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Statistical Inference of NH Distribution under Joint Type-I Hybrid Censoring Scheme

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Abstract: In this paper, we adopted a new insight of testing for Nadarajaha and Haghighi (NH) lifetime products under Type-I hybrid censoring scheme. Hence, NH is an expansion of the exponential distribution, and it provides a more reasonable estimate than the generalized exponential, Weibull, and the gamma distributions, particularly when the observations contain zero values. We built this model and preset some statistical inferences of the model parameters with classical and Bayes methods under different types of loss functions using gamma priors for NH distribution parameters. Next, we consider the point and asymptotic confidence interval estimators with maximum likelihood, and the Bayes estimators are discussed. The joint lifetime data is analyzed for illustration purposes. The Monte Carlo simulation study is built to stand on accuracy of these estimators.

Keywords: Bayesian, Gibbs, Joint Type-I hybrid censoring, Maximum likelihood estimation, MCMC, Nadarajaha and Haghighi distribution.

1 Introduction

The problem of determining the relative merits of manufactured products, specially manufactured products from different production lines, has been considered in the last few years. Studying the reliability of these products needs to implement some life testing for the joint set of product units known with a jointly censoring scheme. More precisely, consider that the two lines of manufactured products Γ_1 and Γ_2 have the same facility. From line Γ_1, we randomly select a sample of size κ_1 as well as κ_2 from the line Γ_2, respectively. The total sample is placed under life testing, so the samples are collected from two lines. The failure time and the type of unit are recorded over all steps test. Then, the observed data obtained from these testing are called joint samples because they are obtained from different lines. However, under some restrictions of times and cost, the experimenter may be a terminate test after a fixed number of failures occur. This problem was discussed by different authors, for example, [1] and [2]. The inference with the exact likelihood and bootstrap algorithms discussed with [3], [4] and [5]. For Rayleigh lifetime distributions, see [6], for accelerated Rayleigh life model, [7] and for compound Rayleigh lifetime distributions, see [8]. NH distribution which is used as an alternative to the generalized exponential, Weibull, and the gamma distributions. The main interest is that the model has zero modes; this feature might explain constant hazard rate functions. Moreover, NH distribution is a particular case of the three-parameter generalized power Weibull distribution; see [9] for more details. In 2013, NH distribution was extended by Lemonte [10], using the idea of [11], which is a particular case of the exponential distribution and the generalized exponential distribution, which can be used in modeling censored data. The real-life data obtained from the life testing experiments are censored or complete. Censored data appears when some but not all failure times are observed. Furthermore, the data are used when the failure time of all units under test is observed. The oldest commonly censoring schemes in life test experiments are called Type-I censoring and Type-II censoring schemes. The test time is pre-fixed in the Type-I censoring scheme, but the number of failure units is pre-fixed in the Type-II censoring scheme. These two types of censoring do not allow removing units from the test other than the final point. The progressive censoring scheme has the property that units can be removed at any step of the experiment, see [12]. Also, Childs *et al.* [13] investigated a new approach for analysing the exact likelihood based on hybrid censored

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samples from the exponential distribution. Moreover, Algarni *et al*. [14] analysed joint Type-I hybrid censoring for estimating parameters from Weibull probability density function. Górny *et al*. [15] suggested uniformly distribution to study Type-I hybrid censoring. In addition, Chakrabarty *et al*. [16] studied this type of data under warranty. In practice, the experimenter has to run the experiment under a mixture of Type-I and Type-II censoring schemes known as the hybrid censoring scheme (HCS). There are different types of hybrid censoring schemes described, as follows: Before the experiment runs, the ideal test time and a fair number of failure units needing statistical inference are pre-fixed and denoted by (m,η). In Type-I HCS, the experiment is removed at the min (T_m,η). However, in Type-II HCS, the experiment is terminated at max (T_m,τ). Also, these two types of censoring schemes (Type-I HCS and Type-II HCS) are generalized in progressive hybrid cases that allow the removal of units from the test over the experiment's total time. For more details of hybrid censoring scheme, see [17], and [18]. Different life models, such as Weibull, gamma, and generalized exponential distributions, are used to analyze the lifetime data. The NH distribution has the property that the random variable can take the values of zero. It is the better fit for the data that contain zero values other than the other lifetime distributions. NH distribution with random variable T has probability distribution (PDF) given by

$$g(t) = \beta\theta(1+\theta t)^{\beta-1}\exp\left\{1-(1+\theta t)^{\beta}\right\}, \quad t>0,\ (\beta,\theta)>0, \quad (1)$$

where θ is the scale, and β is the shape parameters. Also, the corresponding cumulative probability (CPD), survival function $S(.)$ and hazard failure rate function $H(.)$ are respectively given by

$$G(t) = 1-\exp\left\{1-(1+\theta t)^{\beta}\right\},\ t>0,\ (\beta,\theta)>0, \quad (2)$$

$$S(t) = \exp\left\{1-(1+\theta t)^{\beta}\right\},\ t>0,\ (\beta,\theta)>0, \quad (3)$$

and

$$H(t) = \beta\theta(1+\theta t)^{\beta-1},\ t>0,\ (\beta,\theta)>0, \quad (4)$$

note that for NH distribution

$$(1+\theta t)g(t) = \beta\theta[1-\ln\bar{G}(t)]\bar{G}(t) \quad (5)$$

where

$$\bar{G}(t) = 1-G(t). \quad (6)$$

The NH distribution presented the extension exponential distribution and equal to an exponential distribution with $\beta = 1$. Hence, this distribution is a special case of Gurvich [19] where $G(t) = 1-\exp(-f(\lambda t))$, and $f(\lambda t)$ is a monotonically increasing function of t with the only limitation $f(\lambda t)\geq 0$. Also, the NH distribution has a decreasing probability function, and its mode is at zero. The properties of this distribution were presented by [20]. The distribution also provides increasing and decreasing shape as well as constant hazard rates. This paper aims to develop statistical inference methods for industrial products that come from different production lines under the same facility and its unit's life distributed with NH lifetime distribution. The joint Type-I HCS is built, and the observed sample is used to construct the maximum likelihood estimates of model parameters. Also, Bayes estimation is adopted under the MCMC technique. The

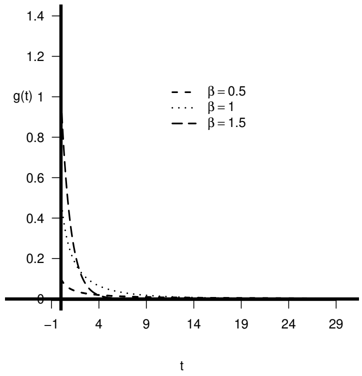

Fig. 1: Plots of the density function (a), the cumulative probability (b), the survival and hazard failure rate functions(c) and (d) with parameters $\theta = 1$ and $\beta = 0.1, 0.5, 1$.

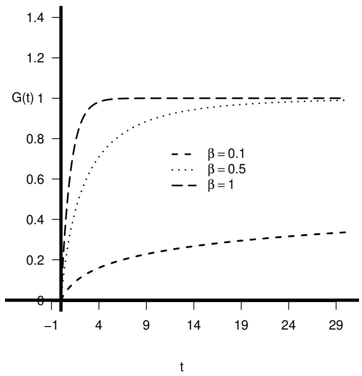

Fig. 2: Plots of the density function (a), the cumulative probability (b), the survival and hazard failure rate functions(c) and (d) with parameters $\theta = 1$ and $\beta = 0.1, 0.5, 1$.

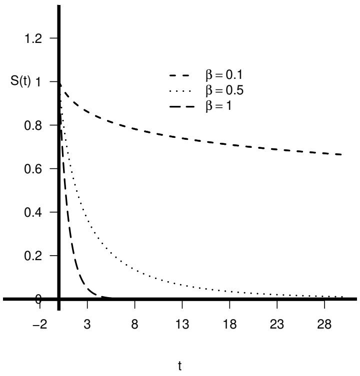

Fig. 3: Plots of the density function (a), the cumulative probability (b), the survival and hazard failure rate functions(c) and (d) with parameters $\theta = 1$ and $\beta = 0.1, 0.5, 1$.

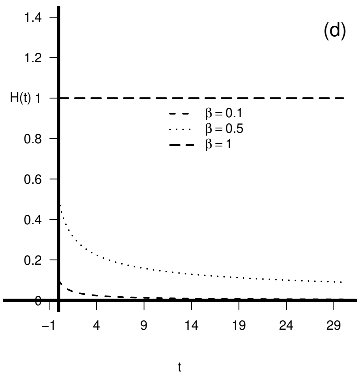

Fig. 4: Plots of the density function (a), the cumulative probability (b), the survival and hazard failure rate functions(c) and (d) with parameters $\theta = 1$ and $\beta = 0.1, 0.5, 1$.

results of point and interval estimation are discussed through the analysis of the set of data. Moreover, the quality of these estimators is assessed and compared using the Monte Carlo simulation study. The results are measured with mean squared error and probability coverage. Figure 1-4 shows the plots of the the properties of the NH distribution. It shows that the density distribution has a decreasing probability function. For smaller values of β, the density becomes faster flat. As β increases, the density has an infinite mode at zero. The cumulative probability increases faster as β increases–see Figure 2, and reaches 1 quickly. The density allows for decreasing survival and hazard failure rate functions–see Figure 3 and 4 where hazard function reaches 1 quickly as β increases.

The paper is organized, as follows: The joint likelihood function of jointly Type-I HCS is formulated in Section 2. The point and interval MLE are discussed in Section 3. Bayesian approach under MCMC methods is adopted in Section 4. A numerical example of jointly Type-I HC data is analysed in Section 5. Finally, results are assessed with Monte Carlo studying in Section 6.

2 Joint Type-I HCS Model

Let, $\kappa_1 + \kappa_2$ be a random sample selected from the two lines of production, κ_1 from the line Γ_1 and κ_2 from the line Γ_2 is put under test. The priors, integers m and time η are selected. The joint sample has identical distributed (i.d) lifetimes $T_1, T_2, \ldots T_{\kappa_1}$ and $T_1^*, T_2^*, \ldots T_{\kappa_2}^*$ respectively. These samples are distributed with PDFs and CDFs given respectively by $g_j(.)$ and $G_j(.), j = 1, 2$. After the test begins, the lifetime and the type of the failure unit is recorded until the minimum time (T_m, η) is observed. Then, the joint lifetime sample $(T_1, T_2, \ldots T_\kappa)$ is constructed from the sample $\{T_1, T_2, \ldots T_{\kappa_1}, T_1^*, T_2^*, \ldots T_{\kappa_2}^*\}$, where $k = k_1 + k_2$ and

$$k = \begin{cases} m, & T_m \le \eta \\ k < m, & T_m > \eta. \end{cases} \tag{7}$$

Therefore, the joint Type-I HC sample presented by $\mathbf{T} = \{(T_1, \rho_1), (T_2, \rho_2), dots, (T_k, \rho_k)\}$ and the value of ρ_i, $i = 1, 2, \ldots, k$ the the value (1 or 0) depends on $\mathbf{T}$ or $\mathbf{T}^*$ failure. Then, we propose that number of failure from the line Γ_1 is defined by $m_1 = \sum_{i=1}^{k} \rho_i$ and from the from the line Γ_2 is defined by $m_2 = \sum_{i=1}^{k} (1 - \rho_i)$. The likelihood function of joint observed sample $\mathbf{t} = \{(t_1, \rho_1), (t_2, \rho_2), \ldots, (t_k, \rho_k)\}$ is given by

$$f_{1,2,\ldots,k}(t) = \frac{\kappa_1!\kappa_2![S_1(\zeta)]^{\kappa_1 - m_1}[S_2(\zeta)]^{\kappa_2 - m_2}}{(\kappa_1 - m_1)(\kappa_2 - m_2)} \times \left[\prod_{i=1}^{k}[g_1(t_i)]^{\rho_i}[g_2(t_i)]^{1-\rho_i}\right], \tag{8}$$

where

$$\zeta = \begin{cases} t_m, & T_m \le \eta \\ \eta, & T_m > \eta, \end{cases} \tag{9}$$

where $S_j(.)$, $j = 1, 2$ denotes reliability functions. The NH distribution under two lines Γ_1 and Γ_2 distributed with CDF given by

$$G(t) = 1 - \exp\left\{1 - (1 + \theta_{ij} t)^{\theta_{ij}}\right\}, \; i, j = 1, 2, \tag{10}$$

where $G(t)$ is the cumulative probability distribution.

3 ML Estimation

From the distribution given by (10) and observed joint Type-I HCS data $\mathbf{t} = \{(t_1,\rho_1),(t_2,\rho_2),\ldots,(t_k,\rho_k)\}$, the function (8) without normalized constant reduces to

$$L(\beta_1,\beta_2,\theta_1,\theta_2|\mathbf{t}) = (\beta_1\theta_1)^{m_1}(\beta_2\theta_2)^{m_2}$$
$$\exp\left\{(\beta_1-1)\sum_{i=1}^{k}\rho_i\log(1+\theta_1 t_i) - \sum_{i=1}^{k}\rho_i(1+\theta_1 t_i)^{\beta_1}\right.$$
$$+(\beta_2-1)\sum_{i=1}^{k}(1-\rho_i)\log(1+\theta_2 t_i)$$
$$-\sum_{i=1}^{k}(1-\rho_i)\log(1+\theta_2 t_i)^{\beta_2}$$
$$\left.-(\kappa_1-m_1)(1+\theta_1\zeta)^{\beta_1}-(\kappa_2-m_2)(1+\theta_2\zeta)^{\beta_2}\right\}. \quad (11)$$

Then, the log-likelihood is given by

$$\ell(\beta_1,\beta_2,\theta_1,\theta_2|\mathbf{t}) = m_1\log(\beta_1\theta_1)+m_2\log(\beta_2\theta_2)$$
$$+(\beta_1-1)\sum_{i=1}^{k}\rho_i\log(1+\theta_1 t_i)$$
$$-\sum_{i=1}^{k}\rho_i(1+\theta_1 t_i)^{\beta_1}+(\beta_2-1)\sum_{i=1}^{k}(1-\rho_i)\log(1+\theta_2 t_i)$$
$$-\sum_{i=1}^{k}(1-\rho_i)\log(1+\theta_2 t_i)^{\beta_2}-(\kappa_1-m_1)(1+\theta_1\zeta)^{\beta_1}$$
$$-(\kappa_2-m_2)(1+\theta_2\zeta)^{\beta_2}. \quad (12)$$

3.1 Point estimation

After taking the first partial derivatives of (12) with respect to parameters vector $\omega = (\beta_1,\beta_2,\theta_1,\theta_2)$, the likelihood equations are presented by

$$\frac{m_1}{\beta_1}+\sum_{i=1}^{k}\rho_i(1+\theta_1 t_i)-$$
$$\sum_{i=1}^{k}\rho_i(1+\theta_1 t_i)^{\beta_1}\log(1+\theta_1 t_i)$$
$$-(\kappa_1-m_1)(1+\theta_1\zeta)^{\beta_1}\log(1+\theta_1\zeta)=0, \quad (13)$$

$$\frac{m_2}{\beta_2}+\sum_{i=1}^{k}(1-\rho_i)(1+\theta_2 t_i)$$
$$-\sum_{i=1}^{k}(1-\rho_i)(1+\theta_2 t_i)^{\beta_2}\log(1+\theta_2 t_i)$$
$$-(\kappa_2-m_2)(1+\theta_2\zeta)^{\beta_2}\log(1+\theta_2\zeta)=0, \quad (14)$$

$$\frac{m_1}{\theta_1}+(\beta_1-1)\sum_{i=1}^{k}\frac{\rho_i t_i}{1+\theta_1 t_i}$$
$$-\beta_1\sum_{i=1}^{k}\rho_i t_i(1+\theta_1 t_i)^{\beta_1-1}-$$
$$(\kappa_1-m_1)\beta_1\zeta(1+\theta_1\zeta)^{\beta_1-1}=0, \quad (15)$$

and

$$\frac{m_2}{\theta_2}+(\beta_2-1)\sum_{i=1}^{k}\frac{(1-\rho_i)t_i}{1+\theta_2 t_i}$$
$$-\beta_2\sum_{i=1}^{k}(1-\rho_i)t_i(1+\theta_2 t_i)^{\beta_2-1}$$
$$-(\kappa_2-m_2)\beta_2\zeta(1+\theta_2\zeta)^{\beta_2-1}=0. \quad (16)$$

The equations from (13) to (14) show that the likelihood equations reduce to four of non-linear equations which can be solved with iteration method such as, Newton Raphson to obtain $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\theta}_1$ and $\hat{\theta}_2$.

Remark: If $m_1 = 0$ in (13) and (15), then β_1 and θ_1 do not exist. Also, if $m_2 = 0$ in (14) and (16), then β_2 and θ_2 do not exist, see [21].

3.2 Interval estimation

The Fisher information matrix is defined as the expectation of minus second partial derivative of a log-likelihood function as

$$\prod = -\mathrm{E}\left(\frac{\partial^2\ell(\beta_1,\beta_2,\theta_1,\theta_2|\mathbf{t})}{\partial\omega_i\partial\omega_j}\right),\quad i,j=1,2,3,4. \quad (17)$$

In practice, the expectation of minus second partial derivative is more difficult so, the approximate information matrix is used as the approximation form to Fisher information matrix as

$$\prod\nolimits_0 = -\mathrm{E}\left(\frac{\partial^2\ell(\beta_1,\beta_2,\theta_1,\theta_2|\mathbf{t})}{\partial\omega_i\partial\omega_j}\right)\Bigg|_{\omega=\hat{\omega}},\quad i,j=1,2,3,4. \quad (18)$$

Hence, the approximate confidence intervals of model parameters $\omega = (\beta_1,\beta_2,\theta_1,\theta_2)$ is obtained under approximate normality distribution of $\hat{\omega} = (\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}_1,\hat{\theta}_2)$ under the large sample approximation

$$\hat{\omega}\rightarrow N(\omega,\prod\nolimits_0^{-1}(\hat{\omega})), \quad (19)$$

where $\prod_0^{-1}(\hat{\omega})$ is the approximate information matrix presented by (18). Then, $100(1-2\alpha)\%$ approximate confidence intervals of $\omega = (\beta_1,\beta_2,\theta_1,\theta_2)$ are given by

$$\begin{cases}\hat{\beta}_1\pm z_\alpha\varepsilon_{11}\\ \hat{\beta}_2\pm z_\alpha\varepsilon_{22}\\ \hat{\theta}_1\pm z_\alpha\varepsilon_{33}\\ \hat{\theta}_2\pm z_\alpha\varepsilon_{44}\end{cases}, \quad (20)$$

where the values ε_{11}, ε_{22}, ε_{33} and ε_{44} are the rote of the diagonal of approximate information matrix and the value z_{γ} is the percentile of the normal (0,1) with right-tail probability α.

4 Bayes Estimation

Bayesian approach that presents the point and interval estimates of model parameters depend on the prior information, and the information satisfies by the data set. The prior information is formulated as independent gamma prior given by

$$\Omega^*(\omega_i) \propto \omega_i^{a_i-1}\exp(-b_i\omega_i),$$
$$\omega_i > 0,\ (a_i,b_i) > 0,\ i = 1,2,3,4, \tag{21}$$

and the information is presented by the likelihood function given by (11) to present the posterior distribution under given joint Type-I HCS, as follows:

$$\Omega(\omega_i|\mathbf{t}) = \frac{L(\omega_i|\mathbf{t})\Omega^*(\omega_i)}{\int_0^\infty L(\omega_i|\mathbf{t})\Omega^*(\omega_i)d\omega_i}. \tag{22}$$

Then, the Bayes estimate of any function $h(\beta_1,\beta_2,\theta_1,\theta_2)$ under posterior distribution (22) is given under squared error loss function (SEL) by

$$E_{\omega_i|\mathbf{t}}(h(\omega_i)) = \frac{\int_0^\infty h(\omega_i)L(\omega_i|\mathbf{t})\Omega^*(\omega_i)d\omega_i}{\int_0^\infty L(\omega_i|\mathbf{t})\Omega^*(\omega_i)d\omega_i}. \tag{23}$$

The ratio (23) of two integrals in general is needed to approximate. Several methods can be used to approximate this integral and the important one called MCMC algorithms is described, as follows: The posterior distribution (22) under (11) and (21) can be formulated by

$$\begin{aligned}\Omega(\beta_1,\beta_2,\theta_1,\theta_2|\mathbf{t}) &\propto \beta_1^{m_1+a_1-1}\theta_1^{m_1+a_2-1}\beta_2^{m_2+a_3-1} \\ &\theta_2^{m_2+a_4-1}\exp\Bigg\{-b_1\beta_1-b_2\theta_1-b_3\beta_2-b_4\theta_2 \\ &+(\beta_1-1)\sum_{i=1}^k\rho_i\log(1+\theta_1t_i)-\sum_{i=1}^k\rho_i(1+\theta_1t_i)^{\beta_1} \\ &+(\beta_2-1)\sum_{i=1}^k(1-\rho_i)\log(1+\theta_2t_i) \\ &-\sum_{i=1}^k(1-\rho_i)(1+\theta_2t_i)^{\beta_2}-(\kappa_1-m_1)(1+\theta_1\zeta)^{\beta_1} \\ &-(\kappa_2-m_2)(1+\theta_2\zeta)^{\beta_2}\Bigg\}.\end{aligned} \tag{24}$$

Then, the full conditional posterior distribution of parameters vector $\omega = (\beta_1,\beta_2,\theta_1,\theta_2)$ is given by

$$\begin{aligned}&\Omega_1(\beta_1|\theta_1,\mathbf{t}) \propto \beta_1^{m_1+a_1-1} \\ &\exp\Bigg\{-\beta_1(b_1-\sum_{i=1}^k\rho_i\log(1+\theta_1t_i))-\sum_{i=1}^k\rho_i(1+\theta_1t_i)^{\beta_1} \\ &-(\kappa_1-m_1)(1+\theta_1\zeta)^{\beta_1}\Bigg\} = \\ &\exp\Bigg\{-\Bigg(\sum_{i=1}^k\rho_i(1+\theta_1t_i)^{\beta_1}+(\kappa_1-m_1)(1+\theta_1\zeta)^{\beta_1}\Bigg)\Bigg\} \\ &\times Gamma\Bigg(m_1+a_1,b_1-\sum_{i=1}^k\rho_i\log(1+\theta_1t_i)\Bigg),\end{aligned} \tag{25}$$

$$\begin{aligned}&\Omega_2(\beta_2|\theta_2,\mathbf{t}) \propto \beta_2^{m_2+a_3-1} \\ &\exp\Bigg\{-b_3\beta_2+(\beta_2-1)\sum_{i=1}^k(1-\rho_i)\log(1+\theta_2t_i) \\ &-\sum_{i=1}^k(1-\rho_i)(1+\theta_2t_i)^{\beta_2}-(\kappa_2-m_2)(1+\theta_2\zeta)^{\beta_2}\Bigg\} = \\ &\exp\Bigg\{-\sum_{i=1}^k(1-\rho_i)(1+\theta_2t_i)^{\beta_2}-(\kappa_2-m_2)(1+\theta_2\zeta)^{\beta_2}\Bigg\} \\ &\times Gamma\Bigg(m_2+a_3,b_3-\sum_{i=1}^k(1-\rho_i)\log(1+\theta_2t_i)\Bigg),\end{aligned} \tag{26}$$

from joint posterior distribution of model parameters, the conditional posterior PDF's

$$\begin{aligned}\Omega_3(\theta_1|\beta_1,\mathbf{t}) &\propto \exp\Bigg\{(\beta_1-1)\sum_{i=1}^k\rho_i\log(1+\theta_1t_i) \\ &-\sum_{i=1}^k\rho_i(1+\theta_1t_i)^{\beta_1}-(\kappa_1-m_1)(1+\theta_1\zeta)^{\beta_1}\Bigg\} \\ &\times Gamma(m_1+a_2,b_2)\end{aligned} \tag{27}$$

and

$$\begin{aligned}\Omega_4(\theta_2|\beta_2,\mathbf{t}) &\propto \theta_2^{-1}\exp\Bigg\{+(\beta_2-1)\sum_{i=1}^k(1-\rho_i) \\ &\times\log(1+\theta_2t_i)-\sum_{i=1}^k(1-\rho_i)(1+\theta_2t_i)^{\beta_2}-(\kappa_2-m_2) \\ &\times(1+\theta_2\zeta)^{\beta_2}\Bigg\}\times Gamma(m_2+a_4,b_4).\end{aligned} \tag{28}$$

Then, the conditional distribution of parameters (25) to (28) reduces to four functions more similar to normal populations. The joint posterior probability for the parameters β_1,β_2,θ_1, and θ_2 given data is shown in (24). Where, the conditional posterior distribution shows that β_2 is based on the parameter θ_1 and contains two parts exponential distribution multiply by gamma distribution with shape $m_2 + a_3$, and scale $b_3 - \sum_{i=1}^{k}(1-\rho_i)\log(1+\theta_2 t_i)$. Also, the conditional distribution of θ_1 given β_1 in (27) and the data is combined between two distributions. The first is exponential distribution and the other is gamma with parameters $m_1 + a_2$, and b_2. Hence, this process can be seen in the conditional distributions of β_2 and θ_2. Hence, the Metropolis-Hasting (MH) [22] under Gibbs algorithms, as follows:

Step 1: Put $\omega^{(0)} = (\beta_1^{(0)}, \beta_2^{(0)}, \theta_1^{(0)}, \theta_2^{(0)})$ and I = 1.
Step 2: Generate $\theta_1^{(I)}$ from (25).
Step 3: Generate $\beta_2^{(I)}$ from (27).
Step 4: Generate $\beta_2^{(I)}$ from (26).
Step 5: Generate $\theta_2^{(I)}$ from (28).
Step 6: Record the vector $\omega^{(I+1)} = (\beta_1^{(I)}, \beta_2^{(I)}, \theta_1^{(I)}, \theta_2^{(I)})$.
Step 7: Put I = I + 1.
Step 8: Steps from 2–7 are repeated N times.
Step 9: If N^* is the MCMC number that is needed to achieved the stationary distribution (burn-in), then the Bayes MCMC point estimate of ω is given by

$$\hat{\omega}_B = \frac{1}{N-N^*}\sum_{i=N^*-1}^{N}\omega^{(i)}, \tag{29}$$

and the corresponding posterior variance of ω is given by

$$\hat{V}(\omega) = \frac{1}{N-N^*}\sum_{i=N^*-1}^{N}(\omega^{(i)} - \hat{\omega}_B)^2. \tag{30}$$

Step 10: The corresponding credible $100(1-2\alpha)\%$ interval of ω is given by

$$(\omega_{\alpha(N-N^*)}, \phi_{(1-\alpha)(N-N^*)}). \tag{31}$$

where $\omega_{(I)}$ denotes ordered value of $\omega^{(I)}$ and $\omega = (\beta_1, \beta_2, \theta_1, \theta_2)$. In MCMC algorithms, it is difficult to estimate several parameters because the evaluation at each step. Thus, several techniques technique can be used to estimate the unknown parameters, such as jumping. These tools can be found in Aykroyd, [23], who explained the method. Also, Gelman *et al.* [24] described a new tool conceited with the variance of the proposed new value during the loop of MCMC algorithms. The jumper of the estimations of the unknown parameters are used to generate sample from the posterior distribution and then the chain is in equilibrium. Figure 5 shows the Gibbs algorithms jumping rules, over which the samples can be generated to approximate a target distribution. It indicates the parameter θ_1 is proposed at each step and the parameter θ_2 is proposed at steps 1, 3, 5,..., etc. The parameter β_1 is proposed at steps 1, 4, 7,..., etc. and the parameter β_2 is proposed at steps 1, 5, 9,..., etc. More precisely, the parameters θ_1 is proposed and evaluated at each step, where β_2 is proposed after 4 steps. This procedure, as shown in Figure 5, accelerates obtaining the target solution.

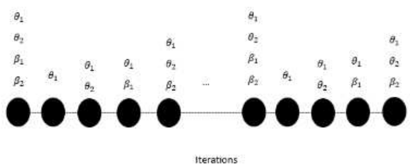

Fig. 5: Diagrammatic representation of the the procedure of Gibbs algorithms.

5 Illustrative Example

In this section, the estimation procedure developed in this article has been discussed through numerical example. The proposed methodology is studied through extensive simulations and it evaluates the performance of several methods provided above. Then, we simulate a set of data with $n = 60(\kappa_1 = \kappa_2 = 30)$ and the effect sample size $m = 40$. The data is generated from NH distribution with parameters vectors $\omega = (3.0, 0.5, 3.0, 1.0)$ with test time $\eta = 1.0$ and it is summarized in Table 1. The expectation is computed based on the unknown parameters distributions. The number of iterations is 10000 to estimate the mean for unknown parameters. Table 1

Table 1: The generated joint Type-I HCS data of NH distributions.

Lifetime	0.0082	0.0106	0.0168	0.0173	0.0195	0.0227	0.0260	0.0332	0.0373	0.0488
Cause	0	0	0	0	1	0	0	0	0	1
Lifetime	0.0488	0.0537	0.0546	0.0632	0.0767	0.1348	0.1357	0.1402	0.1511	0.1517
Cause	0	1	1	1	0	0	0	0	0	1
Lifetime	0.1591	0.1642	0.1681	0.1845	0.1895	0.2155	0.2158	0.2250	0.2354	0.2380
Cause	1	1	0	1	0	0	0	1	1	0
Lifetime	0.2394	0.2398	0.2543	0.2877	0.2938	0.2982	0.2983	0.2989	0.3109	0.3380
Cause	1	0	1	0	1	0	0	1	0	0

shows the sample obtained from NH distribution where 0 indicates the sample collected from line Γ_1 and 1 indicates the sample collected from line Γ_2. Table 2 shows the results of the comparison between ML CIs and MCMC CIs with confidence level of 95%. The line Γ_2 achieves a better results than line Γ_1. Table 2 presents slight difference between the expectation of these methods. The results of Gibbs algorithms are illustrated in Figures 6-9.

Table 2: MLEs, Bays estimate and 95% approximate intervals estimate.

Pa.s	$(.)_{\text{ML}}$	$(.)_{\text{MCMC}}$	95% CI_{ML}	95% CI_{MCMC}
$\beta_1 = 3.0$	4.6940	2.6521	(0.4171, 7.4572)	(0.5684, 63245)
$\theta_1 = 0.5$	0.3511	0.6215	(0.00214, 2.147)	(0.1214, 2.2310)
$\beta_2 = 3.0$	2.849	2.7123	(1.2453, 6.2143)	(1.4218, 7.0001)
$\theta_2 = 1.0$	1.227	1.2324	(0.2147, 3.1254)	(0.2524, 3.3002)

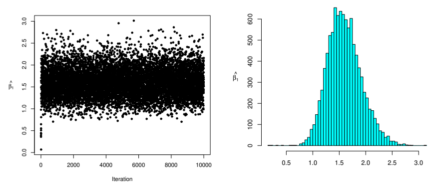

Fig. 6: Simulation and histogram number of β_1 generated by MCMC method.

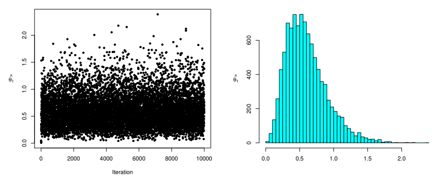

Fig. 7: Simulation and histogram number of θ_1 generated by MCMC method.

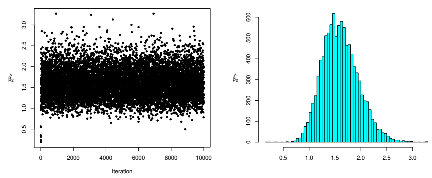

Fig. 8: Simulation and histogram number of β_2 generated by MCMC method.

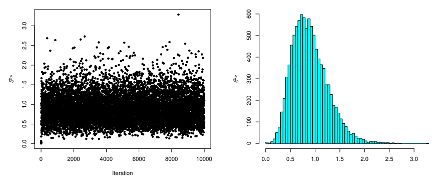

Fig. 9: Simulation and histogram number of θ_2 generated by MCMC method.

6 Simulation Studies

The developed results discussed in this paper for the ML and Bayes estimators are compared and assessed through building simulation studies. Thus, we adopt different combinations of sample size $(\kappa_1 + \kappa_2)$ and the effected sample size m as well as different censoring time η. Then, for the two samples of the model's parameters, we measure the estimators average $\bar{\omega}$ and estimators mean square error (MSE) described by

$$E(\hat{\omega}) = \frac{1}{\kappa}\sum_{i=1}^{\kappa}\hat{\omega}^{(i)}, \tag{32}$$

and

$$\text{MSE} = \frac{1}{\kappa}\sum_{i=1}^{\kappa}\left(\bar{\omega}^{(i)} - \bar{\omega}\right)^2, \tag{33}$$

where $\omega = (\beta_1, \beta_2, \theta_1, \theta_2)$ is the parameters vector. For

Table 3: The average of $\hat{\omega}$ and MSE of $\omega = \{1.0, 1.5, 2.0, 2.5\}$.

(κ_1,κ_2)	(m,η)	Pa.s	$(.)_{\text{ML}}$ $E(\hat{\omega})$	MSEs	$(.)_{\text{MCMC}_{P_0}}$ $E(\hat{\omega})$	MSEs	$(.)_{\text{MCMC}_{P_1}}$ $E(\hat{\omega})$	MSEs
(30,30)	(30,0.4)	β_1	1.841	0.421	1.782	0.399	1.421	0.231
		β_2	2.011	0.624	1.999	0.607	1.840	0.400
		θ_1	2.412	0.821	2.377	0.798	2.311	0.542
		θ_2	2.823	0.990	2.799	0.910	2.669	0.711
(30,30)	(50,0.4)	β_1	1.642	0.328	1.615	0.301	1.400	0.202
		β_2	1.852	0.542	1.810	0.514	1.777	0.350
		θ_1	2.321	0.699	2.311	0.684	2.300	0.498
		θ_2	2.784	0.841	2.790	0.830	2.615	0.600
(50,50)	(50,0.4)	β_1	1.671	0.314	1.609	0.302	1.395	0.198
		β_2	1.815	0.511	1.803	0.507	1.751	0.332
		θ_1	2.301	0.670	2.297	0.661	2.289	0.481
		θ_2	2.760	0.822	2.765	0.813	2.607	0.597
(50,50)	(80,0.4)	β_1	1.555	0.277	1.541	0.269	1.315	0.101
		β_2	1.785	0.480	1.777	0.461	1.707	0.300
		θ_1	2.289	0.600	2.271	0.602	2.225	0.401
		θ_2	2.710	0.741	2.720	0.725	2.599	0.560
(30,30)	(30,0.8)	β_1	1.790	0.370	1.730	0.348	1.390	0.200
		β_2	1.960	0.580	1.954	0.561	1.802	0.350
		θ_1	2.380	0.782	2.340	0.756	2.280	0.504
		θ_2	2.790	0.949	2.782	0.912	2.631	0.640
(30,30)	(50,0.8)	β_1	1.607	0.270	1.590	0.266	1.375	0.162
		β_2	1.801	0.511	1.780	0.502	1.740	0.311
		θ_1	2.284	0.657	2.290	0.644	2.298	0.463
		θ_2	2.725	0.807	2.714	0.801	2.599	0.554
(50,50)	(50,0.8)	β_1	1.625	0.274	1.611	0.264	1.364	0.161
		β_2	1.800	0.482	1.811	0.472	1.722	0.308
		θ_1	2.250	0.642	2.266	0.635	2.245	0.457
		θ_2	2.714	0.800	2.707	0.792	2.592	0.571
(50,50)	(80,0.8)	β_1	1.514	0.251	1.514	0.244	1.300	0.098
		β_2	1.752	0.443	1.753	0.441	1.694	0.261
		θ_1	2.263	0.570	2.245	0.568	2.201	0.385
		θ_2	2.701	0.713	2.710	0.700	2.544	0.532

interval estimation, we use the probability coverage (PC) as well as average interval length (AIL). In our paper, the two set of parameters vector are given by $\omega^1 = (1.0, 1.5, 2.0, 2.5)$ and $\omega^2 = (0.5, 0.8, 1.0, 1.5)$. In Bayesian approach, we selected the prior parameters as $P_0 = \{a_1 = 0.0001,\ b_1 = 0.0001,\ a_2 = 0.0001,\ b_2 = 0.0001,\ a_3 = 0.0001,\ b_3 = 0.0001,\ a_4 = 0.0001,\ b_4 =$

Table 4: The PCs and AILs of $\omega = \{1.0, 1.5, 2.0, 2.5\}$.

(κ_1, κ_2)	(m, η)	Pa.s	$(.)_{ML}$		$(.)_{MCMC_{P_0}}$		$(.)_{MCMC_{P_1}}$	
			PCs	AILs	PCs	AILs	PCs	AILs
(30,30)	(30,0.4)	β_1	0.88	3.002	0.89	2.987	0.90	2.911
		β_2	0.87	4.014	0.89	3.985	0.89	3.924
		θ_1	0.89	4.875	0.89	4.825	0.89	4.762
		θ_2	0.85	5.129	0.87	5.103	0.90	5.005
(30,30)	(50,0.4)	β_1	0.89	2.985	0.90	2.977	0.90	2.870
		β_2	0.90	3.879	0.89	3.865	0.91	3.900
		θ_1	0.89	4.811	0.90	4.801	0.90	4.735
		θ_2	0.89	5.095	0.90	5.083	0.92	4.887
(50,50)	(50,0.4)	β_1	0.89	2.970	0.90	2.964	0.92	2.831
		β_2	0.91	3.871	0.91	3.849	0.91	3.872
		θ_1	0.89	4.803	0.90	4.795	0.90	4.701
		θ_2	0.88	5.087	0.91	5.069	0.90	4.869
(50,50)	(80,0.4)	β_1	0.90	2.940	0.91	2.936	0.96	2.810
		β_2	0.91	3.843	0.92	3.835	0.91	3.814
		θ_1	0.90	4.780	0.90	4.762	0.92	4.680
		θ_2	0.92	5.044	0.93	5.041	0.91	4.813
(30,30)	(30,0.8)	β_1	0.89	2.950	0.91	2.939	0.91	2.870
		β_2	0.89	3.986	0.89	3.964	0.89	3.901
		θ_1	0.90	4.841	0.89	4.811	0.92	4.724
		θ_2	0.88	5.101	0.90	5.009	0.91	4.975
(30,30)	(50,0.8)	β_1	0.90	2.944	0.93	2.939	0.95	2.822
		β_2	0.91	3.852	0.90	3.847	0.92	3.760
		θ_1	0.90	4.781	0.90	4.762	0.92	4.700
		θ_2	0.89	5.061	0.91	5.054	0.90	4.832
(50,50)	(50,0.8)	β_1	0.91	2.942	0.92	2.925	0.92	2.801
		β_2	0.91	3.833	0.91	3.828	0.93	3.800
		θ_1	0.92	4.789	0.91	4.772	0.95	4.685
		θ_2	0.89	5.053	0.93	5.044	0.97	4.827
(50,50)	(80,0.8)	β_1	0.93	2.904	0.92	2.902	0.95	2.774
		β_2	0.91	3.803	0.92	3.811	0.91	3.784
		θ_1	0.92	4.751	0.93	4.746	0.92	4.642
		θ_2	0.92	5.013	0.93	5.009	0.90	4.797

Table 6: The PCs and ALs of $\omega = \{0.5, 0.8, 1.0, 1.5\}$.

(κ_1, κ_2)	(m, η)	Pa.s	$(.)_{ML}$		$(.)_{MCMC_{P_0}}$		$(.)_{MCMC_{P_1}}$	
			PCs	AILs	PCs	AILs	PCs	AILs
(30,30)	(30,0.4)	β_1	0.88	1.899	0.90	1.875	0.90	1.652
		β_2	0.87	2.423	0.90	2.415	0.91	2.241
		θ_1	0.90	3.125	0.90	3.001	0.92	2.985
		θ_2	0.89	3.583	0.89	3.520	0.90	3.324
(30,30)	(50,0.4)	β_1	0.89	1.830	0.90	1.809	0.92	1.600
		β_2	0.89	2.295	0.90	2.280	0.91	2.201
		θ_1	0.90	2.990	0.91	2.974	0.92	2.715
		θ_2	0.91	3.501	0.90	3.503	0.94	3.280
(50,50)	(50,0.4)	β_1	0.90	1.811	0.93	1.802	0.97	1.570
		β_2	0.89	2.282	0.90	2.277	0.92	2.181
		θ_1	0.91	2.976	0.91	2.964	0.92	2.680
		θ_2	0.91	3.482	0.93	3.491	0.90	3.261
(50,50)	(80,0.4)	β_1	0.92	1.762	0.91	1.754	0.92	1.420
		β_2	0.89	2.235	0.92	2.228	0.92	2.127
		θ_1	0.90	2.944	0.91	2.933	0.93	2.651
		θ_2	0.93	3.470	0.90	3.465	0.91	3.232
(30,30)	(30,0.8)	β_1	0.89	1.870	0.90	1.861	0.91	1.641
		β_2	0.87	2.411	0.90	2.402	0.90	2.232
		θ_1	0.90	3.111	0.90	2.981	0.89	2.972
		θ_2	0.90	3.560	0.90	3.507	0.91	3.302
(30,30)	(50,0.8)	β_1	0.90	1.817	0.92	1.808	0.92	1.591
		β_2	0.91	2.282	0.90	2.277	0.93	2.187
		θ_1	0.90	2.975	0.91	2.970	0.92	2.702
		θ_2	0.92	3.485	0.92	3.475	0.96	3.241
(50,50)	(50,0.8)	β_1	0.91	1.790	0.93	1.782	0.97	1.539
		β_2	0.91	2.275	0.90	2.262	0.92	2.149
		θ_1	0.91	2.959	0.91	2.964	0.92	2.671
		θ_2	0.91	3.470	0.93	3.465	0.91	3.226
(50,50)	(80,0.8)	β_1	0.92	1.748	0.92	1.742	0.92	1.403
		β_2	0.92	2.221	0.95	2.214	0.92	2.112
		θ_1	0.91	2.925	0.93	2.917	0.93	2.639
		θ_2	0.93	3.459	0.90	3.449	0.94	3.211

Table 5: The average of $\hat{\omega}$ and MSE of $\omega = \{0.5, 0.8, 1.0, 1.5\}$.

(κ_1, κ_2)	(m, η)	Pa.s	$(.)_{ML}$		$(.)_{MCMC_{P_0}}$		$(.)_{MCMC_{P_1}}$	
			$E(\hat{\omega})$	MSEs	$E(\hat{\omega})$	MSEs	$E(\hat{\omega})$	MSEs
(30,30)	(30,0.4)	β_1	0.745	0.167	0.721	0.155	0.654	0.102
		β_2	0.982	0.199	0.973	0.191	0.885	0.145
		θ_1	1.482	0.368	1.472	0.359	1.400	0.298
		θ_2	1.824	0.412	1.811	0.107	1.784	0.387
(30,30)	(50,0.4)	β_1	0.714	0.141	0.700	0.131	0.622	0.097
		β_2	0.954	0.171	0.947	0.166	0.852	0.117
		θ_1	1.460	0.341	1.451	0.318	1.385	0.271
		θ_2	1.813	0.398	1.801	0.399	1.764	0.359
(50,50)	(50,0.4)	β_1	0.707	0.132	0.702	0.118	0.613	0.079
		β_2	0.942	0.154	0.938	0.160	0.847	0.114
		θ_1	1.440	0.327	1.435	0.314	1.366	0.258
		θ_1	1.800	0.379	1.811	0.381	1.748	0.330
(50,50)	(80,0.4)	β_1	0.713	0.112	0.715	0.109	0.607	0.055
		β_2	0.929	0.142	0.931	0.151	0.836	0.107
		θ_1	1.427	0.314	1.412	0.302	1.351	0.224
		θ_2	1.792	0.344	1.780	0.361	1.722	0.321
(30,30)	(30,0.8)	β_1	0.761	0.151	0.721	0.147	0.647	0.098
		β_2	0.977	0.181	0.968	0.180	0.875	0.129
		θ_1	1.483	0.351	1.466	0.348	1.401	0.282
		θ_2	1.813	0.401	1.803	0.100	1.771	0.376
(30,30)	(50,0.8)	β_1	0.698	0.130	0.701	0.125	0.614	0.078
		β_2	0.953	0.155	0.942	0.147	0.808	0.103
		θ_1	1.449	0.328	1.439	0.322	1.371	0.262
		θ_2	1.809	0.381	1.813	0.372	1.749	0.339
(50,50)	(50,0.8)	β_1	0.690	0.114	0.712	0.109	0.607	0.061
		β_2	0.928	0.139	0.925	0.139	0.826	0.101
		θ_1	1.433	0.314	1.440	0.307	1.351	0.246
		θ_2	1.785	0.358	1.810	0.348	1.725	0.311
(50,50)	(80,0.8)	β_1	0.725	0.102	0.711	0.099	0.611	0.048
		β_2	0.914	0.124	0.918	0.135	0.830	0.098
		θ_1	1.418	0.304	1.400	0.297	1.328	0.213
		θ_2	1.760	0.331	1.782	0.328	1.711	0.309

$0.0001\}$, $P_1 = \{a_1 = 2,\ b_1 = 2,\ a_2 = 3,\ b_2 = 2,\ a_3 = 4,\ b_3 = 2.5,\ a_4 = 4,\ b_4 = 1.5\}$ and $P_2 = \{a_1 = 2,\ b_1 = 2,\ a_2 = 3,\ b_2 = 2,\ a_3 = 4,\ b_3 = 2.5,\ a_4 = 4,\ b_4 = 1.5\}$ respectively. In the simulation study, we generate 1000 random samples of NH distribution for each sample, compute the point and interval estimate, and the corresponding tools. We chose the sample size (κ_1, κ_2) by (30,30) and (50,50), while m and time η are selected by (30,0.4), (50,0.4), (80,0.4), (30,0.8), (50,0.8), (80, 0.8). For MCMC approach, we run the posterior distribution 10000 times with 1000 iterations as burn-in. The results of $E(\hat{\omega})$, MSEs, PCs, and AILs of estimates are computed and presented in Tables 3-6. Table 6 shows the results of parameters estimation using MLEs and MCMC methods. The results of MLEs are slightly larger than the corresponding results of MCMC methods. Also, the results of MCMC_{p_1} of parameters are closer to the true value and the corresponding MSEs decrease except for some cases of the parameter θ_2. The reason might be that the true value of θ_2 is bigger than other parameters. Hence, from Table 5, the estimation of the parameters are closer to the true value. It is not surprising since the parameter β_2 in Table 5 is smaller than β_2 in Table 3. According to Table 3, the results of estimation, in the case $\kappa = 30, \kappa = 30, m = 30, \eta = 0.4$, are larger than the case $\kappa = 30, \kappa = 30, m = 50, \eta = 0.4$, because the number of failures increases. Also, Table 7 shows the results of PCs and AILs. The results of PCs using MLEs are smaller than MCMC methods, while the results of AILs are closer in Table 6. Finally, as the integers m and η, increase the corresponding results of PCs slightly increase.

7 Conclusion

In this paper, new approach was introduced to estimate NH distribution under joint two groups of population. The proposed method provides knowledge concerning dealing with different types of set data. Where, the problem of determining the quality of the products with different product lines under the same facility required some tests about the product life to define the relative merits of these products. Moreover, we considered the products whose life is distributed with NH distribution. Then, the lifetime data under a hybrid Type-I censored scheme was collected to be used in the operation of statistical inference about the unknown distribution. The theoretical results under ML and Bayes estimation were developed and measured with numerical computation. From the numerical results, we can report the following results.

–The results have shown that joint Type-I HCS presents a suitable scheme for modeling two populations.
–The results of classical MLE, as well as Bayes estimation under population P0, are more closed to others, since the hyper-parameters start close to the zero.
–The estimation results under informative priors P_1 and P_2 are better than MLE or non-informative Byes prior P_0.
–The quality of the point and interval estimation are better for increasing sample size $\kappa_1 + \kappa_2$ or effecting sample size m.
–The results perform better for the large value of censoring time η.
–The results showed that the proposed methods are more valid for different model parameters and suitable for joint different populations.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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