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A Note on Modules and Submodules over Polynomial Rings

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Abstract: In this paper, we consider R a commutative ring with identity non-zero and the R-module $I(G)$, which is the edge ideal of a graph simple and finite G, with no isolated vertices. A submodule N of $I(G)$ is called an edge dense submodule if $\text{Hom}_R(I(G)/N, \text{E}_R(I(G))) = 0$, where $\text{E}_R(I(G))$ is the injective hull of $I(G)$. The R-module $I(G)$ is said to be edge monoform if any nonzero submodule of $I(G)$ is an edge dense submodule. Here in this paper, we presented some results which involve the definition of the module $I(G)$ to be edge monoform.

Keywords: edge dense submodule, prime module, edge monoform module, edge multiplication module, edge ideal of a graph.

1 Introduction

Throughout this paper, R is a commutative ring with non-zero identity.

Here, we consider the R-module $I(G)$ and the injective hull of $I(G)$ is denoted by $\text{E}_R(I(G))$. A submodule N of $I(G)$ is called a proper submodule if $N \neq I(G)$.

For an R-module M any, the annihilator of M is denoted by $\text{Ann}_R(M)$ and for any $x \in M$ the annihilator of Rx is denoted by $\text{Ann}_R((x))$.

In this paper, we consider submodules which involve the theory of graphs, together with the edge ideal of a graph. A submodule N of $I(G)$ is called *edge dense submodule*, and is written $N \leq_d I(G)$, if for any f, $g \in I(G)$ with $f \neq 0$ there exists $p \in R = K[v_1, \ldots, v_s]$ such that $pf \neq 0$ and $pg \in N$. The R-module $I(G)$ is said to be *edge monoform* if any nonzero submodule of $I(G)$ is an edge dense submodule.

A proper submodule N of an R-module M is called prime submodule if for $r \in R$ and $x \in M$, we have that

$$rx \in N \text{ implies that } x \in N \text{ or } rM \subseteq N.$$

Also, M is called prime module if the submodule 0 of M is prime.

If N is a submodule of an R-module M, then $(N :_R M)$ denotes the ideal $\text{Ann}_R(M/N)$ of R, that is, we have

$$(N :_R M) = \{r \in R : rM \subseteq N\}.$$

In the Section 2, we put some definitions and prerequisites for a better understanding of the theory and results. We introduce preliminaries of the theory of graphs which involving the edge ideal of a graph G; associated to the graph G is a monomial ideal, defined by

$$I(G) = (v_i v_j \mid v_i v_j \text{ is an edge of } G),$$

with $v_i v_j = v_j v_i$ and with $i \neq j$, in the polynomial ring $R = K[v_1, v_2, \ldots, v_s]$ over a field K, called the **edge ideal** of G. The preliminaries of the theory of graphs were introduced in this Section 2 together with the concepts suitable for the work in question.

In the Section 3, we prove some properties of modules and submodules with respect to theory in question, properties that involve the edge ideal of a graph G, which is a graph simple and finite, with no isolated vertices. In this section we presented the definitions that we use in the results, and we put some results in the context of the edge ideal.

In the Section 4, we presented a relationship between edge multiplication module and edge dense module; we

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put also, the definition of $I(G)$ to be edge faithful, which also will be used.

Throughout of the paper, we mean by a graph G, a finite simple graph with the vertex set $V(G)$ and with no isolated vertices.

Here, we use properties of commutative algebra and homological algebra for the development of the results (see [1], [2] and [3]).

2 Prerequisites of the graphs theory

Let us present in this section the concepts of the graphs theory that we will use in the course of this paper.

2.1 Edge ideal of a graph

This section is in accordance with [4] and [5].

Let $R = K[v_1,\ldots,v_s]$ be a polynomial ring over a field K, that was fixed, and let $Z = \{z_1,\ldots,z_q\}$ be a finite set of monomials in R.

The **monomial subring** spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1,\ldots,z_q] \subset R.$$

In general, it is very difficult to certify whether $K[Z]$ has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, we consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1,\ldots,v_s\}$.

Let Z be the set of all monomials $v_iv_j = v_jv_i$, with $i \neq j$, in $R = K[v_1,\ldots,v_s]$, such that $\{v_iv_j\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

Definition 21A **walk** of length s in G is an alternating sequence of vertices and edges $w = \{v_1,z_1,v_2,\ldots,v_{s-1},z_h,v_s\}$, where $z_i = \{v_{i-1}v_i\}$ is the edge joining v_{i-1} and v_i.

A other definition is the following.

Definition 22A walk is **closed** if $v_1 = v_s$. A walk may also be denoted by $\{v_1,\ldots,v_s\}$, the edges being evident by context. A **cycle** of length s is a closed walk, in which the points $v_1,\ldots,v_s$ are distinct.

A **path** is a walk with all the points distinct. A **tree** is a connected graph without cycles and a graph is **bipartite** if all its cycles are even. A vertex of degree one will be called an **end point**.

Definition 23A subgraph $G' \subseteq G$ is called **induced** if $v_iv_j = v_jv_i$, with $i \neq j$, is an edge of G' whenever v_i and v_j are vertices of G' and v_iv_j is an edge of G.

The **complement** of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_iv_j = v_jv_i$, with $j \neq i$, is an edge of G^c if and only if it is not an edge of G. Finally, let C_k be denote the cycle on k vertices; a **chord** is an edge which is not in the edge set of C_k. A cycle is called **minimal** if it has no a chord.

If G is a graph without isolated vertices, simple and finite, then let R denote the polynomial ring on the vertices of G over some fixed field K. The next definition is important for the sequel of the article.

Definition 24([4]) According to the previous context, the **edge ideal** of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with $v_iv_j = v_jv_i$, and with $i \neq j$.

3 Results main involving submodules of the edge ideal of a graph simple

In this section, we presented some results about the modules and submodules, over the polynomial ring $R = K[v_1,v_2,\ldots,v_s]$, where we have that K is a field any, which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and with no isolated vertices. Here, we take K a fixed field and we consider $K[v_1,v_2\ldots,v_s]$ the ring polynomial over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1,\ldots,v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis).

Therefore, the new discovery in the research is to face the edge ideal of a graph as a module, and thus to present results involving this object.

And, in addition, to use the techniques already known from the theory of modules in general for this specific case of graph theory.

Remark 31 By the previous context, $R = K[v_1,v_2\ldots,v_s]$ is a Noetherian ring. Therefore, the edge ideal $I(G)$ is a finitely generated ideal. Thus, the edge ideal $I(G)$ is an R-module, which is finitely generated, and thus $I(G)$ is a Noetherian R-module. So, any submodule of $I(G)$ is finitely generated. In this context, we can to get characterizations for this module under certain hypothesis.

First, we put a proposition about edge dense submodules, which also applies to modules in general, which is used widely in the sequel.

Proposition 32([6, Proposition 8.6]) *Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge* *ideal in R of a finite simple graph G, with no isolated vertices. Let N be a nonzero submodule of $I(G)$. Then the following are equivalent:*

(1.)$N \leq_d I(G)$.
(2.)$\mathrm{Hom}_R(I(G)/N, \mathrm{E}_R(I(G))) = 0$.
(3.)*For any submodule P such that $N \leq P \leq I(G)$,* $\mathrm{Hom}_R(P/N, I(G)) = 0$.

We put now the following definition.

Definition 33 Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. The R-module $I(G)$ is called **edge monoform** if any nonzero submodule of $I(G)$ is an edge dense submodule.

The support of an R-module M is the set of all prime ideals $\mathfrak{p}$ of R such that $M_{\mathfrak{p}} \neq 0$, and it is denoted by $\mathrm{Supp}_R(M)$. Also, let $\mathfrak{p}$ be a prime ideal of R. Thus, $\mathfrak{p}$ is said to be an associated prime ideal of M if $\mathfrak{p}$ is the annihilator of some $x \neq 0$ of M. The set of associated primes of M is denoted by $\mathrm{Ass}_R(M)$. If $\mathfrak{a}$ is an ideal of R, then $\mathrm{V}(\mathfrak{a})$ is the set of all prime ideals of R which contains the ideal $\mathfrak{a}$.

Definition 34 Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. The R-module $I(G)$ is said to be **edge coretractable** if for any proper submodule N of $I(G)$, we have that $\mathrm{Hom}_R(I(G)/N, I(G)) \neq 0$.

We have then the following proposition.

Proposition 35*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Then $I(G)$ has no proper edge dense submodule if and only if $I(G)$ is edge coretractable.*

*Proof.*We have that if $I(G)$ is edge coretractable module, then has no proper edge dense submodule.

Conversely, suppose that $I(G)$ has no proper edge dense submodule and let N be a proper submodule of $I(G)$. Then, we have that

$$\mathrm{Hom}_R(I(G)/N, \mathrm{E}_R(I(G))) \neq 0.$$

So, we have that

$$\mathrm{Supp}_R(I(G)/N) \cap \mathrm{Ass}_R(\mathrm{E}_R(I(G))) = \mathrm{Supp}_R(I(G)/N) \cap \mathrm{Ass}_R(I(G)) \neq \emptyset.$$

Therefore, $\mathrm{Hom}_R(I(G)/N, I(G)) \neq 0$, and hence $I(G)$ is an R-module edge coretractable.

Now, we have the following definition.

Definition 36 Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. The R-module $I(G)$ is called **edge quasi-dedekind** if for any nonzero submodule N of $I(G)$, we have that $\mathrm{Hom}_R(I(G)/N, I(G)) = 0$.

Definition 37 Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. The R-module nonzero $I(G)$ is called **edge uniform** if any two nonzero submodules of $I(G)$ intersect nontrivially.

We presented now the following result.

Theorem 38*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Then the following statements are hold:*

(1)*We have that if the R-module $I(G)$ is edge uniform module then $I(G)$ is primary.*
(2)*The R-module $I(G)$ is edge monoform if and only if it is edge uniform prime.*

Proof.(1) By [2, Corollary 9.35] and [6, Lemma 3.59] we have that, the set of associated primes of the R-module $I(G)$ is such that $|\mathrm{Ass}_R(I(G))| = 1$. Thus, it follows that $I(G)$ is primary, from [9, Theorem 6.6].

(2) ($\Rightarrow$) Let N be a nonzero submodule of the R-module $I(G)$ and $r \in \mathrm{Ann}_R(N)$. Then the map $r : I(G) \to I(G)$ by $x \mapsto rx$ is an R-homomorphism and $N \subseteq \mathrm{Ker}(r)$. So, there exists an R-homomorphism

$$\phi : I(G)/N \to I(G) \text{ such that } \phi(x+N) = rx.$$

From Proposition 32, we have that $\phi = 0$ and this results that we have $r \in \mathrm{Ann}_R(I(G))$.

($\Leftarrow$) Let N be a nonzero submodule of the R-module $I(G)$. Let $x, y \in I(G)$ with $x \neq 0$. We first suppose that $y = 0$. Since $I(G)$ is edge uniform, we have that there exists $f \in R$ such that we have

$$0 \neq fx \in N \text{ and } 0 = fy \in N.$$

Now, let $y \neq 0$. Then, we have that there exists $g \in R$ such that we have

$$0 \neq gy \in N \text{ and } gx \neq 0,$$

since, by hypothesis, we have that $I(G)$ is an R-module prime. So, we conclude that $N \leq_d I(G)$.

The following lemma is used in the sequel of this paper.

Lemma 39*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. In this context, let N be a (nonzero) submodule of the R-module $I(G)$ such that we have, for any $0 \neq x \in I(G)$, $\mathrm{Ann}_R(I(G)/N)$ is not contained in $\mathrm{Ann}_R((x))$. Then, we have that $N \leq_d I(G)$.*

*Proof.*Let $x, y \in I(G)$ with $x \neq 0$. Then, we have that $\mathrm{Ann}_R(I(G)/N)$ is not contained in $\mathrm{Ann}_R((x))$. So, there exists $f \in R$ such that $fI(G) \subseteq N$ and $fx \neq 0$. It follows that $fy \in N$. Then, we have that $N \leq_d I(G)$.

Let N be a submodule of an R-module M and let $\mathfrak{a}$ be an ideal of R. The residual submodule of N by $\mathfrak{a}$ is defined by

$$(N :_M \mathfrak{a}) = \{x \in M \mid x\mathfrak{a} \subseteq N\}.$$

It is a submodule of M containing N (see [2, Definition 6.20]).

Theorem 310 Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. If $\mathfrak{a}$ is an ideal of R such that $(0 :_{I(G)} \mathfrak{a}) = 0$, then we have that $\mathfrak{a}I(G)$ is an edge dense submodule of $I(G)$.*

*Proof.*Let $\mathrm{Ann}_R(I(G)/\mathfrak{a}I(G)) \subseteq \mathrm{Ann}_R((x))$ for some $0 \neq x \in I(G)$. Then, we have that $\mathfrak{a}x = 0$, a contradiction. So, $\mathfrak{a}I(G)$ is an edge dense submodule of $I(G)$, by the Lemma 39.

We have, now, the following result which is valid since $R = K[v_1,\ldots,v_s]$ is a Noetherian ring.

Theorem 311*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Thus, we have that the R-module $I(G)$ is a prime R-module and for any nonzero submodule N of $I(G)$, we have that $\mathrm{Ann}_R(I(G)/N) \neq \mathrm{Ann}_R(I(G))$ if, and only if, $I(G)$ is an edge monoform R-module.*

*Proof.*Let N be a nonzero submodule of $I(G)$ and $0 \neq x \in I(G)$. Then, we have that

$$\mathrm{Ann}_R(I(G)/N) \text{ is not contained in } \mathrm{Ann}_R(I(G)) = \mathrm{Ann}_R((x)).$$

Therefore, we have $N \leq_d I(G)$ by Lemma 39, and hence $I(G)$ is an edge monoform R-module.

Conversely, since R is a Noetherian ring and $I(G)$ is a finitely generated edge monoform R-module we have, by theorem 38 (2), that $I(G)$ is prime. Suppose on the contrary that there exists a submodule N of $I(G)$ such that

$$\mathrm{Ann}_R(I(G)/N) = \mathrm{Ann}_R(I(G)).$$

Then, we have that

$$\mathrm{Ann}_R(I(G)/N) = \mathrm{Ann}_R((x))$$

for all $0 \neq x \in I(G)$. It follows that

$$\mathrm{Ann}_R(I(G)/N) \in \mathrm{Supp}_R(I(G)/N) \cap \mathrm{Ass}_R(I(G)) \neq \emptyset$$

which is a contradiction with results of the theory of modules in general. This completes the proof.

4 Edge dense submodules of edge multiplication modules

In this section we consider the same context of the previous section. Then we put some applications.

We presented now the following definition.

Definition 41 Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. The R-module $I(G)$ is called an **edge multiplication module** if for each submodule N of $I(G)$, we have that $N = \mathfrak{a}I(G)$ for some ideal $\mathfrak{a}$ of R.

Thus, if $I(G)$ is an edge multiplication module, for each submodule N of $I(G)$, we have that $N = (N :_R I(G))I(G)$.

Definition 42 Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. The R-module $I(G)$ is called **edge faithful** if $\mathrm{Ann}_R(I(G)) = 0$.

We presented now, the following theorem which uses the previous concept.

Theorem 43*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that the R-module $I(G)$ is edge faithful and edge multiplication. Let $\mathfrak{a}$ be an ideal of R. Then $\mathfrak{a}$ is an edge dense ideal of R if and only if $\mathfrak{a}I(G)$ is an edge dense submodule of $I(G)$.*

*Proof.*Let $\mathfrak{a}$ be an edge dense ideal of R. By [6, Examples 8.3(4)], we have that $(0 :_R \mathfrak{a}) = 0$. So, in view of [8, Lemma 2.1(1)] we have that

$$(0 :_{I(G)} \mathfrak{a}) = (0 :_R \mathfrak{a})I(G) = 0.$$

Now, by the Theorem 310 we obtain that $\mathfrak{a}I(G)$ is an edge dense submodule of $I(G)$.

Conversely, we assume that $\mathfrak{a}I(G)$ is an edge dense submodule of $I(G)$. In view of [6, Examples 8.3 (4)], it is enough to show that $\mathrm{Ann}_R(\mathfrak{a}) = 0$. Suppose on the contrary that $0 \neq r \in \mathrm{Ann}_R(\mathfrak{a})$. Since $I(G)$ is edge faithful, we have that $rI(G) = 0$. Let $x \in I(G)$ and with $rx \neq 0$. Since $\mathfrak{a}I(G)$ is edge dense submodule, there exists $f \in R$ such that $f(rx) \neq 0$ and $fx \in \mathfrak{a}I(G)$. We have that

$$fx = f_1x_1 + \ldots + f_kx_k$$

for some $f_1,\ldots,f_k$ in $\mathfrak{a}$ and $x_1,\ldots,x_k$ in $I(G)$. It follows that we have then $f(rx) = 0$, a contradiction. This completes the proof.

We finalize the paper with a proposition which involve the previous concepts. And let's use the following: for the R-module $I(G)$, we consider $f \in R$; for convenience, we simply we denote

$$(0 :_R Rf) \text{ and } (0 :_{I(G)} Rf)$$

by, respectively,

$$(0 :_R f) \text{ and } (0 :_{I(G)} f).$$

Proposition 44*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that the R-module $I(G)$ is edge faithful and edge multiplication. Let $\mathfrak{a}$ be an ideal of R, and let $f \in R$. Then the following statements are hold:*

(1.) $\mathfrak{a} = (0 :_R f)$ *if and only if* $\mathfrak{a}I(G) = (0 :_{I(G)} f)$.
(2.) $(0 :_R \mathfrak{a}) = 0$ *if and only if* $(0 :_{I(G)} \mathfrak{a}) = 0$.
(3.) $(0 :_R (0 :_R \mathfrak{a})) = \mathfrak{a}$ *if and only if* $(0 :_{I(G)} (0 :_R \mathfrak{a})) = \mathfrak{a}I(G)$.

Proof. In view of [8, Lemma 2.1 (1)], we have that

$$(0 :_{I(G)} \mathfrak{a}) = (0 :_R \mathfrak{a})I(G),$$

and so the proof it follows by [7, Theorem 3.1]. This completes the proof.

Applications

It is important to mention to the fact that understanding modules and submodules over polynomial rings, leads to different interesting applications in different fields such as entropy [10]- [13], variable thermal conductivity, thermal stability and crystallization kinetics of the semiconducting [14]- [28]. For example, once can improve the controlled model to study the influence of annealing temperature on properties of nanocrystalline CdO thin films synthesized via thermal oxidation process, optical, morphological and thermal properties of spray coated polypyrrole film, the Structural Properties of thermally evaporated aluminium thin films on different polymer substrates etc.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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