v_e. This leads to the vertex $v_0=v_{\frac{p-1}{2}+7}$, i.e. $p=13$. Otherwise, v_0 does not lie on any $v_1-v_{\frac{p-1}{2}+5}$, $v_1-v_{\frac{p-1}{2}-2}$ geodetic.
If the vertex $v_e\in S_2$, $v_0-S_3-v_e$, then $v_e=v_{\frac{p-1}{2}-3}$. Otherwise, not all vertices in S_3 are adjacent to v_e. Thus to the vertex v_2 does not lie on any geodetic set or $v_2=v_{\frac{p-1}{2}-2}$, i.e. $p=9$ which is a contradiction.

–Case 2: If $\frac{p-1}{2}$ is odd, then consider $S=\{v_0,v_{\frac{p-1}{2}-2},v_{\frac{p-1}{2}+3},v_1,v_e\}$ with v_1 is not adjacent to v_e where $v_e\in\{S_1,S_4\}\setminus\{v_{\frac{p-1}{2}-1}, v_{\frac{p-1}{2}+2}\}$.
Now, if the vertex $v_e\in S_4$. So, $v_1-S_2-v_e$, then $v_e=v_{\frac{p-1}{2}+4}$. Otherwise, not all vertices in S_2 are adjacent to v_e. Thus to the vertex $v_0=v_{\frac{p-1}{2}+7}$, i.e. $p=13$. Otherwise, v_0 does not lie on any geodetic set. If the vertex $v_e\in S_1$. So, $v_1-S_3-v_e$, then $v_e=v_{\frac{p-1}{2}-1}$. Otherwise, not all vertices in S_3 adjacent to v_e. This leads to the vertex $v_3=v_{\frac{p-1}{2}-2}$ i.e $p=13$, which is a contradiction.

Finally, we discuss the case when the geodetic number is 6.

Lemma 8. *For the circulant graphs $C_m(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\})$. If $m=p$ where p is a prime number greater than 13, then $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=6$.*

Proof. By using previous lemma, we get $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))\geq 6$ when $p>13$.
Now, we consider

–Case 1: If $\frac{p-1}{2}$ is even, then consider $S=\{v_1,v_{\frac{p-1}{2}-2},v_{\frac{p-1}{2}+5},v_0, v_{\frac{p-1}{2}-3}, v_{\frac{p-1}{2}+4}\}$. Using Lemma 7, the $v_0-v_{\frac{p-1}{2}-3}$ geodesics cover all the vertices S_3, $v_0-v_{\frac{p-1}{2}+4}$ geodesics cover all the vertices S_1, $v_1-v_{\frac{p-1}{2}-2}$ geodesics cover all the vertices S_4, and $v_1-v_{\frac{p-1}{2}+5}$ geodesics cover all the vertices S_2. Hence S is a geodetic set and $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=6$.

–Case 2: If $\frac{p-1}{2}$ is odd, then consider $S=\{v_1,v_{\frac{p-1}{2}-1},v_{\frac{p-1}{2}+4},v_0,v_{\frac{p-1}{2}-2}, v_{\frac{p-1}{2}+3}\}$. Using Lemma 7, the $v_0-v_{\frac{p-1}{2}-2}$ geodesics cover all the vertices S_4, $v_0 - v_{\frac{p-1}{2}+3}$ geodesics cover all the vertices S_1, $v_1 - v_{\frac{p-1}{2}-1}$ geodesics cover all the vertices S_3, and $v_1 - v_{\frac{p-1}{2}+4}$ geodesics cover all the vertices S_2. Hence S is a geodetic set and $g(C_p(\{2,4,...,\frac{p-1}{2}-1,\frac{p-1}{2}\}))=6$.

4 Conclusion

In this paper, we determined the geodetic number of circulant graphs $C_m(\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$. We sum up our calculations in the following theorem.

Theorem 1. *If $m=2p$ or $m=p$ where p is a prime number, then*

$$g(C_m(D)) = \left\{ \begin{array}{l} p \ , \ if \ m=2p, \\ 3 \ , \ if \ m=5 \ or \ m=7, \\ 5 \ , \ if \ m=11 \ or \ m=13, \\ 6 \ , \ if \ m>13. \end{array} \right\}$$

where $D=\{2,4,...,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\}$.

5 Perspective

In this paper, we find the geodetic set and geodetic number of circulant graphs $C_m(\{2,4,\cdots,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$ where $m=2p$ or $m=p$..

One can ask the following questions:

–**(1)** What is the geodetic number of complement of the circulant graphs $C_m(\{2,4,\cdots,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$?

–**(2)** What is the geodetic number of the circulant graphs $C_m(\{2,4,\cdots,\lfloor\frac{m}{2}\rfloor-1,\lfloor\frac{m}{2}\rfloor\})$ when $m=2p+1$?

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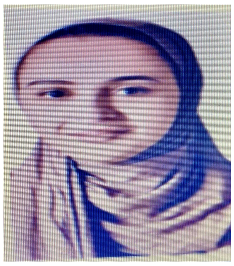

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