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The Critical Exponent to Cauchy Problem for a Time Fractional Semi-Linear Equation with a Structural Damping

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Abstract: In this paper, we study the semilinear equation with a time fractional structural damping

$$\mathbf{D}^{\beta}_{0|t}u(t,x) - 2\Delta\mathbf{D}^{\alpha}_{0|t}u(t,x) + \Delta^2 u(t,x) = |u(t,x)|^p \qquad t>0,\ x\in\Omega,$$

where $p>1$, $\frac{1}{2}<\alpha<1<\beta<2$ and $\mathbf{D}^{\alpha}_{0|t}$ is the Caputo fractional derivative. We obtain the blow- up result under some positive data when $1<p<1+\frac{2\alpha}{N-2\alpha+1}$. Whereas, if $p\geqslant 1+\frac{2\alpha}{N-2\alpha+1}$ and $\|u_0\|_{L^{2q_c}(\Omega)}$, $q_c = N(p-1)/4$ is sufficiently small, we prove the existence of global solution.

Keywords: Cauchy problem, Critical exponent, Structural damping, Time fractional derivatives.

1 Introduction

We consider the following Cauchy problem

$$\begin{cases} \mathbf{D}^{\beta}_{0|t}u - 2\Delta\mathbf{D}^{\alpha}_{0|t} + \Delta^2 u = |u|^p & (t,x)\in(0,\infty)\times\Omega, \\ \Delta u(t,x) = u(t,x) = 0 & (t,x)\in(0,\infty)\times\partial\Omega, \\ u(0,x)=u_0, u_t(t,x)|_{t=0}=0 & x\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain $\Omega\subset\mathbb{R}^N$ $(1\leqslant N\leqslant 4)$, $p>1$, $\frac{1}{2}<\alpha<1$, $1<\beta<2$ and Δ denotes the Laplacian operator with respect to the x variable. The operator $\mathbf{D}^{\alpha}_{0|t}u = I^{1-\alpha}_{0|t}u_t$, $I^{1-\alpha}_{0|t}$ is the Riemann-Liouville fractional integral of order $1-\alpha$ which is defined for $u\in C(0,t)$, as follows

$$I^{1-\alpha}_{0|t}u = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{-\alpha}u(\tau)d\tau.$$

The term $\Delta\mathbf{D}^{\alpha}_{0|t}u$ represents a generalized structural damping. The equation (1) is a generalization of well known damped elastic system[1]. The integer derivatives are replaced by a fractional derivatives in the sense of Caputo.

Our target is to find the critical exponents p_c which solutions cannot exist for all time in the subcritical case. Whereas, in the critical and supercritical cases, global small data solutions exist. Moreover, we see how much of the generalized structural damping will be on the blow-up phenomenon. As $\alpha\to 1$, the critical exponents p_c tend to $1+\frac{2}{N-1}$ showed by D'Abbicco [2]. The discussion is based on the semi-group theory, fixed point theorem and the test function method.
Let us first recall some works related to the problem we address.

For the semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = |u(t)|^p & (t,x)\in(0,\infty)\times\Omega, \\ u(0,x)=u_0, \quad u_t(t,x)|_{t=0}=u_1(x) & x\in\Omega. \end{cases} \tag{2}$$

Todorova and Yordanov [3] investigated the global existence of mild solutions to (2). In addition, they proved that the mild solution cannot exist globally when $1<p\leqslant 1+\frac{2}{N}$ and $\int u_i>0, i=0,1$. In fact, these results are coincided with Fujita critical for $u_t-\Delta u=|u|^p$.

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Luong and Tung [4] considered the following Cauchy problem

$$\begin{cases} u_{tt} + \rho A u_t + A^2 u(t) = f(t,u(t)) \\ u(0,x) = u_0 + g(u), \qquad u_t(t,x)|_{t=0} + h(u) = u_1, \end{cases} \tag{3}$$

where A is closed operator. They established the existence of decay mild solutions to (3) using a suitable measure of non-compactness on the space of continuous functions on the half-line.

In [5], Messaoudi considered the nonlinearly damped semilinear Petrovsky equation

$$\begin{cases} u_{tt} + \Delta^2 u_t + a|u|^{q-1}u_t = |u|^p \qquad (t,x) \in (0,\infty)\times\Omega, \\ u(0,x) = u_0, \qquad u_t(t,x)|_{t=0} = u_1. \end{cases} \tag{4}$$

He proved that the solution is global if $p \geqslant m$, while if $p > m$ and the energy is negative, then every solution of problem (4) blows-up in a finite time.

Erhan et al. [6] addressed a more general case and treated the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u_t + \mathbf{D}^{1+\alpha}_{0|t} u = |u|^p \qquad (t,x) \in (0,\infty)\times\Omega, \\ u(0,x) = u_0, \qquad u_t(t,x)|_{t=0} = u_1. \end{cases} \tag{5}$$

where $-1 < \alpha < 0$. They also discussed the nonexistence of global solutions with negative initial energy.

The rest of the paper is organized, as follows: In section 2, we recall some definitions of fractional order calculus. The study of the existence and uniqueness of local mild solution of problem (1) is presented in section 3. In sections 4 and 5, we prove the blow-up and global existence of solutions to (1).

2 Preliminary

In this section, we present some results and basic properties of fractional calculus. For more details, we refer to [7,8].

Let $0 < \alpha < 1$, $a,b \in \mathbb{R}$ and $f \in L^1(a,b)$. The Riemann-Liouville integrals of order α are defined as

$$I^{\alpha}_{a|t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > a,$$

and

$$I^{\alpha}_{t|b} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau, \quad t < b.$$

For $0 < \alpha < 1$ and $T > 0$. If $I^{1-\alpha}_{a|t} f(t)$ and $I^{1-\alpha}_{t|b} f(t) \in AC[a,b]$, then the Riemann-Liouville derivatives of order α are defined as

$$D^{\alpha}_{a|t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau)\, d\tau, \quad t > a,$$

and

$$D^{\alpha}_{t|b} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha} f(\tau)\, d\tau, \quad t < b.$$

For $0 < \alpha < 1$ and $f \in AC[a,b]$. The Caputo derivatives of fractional order α are defined as

$$\mathbf{D}^{\alpha}_{a|t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau)\, d\tau, \quad t > a, \tag{6}$$

and

$$\mathbf{D}^{\alpha}_{t|b} f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} f'(\tau)\, d\tau, \quad t < b. \tag{7}$$

Assume $\mathbf{D}^{\alpha}_{0|t} f \in L^1(a,b)$, $g \in C^1(a,b)$ and $g(T) = 0$. Then we have the following formula of integration by parts

$$\int_a^b g(t)\mathbf{D}^{\alpha}_{0|t} f(t) dt = \int_a^b (f(t) - f(0))\mathbf{D}^{\alpha}_{t|T} g(t) dt. \tag{8}$$

Proposition 1 **([9])***Let* $1 < \alpha + \beta < 2$. *If* $f_t(a) = 0$, *then*

$$\boldsymbol{D}^{\alpha}_{a|t}\boldsymbol{D}^{\beta}_{a|t} f(t) = \boldsymbol{D}^{\alpha+\beta}_{a|t} f(t).$$

Let $X = L^2(\Omega)$ be a Banach space, $A = \Delta$ $D(A) \subset X \to X$ is the infinitesimal generator of C_0 semigroup $T(t)(t > 0)$.

Definition 1.*Let* $u_0 \in X$, $P_\alpha(t)$ *and* $S_\alpha(t)$ *be two operators defined, as follows:*

$$P_\alpha(t)u_0 = \int_0^\infty \Phi_\alpha(\theta) T(t^\alpha \theta) u_0\, d\theta, \tag{9}$$

and

$$S_\alpha(t)u_0 = \alpha \int_0^\infty \theta \Phi_\alpha(\theta) T(t^\alpha \theta) u_0\, d\theta, \tag{10}$$

for $t > 0$ *and* Φ_α *is the Wright type function which was considered by Mainardi [10].*

The operators $P_\alpha(t)$ and $S_\alpha(t)$ satisfy the following properties (see [11])

(1)Let $1 < p \leqslant q < \infty$, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} < \frac{2}{N}$, then

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \leqslant (4\pi t^\alpha)^{\frac{-N}{2r}} \frac{\Gamma(1-N/2r)}{\Gamma(1-\alpha N/2r)} \|u_0\|_{L^p(\mathbb{R}^N)}. \tag{11}$$

(2)Let $1 < p \leqslant q \leqslant \infty$, if $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} < \frac{4}{N}$, then

$$\|S_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \leqslant \alpha (4\pi t^\alpha)^{\frac{-N}{2r}} \frac{\Gamma(2-N/2r)}{\Gamma(1+\alpha-\alpha N/2r)} \|u_0\|_{L^p(\mathbb{R}^N)}. \tag{12}$$

Lemma 1 ([11])*Assume* $f \in L^q((0,T),L^2(\Omega))$. *Let*

$$w(t) = \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds,$$

then, for $\alpha q > 1$, $w \in C([0,T],L^2(\Omega))$.

Also, we need to calculate the Caputo fractional derivative of the following function. Let

$$\varphi_1(t) = \begin{cases} \left(1-\frac{t}{T}\right)^\eta, & 0 < t \leqslant T, \quad \eta \gg 1, \\ 0, & t > T \end{cases} \tag{13}$$

The following results was proved in [12]

$$\mathbf{D}^\alpha_{t|T}\varphi_1(t) = \frac{(1-\alpha+\eta)B(1-\alpha;\eta-1)}{\Gamma(1-\alpha)} T^{-\alpha}\left(1-\frac{t}{T}\right)^{\eta-\alpha},$$

and

$$\mathbf{D}^{n+\alpha}_{t|T}\varphi_1(t) = \frac{(n-\alpha+\eta)\Gamma(\eta+n)}{\Gamma(n-\alpha+\eta)} T^{-(n+\alpha)}\left(1-\frac{t}{T}\right)^{\eta-\alpha-n}.$$

Lemma 2 ([12])*Let* φ_1 *be displayed as in*(13), *for* $\eta > \frac{p}{p-1}\theta - 1$

$$\int_0^T \boldsymbol{D}^\theta_{t|T}\varphi_1 = C_1 T^{1-\theta},$$

and

$$\int_0^T \varphi_1^{-p'/p} |\boldsymbol{D}^\theta_{t|T}\varphi_1|^{p'} = C_2 T^{1-p'\theta},$$

where

$$\theta = \left\{\alpha,\, 2\alpha\right\}, \quad C_1 = \frac{\eta\Gamma(\eta-\theta)}{(\eta-\theta+1)\Gamma(\eta-2\theta+1)} \quad \textit{and}$$

$$C_2 = \frac{\eta^{p'}}{\eta+1-p'\theta}\left[\frac{\Gamma(\eta-\theta))}{\Gamma(\eta+1-2\theta)}\right]^{p'}.$$

Lemma 3 ([4])*Let* $B_R(0) = \left\{x \in \mathbb{R}^N : \quad |x| < R\right\}$ *for large* R, *let* $\Omega_R = \Omega \cap B_R(0)$. *We introduce* φ_2 *the first eigenfunction of* $-\Delta$ *with* λ *the first eigenvalue on* Ω_R

$$\begin{cases} -\Delta\varphi_2(x) = \lambda\varphi_2(x), & \textit{in}\, \Omega_R, \\ \varphi_2(x) > 0, & \textit{in}\, \Omega_R, \\ \|\varphi_2\|_{L^\infty(\Omega_R)} = 1, \end{cases} \tag{14}$$

there exist C_1 *and* C_2 *independent of* R *such that*

$$C_1 R^{-2} \leqslant \lambda \leqslant C_2 R^{-2}. \tag{15}$$

Throughout this paper, we take $\beta = 2\alpha$.

3 The local Cauchy problem

In this section, we apply the Banach fixed point theorem to prove the local existence of a unique mild solution of problem (1).
Consider the following inhomogeneous equation corresponding to (1)

$$\begin{cases} \mathbf{D}^{2\alpha}_{0|t}u - 2\Delta\mathbf{D}^\alpha_{0|t}u + \Delta^2 u = f(t,x), & (t,x) \in (0,\infty)\times\Omega, \\ \Delta u = u = 0, \quad (t,x) \in (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(t,x)|_{t=0} = u_1(x) = 0, \quad x \in \Omega. \end{cases} \tag{16}$$

First, we present the following Lemma that will be used to give the definition of a mild solution to the problem we address.

Lemma 4*Let* $\frac{1}{2} < \alpha < 1$, $u_0 \in L^2(\Omega)$ *and* $v_0 = (\mathbf{D}^\alpha_{0|t}u\big|_{t=0} - \Delta u_0) \in L^2(\Omega)$. *Then, the problem (16) admits a unique mild solution* $u \in C([0,T],L^2(\Omega))$ *given by*

$$u(t,x) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)P_\alpha(s)v_0 ds + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\int_0^s (s-\tau)^{\alpha-1}S_\alpha(s-\tau)f(\tau,x)d\tau ds, \tag{17}$$

where $P_\alpha(t)$ *and* $S_\alpha(t)$ *were defined as in (9) and (10), respectively.*

*Proof.*By Proposition 1, the problem (16) re-write to two abstract Cauchy problems

$$\begin{cases} \mathbf{D}^\alpha_{0|t}v - \Delta v = f(t,x), & (t,x) \in (0,\infty)\times\Omega, \\ v = 0, & (t,x) \in (0,\infty)\times\partial\Omega, \\ v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{18}$$

and

$$\begin{cases} \mathbf{D}^\alpha_{0|t}u - \Delta u = v(t,x), & (t,x) \in (0,\infty)\times\Omega, \\ u = 0, & (t,x) \in (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{19}$$

which means

$$v_0(x) = \mathbf{D}^\alpha_{0|t}u\Big|_{t=0} - \Delta u_0. \tag{20}$$

If $f \in C([0,T],L^2(\Omega))$ and $v_0 \in L^2(\Omega)$, then by [13,11] the problem (18) has a unique mild solution $v \in C([0,T],L^2(\Omega))$ given by

$$v(t,x) = P_\alpha(t)v_0(x) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s,x)ds. \tag{21}$$

Similarly, if $v \in C([0,T],L^2(\Omega))$, then the mild solution of problem (19) is expressed by

$$u(t,x) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)v(s,x)ds. \quad (22)$$

Substituting (21) into (22), we get

$$u(t,x) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)P_\alpha(s)v_0 ds$$
$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) f(\tau,x) d\tau ds. \quad (23)$$

Definition 1*Let $\frac{1}{2} < \alpha < 1$, $u_0 \in L^2(\Omega)$ and $v_0 \in L^2(\Omega)$. We say that u is a mild solution of (1), if $u \in C([0,T],L^2(\Omega))$ and satisfies*

$$u(t,x) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)P_\alpha(s)v_0 ds$$
$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)$$
$$\int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) |u(\tau,x)|^p d\tau ds, \quad (24)$$

where $P_\alpha(t)$, $S_\alpha(t)$ were defined as (9), (10) and v_0 was specified in (20).

Theorem 1.*Let $\frac{1}{2} < \alpha < 1$ and $u_0 \in L^2(\Omega)$. Then there exists $T_{\max} > 0$ such that problem (1) has a unique mild solution $u \in C([0,T_{\max}),L^2(\Omega))$.*

*Proof.*Let

$$E = C([0,T),L^2(\Omega)).$$

For $T > 0$, E is a Banach space endowed with the norm

$$\|u\|_E = \sup_{t\in(0,T)} \|u(t)\|_{L^2(\Omega)},$$

and

$$B_E(R) = \left\{ u \in E \quad : \quad \|u\|_E \leqslant R \right\},$$

for $c_0 > 1$ and $R = 2c_0(\|u_0\|_{L^2(\Omega)} + T^\alpha \|v_0\|_{L^2(\Omega)})$. Define the operator G as

$$Gu(t) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)P_\alpha(t)v_0(x)ds$$
$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)$$
$$\int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau)|u|^p(\tau,x)d\tau ds. \quad (25)$$

For each $u \in B_E(R)$. Then $G(u) \in C([0,T),L^2(\Omega))$(see [11]).

First, we prove G maps $B_E(R)$ into itself. Using (11) and (12), we have

$$\|G(u)(t)\|_{L^2(\Omega)}$$
$$= \|P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)P_\alpha(t)v_0(x)ds$$
$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)$$
$$\int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau)|u|^p(\tau)d\tau ds\|_{L^2(\Omega)}$$
$$\leqslant \|P_\alpha(t)u_0(x)\|_{L^2(\Omega)}$$
$$+ \int_0^t (t-s)^{\alpha-1} \|S_\alpha(t-s)P_\alpha(t)v_0(x)ds\|_{L^2(\Omega)}$$
$$+ \int_0^t (t-s)^{\alpha-1}$$
$$\|S_\alpha(t-s)\int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau)|u|^p(\tau)d\tau ds\|_{L^2(\Omega)}$$
$$\leqslant \|u_0\|_{L^2(\Omega)} + \frac{1}{\Gamma(1+\alpha)} T^\alpha \|v_0\|_{L^2(\Omega)}$$
$$+ \frac{1}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\alpha-1} \| \, |u(\tau)|^p \|_{L^2(\Omega)} d\tau ds$$
$$\leqslant \frac{R}{2} + \frac{1}{\Gamma(2\alpha+1)} T^{2\alpha} R^p.$$

We choose T small enough such that

$$\frac{1}{\Gamma(2\alpha+1)} T^{2\alpha} R^{p-1} \leqslant \frac{1}{2}.$$

Second, we show that G is a contraction map. For $u,v \in B_E(R)$, we have

$$\|G(u)(t) - G(v)(t)\|_{L^2(\Omega)}$$
$$= \| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)$$
$$\int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau)\big(|u|^p(\tau) - |v|^p(\tau)\big)d\tau ds\|_{L^2(\Omega)}$$
$$\leqslant \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\alpha-1}$$
$$\|S_\alpha(s-\tau)\big(|u|^p(\tau) - |v|^p(\tau)\big)d\tau ds\|_{L^2(\Omega)}$$
$$\leqslant \int_0^t (t-s)^{\alpha-1}$$
$$\int_0^s (s-\tau)^{\alpha-1} \|\big(|u|^p(\tau) - |v|^p(\tau)\big)d\tau\|_{L^1(\Omega)} ds$$
$$\leqslant \frac{1}{\Gamma(2\alpha+1)} T^{2\alpha} R^{p-1} \|u-v\|_E.$$

Due to following inequality

$$| \; |u(t)|^p - |v(t)|^p \; | \leqslant C(p)|u(t)-v(t)|(|u(t)|^{p-1} + |v(t)|^{p-1}).$$

We choose T such that

$$\frac{1}{\Gamma(2\alpha+1)} T^{2\alpha} R^{p-1} < 1.$$

Therefore, G is a strict contraction on $B_E(R)$. According to the Banach fixed point theorem, problem (1) admits a unique mild solution $u \in C([0,T_{\max}), L^2(\Omega))$, where

$$T_{\max} = \sup\left\{ T > 0 \quad | \quad \text{there exists a mild solution } u \in C([0,T), L^2(\Omega)) \text{ to } (1) \right\}.$$

Next, we give a blow-up result of our problem (1).

4 Blowing up solutions

Theorem 2. *Assume $u_0 \in L^2(\Omega)$ and $u_0(x) \geqslant 0$. If*

$$1 < p < 1 + \frac{2\alpha}{N - 2\alpha + 1},$$

then any solution to (1) blows up in a finite time.

Proof. We prove the nonexistence of global (weak) solutions to (1) using the test function method [14].

The weak local solution $u \in L^p((0,T), L^2(\Omega))$ is given by

$$\int_0^T \int_\Omega u \mathbf{D}^{2\alpha}_{t|T} \varphi - 2\int_0^T \int_\Omega u \Delta \mathbf{D}^{\alpha}_{t|T} \varphi + \int_0^T \int_\Omega u \Delta^2 \varphi = \int_0^T \int_\Omega |u|^p \varphi + \int_0^T \int_\Omega u_0 \mathbf{D}^{2\alpha}_{t|T} \varphi - 2\int_0^T \int_\Omega u_0 \Delta \mathbf{D}^{\alpha}_{t|T} \varphi, \tag{26}$$

for each $\varphi \in C^2([0,T] \times \Omega)$ compactly supported and $\varphi(T,.) = \varphi_t(T,.) = 0$. We say the solution u is global if (26) holds for any $T > 0$. Let

$$\varphi(t,x) = \varphi_1(t)\varphi_2(x).$$

Equality (26) actually reads

$$\int_0^T \int_{\Omega_R} u \varphi_2 \mathbf{D}^{2\alpha}_{t|T} \varphi_1 - 2\int_0^T \int_{\Omega_R} u \Delta \varphi_2 \mathbf{D}^{\alpha}_{t|T} \varphi_1 + \int_0^T \int_{\Omega_R} u \varphi_1 \Delta^2 \varphi_2 = \int_0^T \int_{\Omega_R} |u|^p \varphi + \mathscr{I} + \mathscr{J}, \tag{27}$$

where

$$\mathscr{I} = \int_0^T \int_{\Omega_R} u_0 \varphi_2 \mathbf{D}^{2\alpha}_{t|T} \varphi_1 = CT^{1-2\alpha} \int_{\Omega_R} u_0 \varphi_2,$$

and

$$\mathscr{J} = 2\int_0^T \int_{\Omega_R} u_0 (-\Delta) \varphi_2 \mathbf{D}^{\alpha}_{t|T} \varphi_1 = \lambda C T^{1-\alpha} \int_{\Omega_R} u_0 \varphi_2.$$

Under the condition $u_0 \geqslant 0$, Eq.(27) becomes

$$\int_0^T \int_{\Omega_R} |u|^p \varphi \leqslant \int_0^T \int_{\Omega_R} u \varphi_2 \mathbf{D}^{2\alpha}_{t|T} \varphi_1 + 2\lambda \int_0^T \int_{\Omega_R} u \varphi_2 \mathbf{D}^{\alpha}_{t|T} \varphi_1 + \lambda^2 \int_0^T \int_{\Omega_R} u \varphi_2 \varphi_1 = \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3. \tag{28}$$

Using the Young inequality with parameters p and $p' = \frac{p}{p-1}$, we have

$$\mathscr{I}_1 \leqslant \int_0^T \int_{\Omega_R} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_2 \mathbf{D}^{2\alpha}_{t|T} \varphi_1 \leqslant \frac{1}{6p} \int_0^T \int_{\Omega_R} |u|^p \varphi + \frac{6^{p'-1}}{p'} \int_0^T \int_{\Omega_R} \varphi_2 \varphi_1^{\frac{-p'}{p}} \left| \mathbf{D}^{2\alpha}_{t|T} \varphi_1 \right|^{p'}, \tag{29}$$

$$\begin{aligned}\mathscr{I}_2 &\leqslant 2\lambda \int_0^T \int_{\Omega_R} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_2 \mathbf{D}^{\alpha}_{t|T} \varphi_1 \\ &\leqslant CR^{-2} \int_0^T \int_{\Omega_R} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_2 \mathbf{D}^{\alpha}_{t|T} \varphi_1 \\ &= \int_0^T \int_{\Omega_R} |u| \varphi^{1/p} CR^{-2} \varphi^{-1/p} \varphi_2 \mathbf{D}^{\alpha}_{t|T} \varphi_1 \\ &\leqslant \frac{1}{6p} \int_0^T \int_{\Omega_R} |u|^p \varphi + C \frac{6^{p'-1}}{p'} R^{-2p'} \int_0^T \int_{\Omega_R} \varphi_2 \varphi_1^{\frac{-p'}{p}} \left| \mathbf{D}^{\alpha}_{t|T} \varphi_1 \right|^{p'},\end{aligned} \tag{30}$$

and

$$\begin{aligned}\mathscr{I}_3 &\leqslant \lambda^2 \int_0^T \int_{\Omega_R} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_2 \varphi_1 \\ &\leqslant CR^{-4} \int_0^T \int_{\Omega_R} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_2 \varphi_1 \\ &= \int_0^T \int_{\Omega_R} |u| \varphi^{1/p} CR^{-4} \varphi^{-1/p} \varphi_2 \varphi_1 \\ &\leqslant \frac{1}{6p} \int_0^T \int_{\Omega_R} |u|^p \varphi + C \frac{6^{p'-1}}{p'} R^{-4p'} \int_0^T \int_{\Omega_R} \varphi_2 \varphi_1.\end{aligned} \tag{31}$$

Taking into account the above mentioned relations (29), (30) and (31) in (28), we find

$$\left(1 - \frac{1}{2p}\right) \int_0^T \int_\Omega |u|^p \varphi \leqslant C \int_0^T \int_\Omega \varphi_2 \varphi_1^{\frac{-p'}{p}} \left| \mathbf{D}^{2\alpha}_{t|T} \varphi_1 \right|^{p'} + R^{-2p'} \int_0^T \int_\Omega \varphi_2 \varphi_1^{\frac{-p'}{p}} \left| \mathbf{D}^{\alpha}_{t|T} \varphi_1 \right|^{p'} + R^{-4p'} \int_0^T \int_\Omega \varphi_2 \varphi_1. \tag{32}$$

We take $R = T$ and we introduce the following scaled variables

$$\tau = \frac{t}{T} \quad \text{and} \quad \xi = \frac{|x|}{T}, \quad T \gg 1.$$

It appears that

$$\int_0^T \int_\Omega |u|^p \varphi \leqslant CT^{1-2\alpha p' + N} + CT^{1-(2+\alpha)p'+N} + CT^{1-4p'+N}. \tag{33}$$

Therefore, if a solution of (1) exists globally, then taking $T \to +\infty$, we get

$$\lim_{T \to \infty} \int_0^T \int_\Omega |u|^p \varphi = 0.$$

Consequently, $u \equiv 0$. This leads to a contradiction.

We are now in a position to state and prove the global existence of solutions of (1) in this section.

5 Global existence

Theorem 3. *Let $\frac{1}{2} < \alpha < 1$. If $p \geqslant 1 + \frac{2\alpha}{N-2\alpha+1}$ and $\|u_0\|_{L^{2q_c}(\Omega)}$ sufficiently small, where $q_c = \frac{N(p-1)}{4}$, then the mild solution of (1) exists globally.*

Proof. We apply the contraction mapping principle to prove the global solution of (1).

From $p \geqslant 1 + \frac{2\alpha}{N-2\alpha+1}$, we see that

$$\frac{1}{p-1} - 1 \leqslant \frac{N}{4}, \tag{34}$$

for $\frac{1}{r} = \frac{1}{2q_c} - \frac{1}{2} < \frac{2}{N}$. We can deduce

$$\|P_\alpha(t)u_0\|_{L^2(\Omega)} \leqslant (4\pi t^\alpha)^{\frac{-N}{2r}} \frac{\Gamma(1-N/2r)}{\Gamma(1-\alpha N/2r)} \|u_0\|_{L^{2q_c}(\Omega)} < \infty.$$

Let

$$Y = \left\{ u \in C((0,\infty), L^2(\Omega)) : \quad \sup_{t>0} \|u(t)\|_{L^2(\Omega)} \leqslant R \right\}.$$

We define the operator G as

$$G(u)(t) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) P_\alpha(s) v_0 ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau)|u(s,x)|^p ds,$$

for each $u \in Y$. It is easy to see that the operator G is well defined on Y. According to (12), we get

$$\begin{aligned}
&\left\| G(u)(t) - G(v)(t) \right\|_{L^2(\Omega)} \\
&= \Big\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \\
&\int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau)\left[|u|^p(\tau) - |v|^p(\tau)\right] d\tau ds \Big\|_{L^2(\Omega)} \\
&\leqslant \int_0^t (t-s)^{\alpha-1} \\
&\int_0^s (s-\tau)^{\alpha-1} \left\| S_\alpha(s-\tau)\left[|u|^p(\tau) - |v|^p(\tau)\right] \right\|_{L^2(\Omega)} d\tau ds \\
&\leqslant C \int_0^t (t-s)^{\alpha-1} \\
&\int_0^s (s-\tau)^{\alpha-1-\alpha\frac{N}{4}} \left\| |u|^p(\tau) - |v|^p(\tau) \right\|_{L^1(\Omega)} d\tau ds \\
&\leqslant C R^{p-1} \int_0^t (t-s)^{\alpha-1} s^{\alpha-\alpha\frac{N}{4}} \|u(\tau) - v(\tau)\|_{L^2(\Omega)} ds \\
&\leqslant C R^{p-1} t^{2\alpha-\alpha\frac{N}{4}} \int_0^1 (1-w)^{\alpha-1} w^{\alpha-\alpha\frac{N}{4}} dw \|u(\tau) - v(\tau)\|_{L^2(\Omega)} \\
&\leqslant C R^{p-1} t^{2\alpha-\alpha\frac{N}{4}} \int_0^1 (1-w)^{\alpha-1} w^{\alpha-\frac{\alpha}{2}} dw \|u(\tau) - v(\tau)\|_{L^2(\Omega)} \\
&\leqslant C R^{p-1} \frac{\Gamma(\alpha)\Gamma(\frac{\alpha}{2}+1)}{\Gamma(3\alpha/2+1)} \|u-v\|_Y .
\end{aligned}$$

If we choose R small enough such that $CR^{p-1} < \frac{1}{2}$, then we get

$$\left\| G(u) - G(v) \right\|_Y < \frac{1}{2} \|u-v\|_Y .$$

6 Conclusion

In this paper, we addressed fractional damping elastic system in a Banach space. The model is based on the properties of fractional derivatives and a judicious choice of the test function. We proved that the mild solutions cannot exist globally when $1 < p < \frac{2\alpha}{N-2\alpha+1} + 1$ and $u_0 \geqslant 0$. If $p > \frac{2\alpha}{N-2\alpha+1} + 1$, then the non-trivial solutions exist all time under some conditions.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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