Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150502

On the Numerical Solution of the Nonlocal Elliptic Problem With a p-Kirchhoff-Type Term

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Received: 7 Dec. 2020, Revised: 2 Jun. 2021, Accepted: 12 Jul. 2021
Published online: 1 Sep. 2021

Abstract: This work is devoted to the study of the finite element method for a class of nonlocal elliptic problems associated with p-Kirchhoff-type operator. The convergence and *a priori* error estimates for the discrete formulation are established. Moreover, the finite element formulation is nonlinear, it can then be solved by Newton-Raphson's iterative but the main issue is that the Jacobian matrix of the Newton-Raphson method is full due to the presence of the nonlocal term thereby making computation expensive. To avoid this difficulty, the new formulation whose Jacobian matrix is sparse is given. Finally, the predictions observed theoretically are validated by means of numerical experiments.

Keywords: Galerkin finite element method, Newton-Raphson method, Nonlocal diffusion term, p-Kirchhoff operator, optimal error estimate.

1 Introduction

Let Ω be a bounded open subset in $\mathbb{R}^d$, $d = 2,3$ with smooth boundary $\partial\Omega$. Consider the following problem with nonlocal nonlinearity:

$$\begin{cases} -a(\|\nabla u\|_p^p)\Delta_p u = f \quad \text{in } \Omega\ , \\ u = 0 \ \text{ on } \partial\Omega\ , \end{cases} \tag{1}$$

where Δ_p is the p-Laplace operator:

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$$

and $\|\cdot\|_p$ denotes the L^p-norm, $1 < p < \infty$. The functions $a(\cdot)$ and f are given functions and will be defined in the next section. Due to the presence of an integral over Ω in (1), the equation is not pointwise identity and therefore is called a nonlocal problem.
The boundary value problem (1) is the stationary version of the problem

$$\begin{cases} u_t - a(\|\nabla u\|_p^p)\Delta_p u = f \quad \text{in } \Omega\times(0,T)\ , \\ u(x,t) = 0 \ \text{ on } \partial\Omega\times(0,T)\ , \\ u(x,0) = u_0(x) \ \text{ in } \Omega\ . \end{cases} \tag{2}$$

During the past few years, there has been active ongoing research on the study of problems associated with the p-Laplace operator, which appears in a variety of physical fields (see for instance [1–3]). In particular, a lot of attention has been devoted to nonlocal problems. One of the justifications of such models lies in the fact that in reality the measurements are not made pointwise but through some local average. Some interesting features of nonlocal problems and more motivation are described in [4–7] and in the references therein. There are also some closed problems solved by the folowing authors [8–10].
In the literature, the focus has been on proving well-posedness of the solution. In [11, 12], Correa and Nascimento have established existence results by considering several classes of a and f of the following elliptic problems

$$\begin{cases} -a(\|\nabla u\|_p^p)\Delta_p u = f(x,u) \quad \text{in } \Omega\ , \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{3}$$

Chipot and Savitska [13] have discussed the existence and uniqueness of solutions for problem (2) and also the asymptotic behavior of the solution for large time. They have also investigated the stationary case (1).
However, there are few studies on the numerical solution

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of such problems which are focused in the case $p=2$. In [14], Kumar and Kumar studied the finite element method for problem (3). Peradze [15] proposed and analyzed the spectral method for one-dimensional Kirchhoff string problem.
This paper is concerned with the finite element method for solving the problem (1) with $p \in (1,\infty)$. Since the resultant formulation leads to a system of nonlinear problem, a new formulation is presented. Optimal error estimates are also discussed. The layout of this paper is as follows: In Section 2, the finite element formulation and error analysis of the problem are discussed. Section 3 deals with the new reformulation which is an equivalent problem and some numerical results to illustrate our theoretical analysis.

2 Finite element approximation and error analysis

2.1 Variational formulation

Throughout this paper, $W^{m,p}(\Omega)$ is the usual Sobolev space ($m \in \mathbb{N}$ and $1 < p \leq \infty$) with norm $\|\cdot\|_{m,p}$. $W_0^{m,p}(\Omega)$ is the closure of $\mathscr{C}_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. $W^{-m,q}(\Omega)$ denote the topological dual space of $W_0^{m,p}(\Omega)$ where q is the conjugate of p. That is $\frac{1}{p}+\frac{1}{q}=1$. For notions on Sobolev spaces, we refer to [16, 17].
Let us consider problem (1) under the following hypotheses.

$$f \in W^{-1,q}(\Omega), \quad \frac{1}{p}+\frac{1}{q}=1, \quad 1<p<\infty. \tag{4}$$
$$a \text{ is continuous and there exists } m; M, \quad 0<m\leq a(s)\leq M \quad \text{for all} \quad s\in\mathbb{R}. \tag{5}$$
$$a \text{ is increasing.} \tag{6}$$

Lemma 1. *If a is increasing then*

$$s>0 \mapsto a(s^p)s^{p-1} \quad \textit{is increasing.} \tag{7}$$

Proof. Let $s_1, s_2 \in \mathbb{R}^+$ such that $s_1 < s_2$

$$a(s_2^p)s_2^{p-1}-a(s_1^p)s_1^{p-1} = s_2^p(a(s_2^p)-a(s_1^p))+a(s_1^p)(s_2^{p-1}-s_1^{p-1}) \geq 0.$$

A weak formulation for (1) reads as follows: Find $u \in V \equiv W_0^{1,p}(\Omega)$ such that

$$a(\|\nabla u\|_p^p)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx = \langle f,v\rangle \quad \forall v\in V, \tag{8}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $W^{-1,q}(\Omega)$ and $W^{1,p}(\Omega)$.
Chipot & Saviska [13] proved the following result.

Theorem 1. *Suppose that (4), (5) and (7) hold. Then problem (8) admits a unique solution and there exists a constant $C>0$ such that*

$$\|\nabla u\|_p \leq C.$$

Let $\mathscr{T}_h$ be a regular triangulation of Ω with elements K, in following classical finite element method theory [18, 19]. Thus let $h=\max_{K\in\mathscr{T}_h}\{h_K\}$ denote the mesh size, where $h_K = diam(K) = \max\{\|x-y\|, x,y\in K\}$, and S_h be the finite dimensional subspace of $C(\overline{\Omega})$, which consists of piecewise polynomials of degree one on $\mathscr{T}_h$.

$$V_h = \{v_h \in S_h, \ v_h = 0 \text{ on } \partial\Omega\}.$$

Let $\Pi_h : C(\overline{\Omega}) \to S_h$ be an interpolation operator. Then we have the following lemma.

Lemma 2(cf. Ref. [18]). *For $m\in\{0,1\}$,*
(a) for $q,s\in[1,\infty]$, provided $W^{2,s}(\Omega)\subset W^{m,q}(\Omega)$

$$\|w-\Pi_h w\|_{m,q} \leq Ch^{2-m+d(1/q-1/s)}\|w\|_{2,s} \quad \forall w\in W_0^{2,s}(\Omega), \tag{9}$$

(b) for $q>d$

$$\|w-\Pi_h w\|_{m,q} \leq Ch^{1-m}\|w\|_{1,q} \quad \forall w \in W_0^{1,q}(\Omega), \tag{10}$$

where C is a positive constant that does not depend on h.

The finite element approximation associated to problem (8) is as follows.
Find $u_h \in V_h$ such that

$$a(\|\nabla u_h\|_p^p)\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla v_h dx = \langle f,v_h\rangle \quad \forall v_h\in V_h. \tag{11}$$

Lemma 3. *Problem (11) admits a unique solution $u_h \in V_h$ which satisfies:*

$$\|\nabla u_h\|_p \leq C, \tag{12}$$

where C is a constant which does not depend on h.

The following lemmas will be useful throughout this work.

Lemma 4(cf. Ref. [20, 21]). *For all $p\in(1,\infty)$ and $\tau\geq 0$, there exists positive constants $C_1 = C_1(p,d)$ and $C_2 = C_2(p,d)$ such that for all $\xi,\eta\in\mathbb{R}^d$ with $d\geq 1$ we have*

$$||\xi|^{p-2}\xi-|\eta|^{p-2}\eta| \leq C_1|\xi-\eta|^{1-\tau}(|\xi|+|\eta|)^{p-2+\tau}, \tag{13}$$
$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta) \geq C_2|\xi-\eta|^{2+\tau}(|\xi|+|\eta|)^{p-2-\tau}. \tag{14}$$

Lemma 5(cf. Ref. [21]). *For all $p\in(1,\infty)$ there exists and ε_0 such that for all $x,y,z\geq 0$ and for all $\varepsilon\in(0,\varepsilon_0)$*

$$(x+y)^{p-2}yz \leq \varepsilon(x+y)^{p-2}y^2+C(\varepsilon^{-1})(x+z)^{p-2}z^2. \tag{15}$$

Lemma 6(cf. Ref. [13]). *Let a,b be the nonnegative numbers. Then for all $p\in(1,\infty)$,*

$$|a^p-b^p| \leq p|a-b|(a+b)^{p-1}.$$

Lemma 7. *For all $p \in (1,\infty)$ and for all $\xi,\eta \in \mathbb{R}^d$,*

$$|\eta|^p \geq |\xi|^p + p|\xi|^{p-2}\xi\cdot(\eta-\xi), \tag{16}$$
$$\frac{1}{2}(|\eta|+|\xi|) \leq (|\eta-\xi|+|\xi|) \leq 2(|\eta|+|\xi|). \tag{17}$$

Remark. As consequence of relation (17), we have

$$K_1(|\eta|+|\xi|)^{p-2} \leq (|\eta-\xi|+|\xi|)^{p-2} \leq K_2(|\eta|+|\xi|)^{p-2}, \tag{18}$$

where

$$K_1 = \begin{cases} 2^{p-2} & \text{if } 1<p<2 \\ (\frac{1}{2})^{p-2} & \text{if } p \geq 2 \end{cases} \quad \text{and} \quad K_2 = \begin{cases} (\frac{1}{2})^{p-2} & \text{if } 1<p<2 \\ 2^{p-2} & \text{if } p \geq 2. \end{cases} \tag{19}$$

We will also assume that

$$a \text{ is Lipschitz continuous with the Lipschitz constant} L. \tag{20}$$

2.2 Error analysis

In this section, we present one of the main results of this paper.

Theorem 2 (Convergence rate). *Assume that the hypotheses (4)-(6) and (20) hold. Let u be the unique solution of problem (8) and u_h the unique solution of problem (11). Then there exists a positive constant C which does not depend on h such that:*
(a) *if $p \in (1,2)$ and $u \in W^{2,p}(\Omega)$, then*

$$\|u-u_h\|_{1,p}^2 \leq Ch^p. \tag{21}$$

(b) *If $p \in [2,\infty)$ and $u \in W^{2,p}(\Omega)\cap W^{1,\infty}(\Omega)$, then*

$$\|u-u_h\|_{1,p}^p \leq Ch^2. \tag{22}$$

To prove Theorem 2, the following lemma will be useful.

Lemma 8. *Assume that the hypotheses (4)-(6) and (20) hold. Let u be the unique solution of problem (8) and u_h the unique solution of problem (11). Then there exists a positive constant C which does not depend on h such that for any $v_h \in V_h$*

$$\int_\Omega (|\nabla u|+|\nabla(u-u_h)|)^{p-2}|\nabla(u-u_h)|^2dx \leq \tag{23}$$
$$C\left\{\int_\Omega (|\nabla u|+|\nabla(u-v_h)|)^{p-2}|\nabla(u-v_h)|^2dx + \|\nabla(u-v_h)\|_p^2\right\}.$$

Proof. Let us denote $u-u_h$ by $E = u-u_h \equiv (u-v_h)+(v_h-u_h)$ for all $v_h \in v_h$, from (8) and (11), we have

$$\begin{aligned}
&a(\|\nabla u_h\|_p^p)\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_h|^{p-2}\nabla u_h)\cdot\nabla E dx \\
&\quad + (a(\|\nabla u\|_p^p) - a(\|\nabla u_h\|_p^p))\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla E dx \\
&\quad = a(\|\nabla u_h\|_p^p)\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_h|^{p-2}\nabla u_h)\cdot\nabla(u-v_h)dx \\
&\quad + (a(\|\nabla u\|_p^p) - a(\|\nabla u_h\|_p^p))\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u-v_h)dx \\
&\quad + a(\|\nabla u\|_p^p)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(v_h-u_h)dx \\
&\quad - a(\|\nabla u_h\|_p^p)\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla(v_h-u_h)dx \\
&\quad = a(\|\nabla u_h\|_p^p)\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_h|^{p-2}\nabla u_h)\cdot\nabla(u-v_h)dx \\
&\quad + (a(\|\nabla u\|_p^p) - a(\|\nabla u_h\|_p^p))\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u-v_h)dx.
\end{aligned} \tag{24}$$

Applying the lower bound of $a(\cdot)$ (5), the relation (14) with $\tau = 0$ and the relation (17), we obtain

$$\begin{aligned}
a(\|\nabla u_h\|_p^p)&\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_h|^{p-2}\nabla u_h)\cdot\nabla E dx \\
&\geq mC_2K_1\int_\Omega (|\nabla u|+|\nabla E|)^{p-2}|\nabla E|^2dx.
\end{aligned}$$

On the other hand, by the relation (16),

$$\frac{1}{p}(\|\nabla u\|_p^p - \|\nabla u^h\|_p^p) \leq \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla E dx. \tag{25}$$

Since $a(\cdot)$ is increasing, without loss of generality, we assume that $(a(\|\nabla u\|_p^p) - a(\|\nabla u^h\|_p^p)) \geq 0$ (if $(a(\|\nabla u\|_p^p) - a(\|\nabla u^h\|_p^p)) \leq 0$, we interchange the role of u and u^h), then

$$\begin{aligned}
0 &\leq \frac{1}{p}(a(\|\nabla u\|_p^p) - a(\|\nabla u^h\|_p^p))(\|\nabla u\|_p^p - \|\nabla u^h\|_p^p) \\
&\leq (a(\|\nabla u\|_p^p) - a(\|\nabla u^h\|_p^p))\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla E dx.
\end{aligned}$$

Therefore, the left hand side of (24) is lower bounded by

$$LHS \geq mC_2K_1\int_\Omega (|\nabla u|+|\nabla E|)^{p-2}|\nabla E|^2dx. \tag{26}$$

Using the upper bound of $a(\cdot)$ (5), the relation (13) with $\tau = 0$ the relation (17)and Lemma 5

$$\begin{aligned}
&a(\|\nabla u^h\|_p^p)\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u^h|^{p-2}\nabla u^h)\cdot\nabla(u-v^h)dx \\
&\quad \leq MC_1K_2\int_\Omega (|\nabla u|+|\nabla E|)^{p-2}|\nabla E||\nabla(u-v^h)|dx \\
&\quad \leq MC_1K_2\varepsilon\int_\Omega (|\nabla u|+|\nabla E|)^{p-2}|\nabla E|^2dx \\
&\quad + MC_1K_2C(\varepsilon^{-1})\int_\Omega (|\nabla u|+|\nabla(u-v^h)|)^{p-2}|\nabla(u-v^h)|^2dx.
\end{aligned} \tag{27}$$

By the Lipschitz continuity of $a(\cdot)$ and Holder's inequality,

$$
\begin{aligned}
&(a(\|\nabla u\|_p^p) - a(\|\nabla u^h\|_p^p)) \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u - v^h)dx \\
&\le L|\|\nabla u\|_p^p - \|\nabla u^h\|_p^p| \|\nabla u\|_p^{p-1} \|\nabla(u - v^h)\|_p \\
&\le L(\int_\Omega ||\nabla u|^p - |\nabla u^h|^p|dx) \|\nabla u\|_p^{p-1} \|\nabla(u - v^h)\|_p \quad (28)\\
&\le pL(\int_\Omega (|\nabla u| + |\nabla u^h|)^{p-1} |\nabla E|dx) \|\nabla u\|_p^{p-1} \|\nabla(u - v^h)\|_p \\
&\le pL\left(\int_\Omega (|\nabla u| + |\nabla u^h|)^p dx\right)^{1/2} \\
&\times (\int_\Omega (|\nabla u| + |\nabla u^h|)^{p-2} |\nabla E|^2 dx)^{1/2} \|\nabla u\|_p^{p-1} \|\nabla(u - v^h)\|_p \\
&\le K_2^{1/2} pL\left(\int_\Omega (|\nabla u| + |\nabla u^h|)^p dx\right)^{1/2} \\
&(\int_\Omega (|\nabla u| + |\nabla E|)^{p-2} |\nabla E|^2 dx)^{1/2} \|\nabla u\|_p^{p-1} \|\nabla(u - v^h)\|_p \\
&\le \frac{mC_2K_1}{2} \int_\Omega (|\nabla u| + |\nabla E|)^{p-2} |\nabla E|^2 dx \\
&+ C(\|\nabla u\|_p^2 + \|\nabla u^h\|_p^2) \|\nabla u\|_p^{2p-2} \|\nabla(u - v^h)\|_p^2.
\end{aligned}
$$

Combining (27) and (28) the right hand side of (24) can be bounded as follows

$$
\begin{aligned}
RHS &\le (MC_1K_2\varepsilon + \frac{mC_2K_1}{2}) \int_\Omega (|\nabla u| + |\nabla E|)^{p-2} |\nabla E|^2 dx \\
&+ MC_1K_2C(\varepsilon^{-1}) \int_\Omega (|\nabla u| + |\nabla(u - v^h)|)^{p-2} |\nabla(u - v^h)|^2 dx \qquad (29)\\
&+ C(\|\nabla u\|_p^2 + \|\nabla u^h\|_p^2) \|\nabla u\|_p^{2p-2} \|\nabla(u - v^h)\|_p^2.
\end{aligned}
$$

From (26) and (29), with appropriate choice of ε, we end up with

$$
\begin{aligned}
&\int_\Omega (|\nabla u| + |\nabla E|)^{p-2} |\nabla E|^2 dx \qquad (30)\\
&\le C \int_\Omega (|\nabla u| + |\nabla(u - v^h)|)^{p-2} |\nabla(u - v^h)|^2 dx \\
&+ C(\|\nabla u\|_p^2 + \|\nabla u^h\|_p^2) \|\nabla u\|_p^{2p-2} \|\nabla(u - v^h)\|_p^2.
\end{aligned}
$$

Proof of Theorem 2. Taking $v_h = \Pi_h u$ in (23), we have

$$
\begin{aligned}
&\int_\Omega (|\nabla u| + |\nabla(u - u_h)|)^{p-2} |\nabla(u - u_h)|^2 dx \\
&\le C\left\{\int_\Omega (|\nabla u| + |\nabla(u - \Pi_h u)|)^{p-2} |\nabla(u - \Pi_h u)|^2 dx \right. \qquad (31)\\
&\left. + \|\nabla(u - \Pi_h u)\|_p^2 \right\}.
\end{aligned}
$$

For the **case** $p \in (1, 2)$, we proceed as follows.

$$
\begin{aligned}
&\int_\Omega (|\nabla u| + |\nabla(u - \Pi_h u)|)^{p-2} |\nabla(u - \Pi_h u)|^2 dx \\
&\quad \le \int_\Omega |\nabla(u - \Pi_h u)|^p dx = \|\nabla(u - \Pi_h u)\|_p^p. \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla(u - u_h)\|_p^2 &= \left(\int_\Omega |\nabla(u - u_h)|^p dx\right)^{2/p} \\
&= \left(\int_\Omega (|\nabla u| + |\nabla E|)^{(2-p)p/2} (|\nabla u| + |\nabla E|)^{(p-2)p/2} |\nabla E|^p dx\right)^{2/p} \\
&\le \left((\int_\Omega (|\nabla u| + |\nabla E|)^p dx)^{(2-p)/2} (\int_\Omega (|\nabla u| + |\nabla E|)^{p-2} |\nabla E|^2 dx\right)^{p/2})^{2/p} \\
&\le C(\|\nabla u\|_p, \|\nabla u_h\|_p) \int_\Omega (|\nabla u| + |\nabla E|)^{p-2} |\nabla E|^2 dx. \qquad (33)
\end{aligned}
$$

Combining (32) and (33) and using (31), we arrive at

$$
\|\nabla(u - u_h)\|_p^2 \le C(\|\nabla(u - \Pi_h u)\|_p^p + \|\nabla(u - \Pi_h u)\|_p^2)
$$

and we conclude the error bound (21) by Lemma 2.
In the **case** $p \in [2, \infty)$,

$$
\int_\Omega |\nabla(u - u_h)|^p dx \le \int_\Omega (|\nabla u| + |\nabla E|)^{p-2} |\nabla E|^2 dx, \quad (34)
$$

and

$$
\begin{aligned}
&\int_\Omega (|\nabla u| + |\nabla(u - \Pi_h u)|)^{p-2} |\nabla(u - \Pi_h u)|^2 dx \qquad (35)\\
&\le C(\|\nabla u\|_p)(\|\nabla(u - \Pi_h u)\|^2 + \|\nabla(u - \Pi_h u)\|_p^p).
\end{aligned}
$$

Again combining (34) and (35) and using (31), we arrive at

$$
\|\nabla(u - u_h)\|_p^p \le C(\|\nabla(u - \Pi_h u)\|^2 + \|\nabla(u - \Pi_h u)\|_p^p)
$$

and the desired estimated (22) follows immediately by applying Lemma 2.

3 Numerical method

3.1 Nonlinear iterative process

Let N_p be the dimension and $\{\varphi_j\}_{j=1}^{N_p}$ the canonical basis of V_h associated with the nodes of $\mathscr{T}_h$. The solution $u_h \in V_h$ of problem (11) can be written as

$$
u_h = \sum_{j=1}^{N_p} U_j \varphi_j,
$$

where U_j are degrees of freedom.
When this expression is substitute into (11), we obtain the following nonlinear algebraic problem: Find the vector $U = [U_1, \cdots, U_{N_p}]$ which satisfies the system of N_p nonlinear equations

$$
A(U) = F, \qquad (36)
$$

where the entries $A_i(U)$ and F_i are given by

$$
A_i(U) = a(\|\nabla u_h\|_p^p) \int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla \varphi_i dx \quad \text{and} \quad F_i = \langle f, \varphi_i \rangle, \quad 1 \le i \le N_p.
$$

The nonlinear system (36) is generally large sized and hence it is important to develop efficient iterative methods for its numerical solution [22]. It is known that the

Newton-Raphson iterative method is attractive for solving nonlinear algebraic equations since it is fast convergent, achieves the desired tolerance in a small number of iterations, and thus preserves the finite element order of convergence. But using Newton's method directly to solve (36), the sparsity of Jacobian matrices is lost due to the presence of nonlocal term in the equation. Indeed, any element of the Jacobian matrix J_A takes the form

$$\frac{\partial A_i}{\partial U_j}(u_h) = p/2a'(\|\nabla u_h\|_p^p)$$
$$\times\left(\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla\varphi_j dx\right)\left(\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla\varphi_i dx\right)$$
$$+a(\|\nabla u_h\|_p^p)\left\{(p-2)/2\int_\Omega |\nabla u_h|^{p-4}(\nabla u_h\cdot\nabla\varphi_j)(\nabla u_h\cdot\nabla\varphi_i)dx\right.$$
$$\left.+1/2\int_\Omega |\nabla u_h|^{p-2}\nabla\varphi_j\cdot\nabla\varphi_i dx\right\}$$

and from the first term of the equation above, the sparsity of the Jacobian matrix is lost (see also Figure 1). To avoid this lost, we adopt and extend the technique presented in [23].
The modified method is defined as follows: find $x\in\mathbb{R}$ and $u_h\in V_h$ such that

$$\begin{cases} \|\nabla u_h\|_p^p - x = 0, \quad \forall v_h\in V_h \\ a(x)\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla v_h\,dx = \langle f, v_h\rangle. \end{cases} \tag{37}$$

Problem (37) can be written in the following nonlinear system

$$B(U,x) = b, \tag{38}$$

where

$$B_i(u_h,x) = a(x)\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla\varphi_i\,dx, \quad b_i = \langle f,\varphi_i\rangle,$$
$$B_{N_p+1}(u_h,x) = \|\nabla u_h\|_p^p - x \quad \text{and} \quad b_{N_p+1} = 0 \;\; 1\le i\le N_p.$$

To see the sparsity of the Jacobian matrix $J_B(U)$ for (38), we define

$$S_{ij} := \frac{\partial B_i}{\partial U_j}(u_h,x) =$$
$$a(x)\left\{(p-2)/2\int_\Omega |\nabla u_h|^{p-4}(\nabla u_h\cdot\nabla\varphi_j)(\nabla u_h\cdot\nabla\varphi_i)dx\right.$$
$$\left.+1/2\int_\Omega |\nabla u_h|^{p-2}\nabla\varphi_j\cdot\nabla\varphi_i dx\right\} \tag{39}$$

$$C_{i1} := \frac{\partial B_i}{\partial x}(u_h,x) = a'(x)\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla\varphi_i\,dx, \tag{40}$$

$$D_{1j} := \frac{\partial B_{N_p+1}}{\partial U_j}(u_h,x) = p/2\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla\varphi_j\,dx, \tag{41}$$

$$\delta_{11} := \frac{\partial B_{N_p+1}}{\partial x}(u_h,x) = -1 \quad 1\le i,j\le N_p. \tag{42}$$

Table 1: Convergence results for $p=1.5$ and $p=3$ respectively

	$p=1.5$		$p=3$	
h	$\|u-u_h\|_{1,r}$	Rate	$\|u-u_h\|_{1,r}$	Rate
1/5	3.907e-3		9.742e-3	
1/10	2.403e-3	0.70	6.571e-3	0.57
1/15	1.804e-3	0.71	5.103e-3	0.62
1/20	1.454e-3	0.74	4.204e-3	0.67
1/25	1.234e-3	0.73	3.657e-3	0.63

Therefore, $J_B(U)$ takes the form

$$J_B(U) := \begin{pmatrix} S & C \\ D & -1 \end{pmatrix},$$

where $S=(S_{ij})_{1\le i,j\le N_p}$ is sparse and $C=(C_{i1})_{1\le i\le N_p}$, $D=(D_{1j})_{1\le j\le N_p}$ are full.
The following result which establishes the relation between (11) and (37) can easily be proven.

Theorem 3. *If (u_h,x) is a solution of (37), then u_h is a solution to (11). Conversely, If u_h is a solution of (11), then $(u_h,\|\nabla u_h\|_p^p)$ is a solution of (37).*

3.2 Numerical tests

We perform numerical experiments with known exact solution in order to check the convergence rate of the method. We solve the problem (1) in $\Omega=(0,1)^2$ and specify the right hand side according to the exact solution $u(x,y)=xy(1-x)(1-y)$. The numerical convergence rate is computed by solving (37) using first order (P1-FEM) Lagrange polynomials on a sequence of uniform meshes with parameter $h=1/5, 1/10, 1/15, 1/20, 1/25$. The error has been calculated using the $W_0^{1,p}$-norm. As expected Table 1 shows the convergence rate of $0.7\approx 3/4$ for $p=3/2\in(1,2)$ and also shows the convergence rate of $0.6\approx 2/3$ for $p=3\in(2,\infty)$ which are in accordance with the theoretical analysis.
Figure 1 shows the graphical representation of the Jacobian matrices J_A and J_B respectively for $h=1/10, 1/15$. In these figures, the numbers on the vertical and horizontal lines represent the row and column numbers of the Jacobian matrix respectively. The number *nz* denotes the number of nonzero elements.

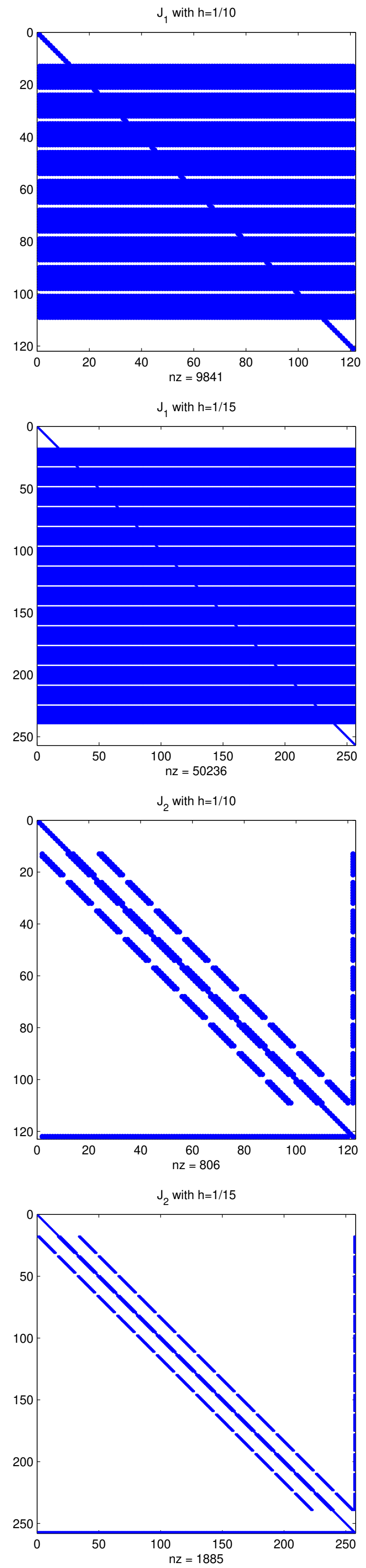

Fig. 1: Jacobian matrices J_A and J_B respectively.

4 Conclusion

A finite element method for a class of nonlinear nonlocal diffusion problems associated with p-Kirchhoff-type operator was presented. From the observation that the Jacobian matrix for the corresponding nonlinear system is full, the new equivalent problem whose Jacobian matrix is sparse was proposed. The numerical experiments have supported the theoretically obtained result. Finally, it is worth pointing out that the techniques used in this article might be applicable to other problems.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

Conflict of Interest

The authors declare that they have no conflict of interest.

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