Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150504

Iterative Methods for Solving the Fractional form of Axisymmetric Squeezing Fluid Flow between Tow Infinite Parallel Plates with Slip Boundary Conditions

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Received: 1 May 2021, Revised: 2 Jun. 2021, Accepted: 20 Aug. 2021
Published online: 1 Sep. 2021

Abstract: We introduce squeezing flow between two infinite parallel plates with slip boundary conditions. By similarity transformations, the system of nonlinear partial differential equations of motion is reduced to fourth order nonlinear ordinary differential equation. The resulting boundary value fractional problem is solved and the velocity profiles are investigated through various techniques like new iterative method, Picard method and Adomain decomposition method. The comparisons of solutions for different values of the fractional order conform that the three methods are identical and are suitable for solving this kind of problems.

Keywords: Fluid flow; Infinite parallel; Slip boundary conditions; Picard method; Adomain decomposition method.

1 Introduction

The squeezing fluid flow between two parallel infinite plates is a fundamental type of flow that is frequently observed in many injection molding, polymer processing, and modeling of lubrication systems are some practical examples of squeezing flows where their usage is found. The first work in squeezing was laid down by Stefan [1] who developed an adhoc asymptotic solution of Newtonian fluids. An explicit solution of the squeeze flow, considering inertial terms, has been established by Thorpe and Shaw [2]. However, P. S. Gupta and A. S. Gupta [3] proved that the solution given in [2] fails to satisfy boundary conditions. Verma [4] and Singh et al. [5] have established numerical solutions of the squeezing flow between parallel plates. Leider and Byron Bird [6] performed theoretical analysis of power-low fluid between parallel disks . The motion of a thin film of lubricant, squeezed flow between two stationary parallel plane surfaces were reported by Tichy and Winner [7] and Wang and Watson [8]. The theoretical and experimental studies of squeezing flows have been conducted by many researchers [9–13]. The mathematical studies of these flows are concerned primarily with the nonlinear partial differential equations which arise from the Navier-Stokes equations, These equations have no general solutions, and only a few exact solutions have been attained by confining some physical aspects of the original problem [12].

The importance of the study of the fractional forms of the differential equations is due to their wide appearing in many of the mathematical, physical, and chemical problems. So the aim of this work is to continue in this study by preparing and using the NIM, PM, ADH where there is not any of the above-mentioned difficulties in the perturbation methods for solving the fractional order form of an unsteady Axisymmetric squeezing fluid flow between two infinite parallel plates with slip boundaries. Also the effects of the fractional order on the solution are studied tabularly and graphically.

The velocity profile is obtained using various analytical techniques like Adomain decomposition method (ADM), new iterative method (NIM), Optimal homotopy asymptotic method (OHAM), and Picard method (PM) [14–23]. The residual of each technique is computed and a comparison is made to assess the efficiency of the above technique. We select NIM and PM for analyzing the velocity profile under different flow parameters.

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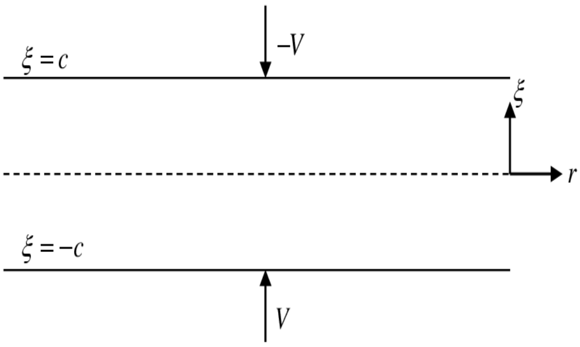
2 Formulation of the Problem

In this section we investigate viscous incompressible Axisymmetric squeezing flow between two infinite parallel plates, separated by a distance $2c$, with density ρ, pressure P and the viscosity η. The plates are moving towards each other with velocity V.



The basic system of equation describing the motion of the fluid is:

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_\xi}{\partial \xi} = 0, \tag{1}$$

$$\frac{\partial \tilde{P}}{\partial r} - \rho \frac{1}{r^2}\frac{\partial \phi}{\partial r} E^2\phi = \frac{-\eta}{r}\frac{\partial}{\partial \xi}E^2\phi, \tag{2}$$

$$\frac{\partial \tilde{P}}{\partial \xi} - \rho \frac{1}{r^2}\frac{\partial \phi}{\partial \xi} E^2\phi = -\frac{\eta}{r}\frac{\partial}{\partial r}E^2\phi, \tag{3}$$

where

$$E^2 = \frac{\partial^2}{\partial r} - \left(\frac{1}{r}\right)\left(\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial \xi^2}, \tag{4}$$

and $\tilde{P} = \frac{\rho}{2}(v_r^2 + v_\xi^2) + P$ is the generalized pressure $\tilde{P}$ from (2) and (3), we get

$$-\rho \frac{\partial(\phi, E^2 \frac{\phi}{r^2})}{\partial(r,\xi)} = \frac{\eta}{r}E^4\phi. \tag{5}$$

The boundary conditions are as follows

$$\begin{aligned} v_r &= \alpha\frac{\partial v_r}{\partial \xi}, \quad v_\xi = -V \quad \text{at} \quad \xi = c, \\ v_\xi &= 0, \qquad \frac{\partial v_r}{\partial v_\xi} = 0 \quad \text{at} \quad \xi = 0. \end{aligned} \tag{6}$$

By virtue of (4) and using the transformation $\phi(r,\xi) = r^2 h(\xi)$, equations (5) and (6) becomes

$$h^{iv}(\xi) + \frac{2\rho}{\eta}h(\xi)h'''(\xi) = 0, \tag{7}$$

$$h(0) = 0, \quad h''(0) = 0, \quad h(c) = \frac{V}{2}, \quad h'(c) = \alpha h''(d). \tag{8}$$

Introducing the following dimensionless parameters

$$h^* = \frac{h}{V/2}, \quad \xi^* = \frac{\xi}{c}, \quad \gamma = \frac{\alpha}{c}, \quad R = \frac{\rho c V}{\eta}, \tag{9}$$

and drooping "$*$" for simplicity, the boundary value problems (**??**)-(9) takes the form

$$\frac{d^4 h}{dF^4} + Rh\frac{d^3 h}{d\xi^3} = 0, \tag{10}$$

with boundary conditions

$$h(0) = 0, \quad h''(0) = 0, \quad h(1) = 1, \quad h'(1) = \gamma h''(1). \tag{11}$$

For more physical explanation and details see [11, 12].

3 Fractional Calculus

In this section, we mention some basic definitions of fractional calculus, which are used in the present work.

Definition 1. The Riemann-Liouville fractional integral operator of order $\beta > 0$, of a function $g(t) \in c_\mu$ and $\mu \geq -1$ is defined as [24]

$$\begin{aligned} J_\xi^\beta g(\xi) &= \frac{1}{\Gamma(\beta)}\int_0^\xi (\xi - v)^\beta g(v)dv, \quad \beta > 0, \xi > 0, \\ J_\xi^0 g(\xi) &= g(\xi). \end{aligned} \tag{12}$$

For the Riemann-Liouville fractional integral operator. J_ξ^β, we obtain

$$J_\xi^\beta \xi^\mu = \frac{\Gamma(\mu+1)\xi^{\mu+\beta}}{\Gamma(\mu+\beta+1)}. \tag{13}$$

Definition 2. The fractional derivative of $g(\xi)$ in the Caputo sense is defined as [25]

$$\begin{aligned} D_\xi^\beta g(\xi) &= J^{m-\beta} D_\xi^m g(\xi) \\ &= \frac{1}{\Gamma - \beta)}\int_0^\xi (\xi - v)^{m-\beta-1} g^{(m)}(v)dv, \end{aligned} \tag{14}$$

for $m-1 < \beta \leq m$, $\xi > 0$. For the Caputo fractional derivative operator D_ξ^β, we obtain

$$D_\xi^\beta \xi^v = \frac{\Gamma(v+1)\xi^{v-\beta}}{\Gamma(v+1-\beta)}. \tag{15}$$

For the Riemann-Liouville fractional integral and Caputo fractional derivative operator of order β, we have the following relation:

$$J_\xi^\beta D_\xi^\beta g(\xi) = g(\xi) - \sum_{k=0}^{m-1} g(k)(0_+)\frac{\xi}{k!}. \tag{16}$$

Remark. According to the previous fractional calculus, (10) can be rewritten in the following fractional order form:

$$\frac{d^\beta h}{d\xi^\beta} + Rh\frac{d^3 h}{d\varepsilon^3},\ 3 < \beta \leq 4. \tag{17}$$

4 Analysis of the Considered Methods

In this section, we discuss the considered methods with preparing them for solving any fractional differential equation.

4.1 New Iterative Method (NIM)

To illustrate the basic idea of this method, we consider the following general functional equations (12, 15, 16):

$$h(\xi) = f(\xi) + N(h(\xi)), \tag{18}$$

where N is a nonlinear operator from a Banach space $B \to B$ and $g(\xi)$ is a known function (element) of a Banach space B. We are looking for a solution $h(\xi)$ having the series form:

$$h(\xi) = \sum_{i=0}^{\infty} h_i(\xi), \tag{19}$$

the nonlinear operator N can be decomposed as:

$$N\left(\sum_{i=0}^{\infty} h_i\right) = N(h_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} h_j\right) - N\left(\sum_{j=0}^{i-1} h_i\right)\right\}. \tag{20}$$

From (19) and (20), Equation (18) is equivalence to:

$$\sum_{i=0}^{\infty} h_i = g + N(h_0) + \sum\left\{N\left(\sum_{j=0}^{i} h_j\right) - N\left(\sum_{j=0}^{i-1} h_j\right)\right\}. \tag{21}$$

The required solution $h(\xi)$ for (18) can be obtained recurrently from the recurrence relation:

$$\begin{aligned} h_0 &= g, \\ h_1 &= N(h_0), \\ h_{r+1} &= N\left(\sum_{i=0}^{r} h_i\right) - N\left(\sum_{i=0}^{r-1} h_i\right),\ r = 1, 2, \ldots. \end{aligned} \tag{22}$$

Then

$$\sum_{i=0}^{\infty} h_i = g + N\left(\sum_{i=0}^{\infty} h_i\right). \tag{23}$$

The r-term approximate solution of (22) is given by

$$h(\xi) = \sum_{i=0}^{r-1} h_i. \tag{24}$$

4.2 solving General Fractional Differential Equation by NIM

To solve any fractional differential equation of arbitrary order $\beta > 0$, we consider the following general fractional differential equation

$$D_\xi^\beta h(\xi) = f(\xi) + L(h(\xi)) + K(h(\xi)), \tag{25a}$$
$$m - 1 < \beta < m,\ m \in N,$$

subject to the initial values

$$\frac{d^k}{d\xi^k} h(0) = d_k,\ k = 0, 1, 2, \ldots, m-1, \tag{25b}$$

where L is a linear operator, K is a nonlinear operator, $f(\xi)$ is a nonhomogeneous term, and D_ξ^β is the fractional differential operator of order $\beta > 0$. In view of the fractional integral operators, the boundary value fractional problem (25a) and (25b) is equivalent to the fractional integral equation:

$$\begin{aligned} h(\xi) &= \sum_{k=0}^{m-1} d_k \cdot \frac{\xi^k}{k!} + J_\xi^\beta[f(\xi)] + J_\xi^\beta[L(h(\xi)) + K(h(\xi))] \\ &= g + N(h), \end{aligned} \tag{26}$$

where $g = \sum_{k=0}^{m-1} d_k(\xi^k/k!) + J_\xi^\beta[f(\xi)]$, $N(h) = J_\xi^\beta [L(h(\xi)) + K(h(\xi))]$, and j_ξ^β is the inverse of D_ξ^β. The required solution $h(\xi)$ for (26) and hence for (25a) and (25b) can be obtained recurrently from the recurrence relation (22).

4.3 Picard Method (PM)

To illustrate the basic idea of this method, we consider the following general fractional differential equation of arbitrary order $\beta > 0$ [26,27]:

$$D_\xi^\beta h(\xi) = F(\xi, h^{(k)}(\xi)), \quad m-1 < \beta \leq m, m \in N, \tag{27a}$$

$$\frac{d^k}{d\xi^k} h(0) = d_k \qquad k = 0, 1, 2, \cdots, m-1, \tag{27b}$$

where D_ξ^β is the fractional differential operator of order $B > 0$. In view of the fractional integral operators, the initial value fractional problems (27a) and (27b) is equivalent to the fractional integral equation:

$$h(\xi) = \sum_{k=0}^{m-1} d_k \frac{\xi^k}{k!}[F(\xi, h_\xi^{(k)}(\xi)] = f + N(h). \tag{28}$$

Where, J_ξ^β is the inverse of D_ξ^β. The required solution $h(\xi)$ for (28) which is solution for (27a and 27b) can be obtained as the limit of a sequence of functions $h_{r+1}(\xi)$ generated by the recurrence relation:

$$\begin{aligned} &h_0 = g, \\ &h_{r+1} = h_0 + N(h_r), \quad r = 0,1,2,\cdots, \end{aligned} \tag{29}$$

where $h(\xi) = \lim_{r\to\infty} h_r(\xi)$.

4.4 Adomian Decomposition Method (ADM)

To illustrate the idea of this method, Let us consider the fractional order problem [12, 14]:

$$D_{\xi}^{\beta} h(\xi) = L(h(\xi)) + N(h(\xi)) + f(\xi), \tag{30a}$$

$m-1 < \beta \leq m, m \in N,$

subject to the initial values

$$\frac{d^k}{d\xi^k} h(0) = d_k,\ k = 0,1,2,\cdots,m-1, \tag{30b}$$

where, L, N are linear and nonlinear operators and $g(\xi)$ is a nonhomogeneous term. The method is based on applying the fractional integral operator J_{ξ}^{β}, the inverse of the fractional differential operator D_{ξ}^{β}, to both sides of (30a) and (30b) to obtain: to obtain:

$$h(\xi) = \sum_{k=0}^{m-1} d_k \cdot \frac{\xi^k}{k!} + J_{\xi}^{\beta}[L(h(\xi)) + N(h(\xi)) + f(\xi)]. \tag{31}$$

The ADM suggests that the solution $h(\xi)$ in the form of infinite series of components:

$$h(\xi) = \sum_{n=0}^{\infty} h_n(\xi), \tag{32}$$

and the nonlinear term $N(h)$ in (30a) is decomposed as:

$$N(h) = \sum_{n=0}^{\infty} A_n, \tag{33}$$

where A_n are the so-called Adomian polynomials given by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\xi^n} N\left(\sum_{k=0}^{n} \xi^k h_k\right)\right]_{\xi=0}. \tag{34}$$

Substituting the decomposition series (32) and (33) into both sides of (31) gives:

$$\sum_{n=0}^{\infty} h_n(\xi) = \sum_{k=0}^{m-1} \cdot \frac{\xi^k}{k!} + J_{\xi}^{\beta}\left[L\sum_{n=0}^{\infty} h_n(\xi) + \sum_{n=0}^{\infty} A_n + f(\xi)\right]. \tag{35}$$

Following the decomposition method, we introduce the recurrence relation as:

$$\begin{aligned} &h_0(\xi) = \sum_{k=0}^{m-1} d_k \cdot \frac{\xi^k}{k!} + J_{\xi}^{\beta}(f(\xi)), \\ &h_{j+1}(\xi) = J_{\xi}^{\beta}[L(h_j(\xi)) + A_j],\ j \geq 0. \end{aligned} \tag{36}$$

This formula is easy to compute. Finally, we approximate the solution $h(\xi)$ by the truncated series:

$$\begin{aligned} &\psi_N(\xi) = \sum_{j=1}^{N-1} h_j(\xi), \\ &\lim_{N\to\infty} \psi_N(\xi) = h(\xi). \end{aligned} \tag{37}$$

5 Applications

In this section, we illustrate the application of the considered methods to solve the nonlinear fractional order ordinary differential equation (17) subject to the boundary conditions (11).

5.1 NIM

According to (26), the boundary value fractional order problem (17)-(11) is equivalent to the fractional integral equation:

$$h(\xi) = a\xi + \frac{b\xi^3}{6} + N(h). \tag{38}$$

Where $N(h) = -J_{\xi}^{\beta}\left[Rh\frac{d^3h}{d\xi^3}\right]$.

Therefore, according to (22), we can obtain the following first few components of the new iterative solution for (35)

$$h_\theta(\xi) = a\xi + \frac{b\xi^3}{6},$$

$$h_1(\xi) = \frac{-6abR\xi^{1+\beta}}{\Gamma(4+\beta)} - \frac{5abR\beta\xi^{1+\beta}}{\Gamma(4+\beta)} - \frac{abR\beta^2\xi^{1+\beta}}{\Gamma(4+\beta)} - \frac{b^2R\xi^{1+\beta}}{\Gamma(4+\beta)},$$

$$\begin{aligned} h_2(\xi) = &-\frac{2^{1-2\beta}a^2b\sqrt{\pi}R^2\xi^{-1+2\beta}}{\Gamma(\beta)\Gamma(\frac{1}{2}+\beta)} \\ &-\frac{2^{1-2\beta}a^2b\sqrt{\pi}R^2\beta\xi^{-1+2\beta}}{\Gamma(\beta)\Gamma(\frac{1}{2}+\beta)} \\ &-\frac{2^{2-2\beta}ab^2\sqrt{\pi}R^2\xi^{1+2\beta}}{3\Gamma(\beta)\Gamma(\frac{3}{2}+\beta)} \\ &+\frac{2^{2-2\beta}ab^2\sqrt{\pi}R^2\beta^2\xi^{1+2\beta}}{3\Gamma(\beta)\Gamma(\frac{3}{2}+\beta)} \end{aligned}$$

$$+\frac{72a^2b^2R^3\beta\xi^{-1+3\beta}\Gamma(2\beta)}{(2+9\beta+9\beta^2)\Gamma(3\beta)\gamma(4+\beta)^2}+\cdots,$$

and so on. Considering the NIM 4^{th} solution, we have the following:

$$h(\xi)=\sum_{i=0}^{3}h_i$$

$$\begin{aligned}&=a\xi+\frac{b\xi^3}{6}-\frac{6abR\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{5abR\beta\xi^{1+\beta}}{\Gamma(4+\beta}-\frac{abR\beta^2\xi^{1+\beta}}{\Gamma(4+\beta)}\\&-\frac{b^2R\xi^{3+\beta}}{\Gamma(4+\beta)}-\frac{2^{1-2\beta}a^2b\sqrt{\pi}R^2\xi^{-1+2\beta}}{\Gamma(\beta)\Gamma(\frac{1}{2}+\beta)}\\&+\frac{2^{1-2\beta}a^2b\sqrt{\pi}R^2\beta\xi^{-1+2\beta}}{\Gamma(\beta)\Gamma(\frac{1}{2}+\beta)}-\frac{2^{2-2\beta}ab^2\sqrt{\pi}R^2\xi^{1+2\beta}}{3\Gamma(\beta)\Gamma(\frac{3}{2}+\beta)}\\&+\frac{2^{2-2\beta}ab^2\sqrt{\pi}R^2\beta^2\xi^{1+2\beta}}{3\Gamma(\beta)\Gamma(\frac{3}{2}+\beta)}+\frac{5a^2b^2R^3\beta^3\xi^{-1+3\beta}}{6(1+2\beta)\Gamma(3\beta)}\\&+\frac{ab^3R^3\beta^3\xi^{1+3\beta}}{18(1+3\beta)\Gamma(3\beta}+\frac{72a^2b^2R^3\beta^3\xi^{-1+3\beta}\Gamma(4+\beta)^2}{(2+9\beta+9\beta^2)\Gamma(4+\beta)^2}\\&+\cdots.\end{aligned}\tag{39}$$

In the special case $\beta=4$, (39) becomes

$$\begin{aligned}h(\xi)=a\xi+\frac{b\xi^3}{6}&-8.33333\times10^{-3}abR\xi^5-1.98413\\&\times10^{-4}b^2R\xi^7+5.9238\times10^{-4}a^2bR^2\xi^7\\&+2.75573\times10^{-5}ab^2R^2\xi^9+1.484648\\&\times10^{-7}a^2b^2R^3\xi^11+1.0706\times10^{-10}ab^3R^3\xi^{13}\\&+\cdots.\end{aligned}$$

The boundary conditions in (11) at $\xi=1$ and $\gamma=1$ are used to get the values of a and b as follows

$$a=0.71897,\qquad b=1.74911.\tag{40}$$

The four-term solution obtained by the NIM for (17)-(11), is therefore

$$\begin{aligned}h(\xi)=&0.71897\xi+0.291518x^3-0.010478\xi^5\\&-0.0000677914\xi^7+0.0000699355\xi^9-4.72198\\&\times10^{-6}\xi^{11}-1.0173\times10^{-6}\xi^{13}-1.41631\\&\times10^{-8}\xi^{15}+1.81369\times10^{-9}\xi^{17}-4.71881\\&\times10^{-11}\xi^{19}-4.6022\times10^{-12}\xi^{21}+2.48384\\&\times10^{-14}\xi^{23}+6.00342\times10^{-15}\xi^25+4.25066\\&\times10^{-17}\xi^{27}-1.84398\times10^{-18}\xi^{29}-2.01692\\&-2.01692\times10^{-20}\xi^{31}.\end{aligned}\tag{41}$$

5.2 PM

Also the boundary value fractional problem (17)-(11), according to (28) and (29), is equivalent to the fractional integral equation:

$$h_{r+1}(\xi)=h_0-J_\xi^\beta\left[Rh\frac{d^3h_r}{d\xi^3}\right],\qquad r=0,1,2,\cdots.\tag{42}$$

Therefore, according to (29), we can obtain the following first few components of Picard solution for (17)-(11):

$$h_0(\xi)=a\xi+\frac{b\xi^3}{6},$$

$$\begin{aligned}h_1(\xi)=&a\xi+\frac{b\xi^3}{6}-\frac{6abR\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{5abR\beta\xi^{\beta+1}}{\Gamma(4+\beta)}\\&-\frac{abR\beta^2\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{b^2R\xi^{3+\beta}}{\Gamma(4+\beta)},\end{aligned}$$

$$\begin{aligned}h_2(\xi)=&a\xi+\frac{b\xi^3}{6}-\frac{2^{1-2\beta}a^2b\sqrt{\pi}R^2\xi^{-1+2\beta}}{\Gamma(\beta)\Gamma(\frac{1}{2}+\beta}\\&+\frac{2^{1-2\beta}a^2b\sqrt{\pi}R^2\beta\xi^{-1+2\beta}}{\Gamma(\beta)\Gamma(\frac{1}{2}+\beta)}\\&-\frac{2^{-2-2\beta}ab^2\sqrt{\pi}R^2\xi^{1+2\beta}}{3\Gamma(\beta)\Gamma(\frac{3}{2}+\beta)}\\&+\frac{72a^2b^2R^3\beta\xi^{-1+3\beta}\Gamma(2\beta)}{(2+9\beta+9\beta^2)\Gamma(3\beta)\Gamma(4+\beta)^2}\\&-\frac{48ab^2R^3\xi^{1+3\beta}\Gamma(2\beta)}{(2+9\beta+9\beta^2)\Gamma(3\beta)\Gamma(4+\beta)^2}\\&-\frac{7b^4R^3\xi^{3+3\beta}\Gamma(2\beta)}{(2+9\beta+9\beta^2)\Gamma(3\beta)\Gamma(4+\beta)^2}+\cdots\end{aligned}$$

and so on. In the same manner, the rest of component can be obtained. The 4-order term solution for (17)-(11) in series form, is given by:

$$\begin{aligned}h(\xi)=&a\xi+\frac{b\xi^3}{6}-\frac{abR\xi^{1+\beta}}{\Gamma(\beta+2)}-\frac{b^2R\xi^{3+\beta}}{\Gamma(\beta+4)}\\&+\frac{5a^2b^2R^3\beta^3\xi^{-1+3\beta}}{6(1+2\beta)\Gamma(3\beta)}\\&\frac{12a^2b^2R^3\xi^{-1+3\beta}}{(1+\beta)(2+\beta)(3+\beta)(1+2\beta)\Gamma(3\beta)}+\cdots.\end{aligned}\tag{43}$$

In the special case $\beta=4$, (43) becomes

$$\begin{aligned}h_3(\xi)=&a\xi+\frac{b\xi^3}{6}-8.33333\times10^{-3}abR\xi^5-1.98413\\&\times10^{-4}b^2R\xi^7+1.48616\times10^{-7}a^2b^2R^3\xi^{11}+\cdots.\end{aligned}$$

On using (40), we get

$$h(\xi)=0.71897\xi+0.291518\xi^3-0.0104796\xi^5$$

$$-0.000068838\xi^7+0.0000701138\xi^9-4.2.198\times 10^{-6}\xi^{11}-1.0173\times 10^{-6}\xi^{13}-1.41631\times 10^{-8}\xi^{15}+1.81369\times 10^{-9}\xi^{17}-4.71881\times 10^{-11}\xi^{19}-4.6022\times 10^{-12}\xi^{21}+2.48384\times 10^{-14}\xi^{23}+6.00352\times 10^{-15}\xi^{25}+4.25067\times 10-17\xi^{27}-1.84398\times 10^{-18}\xi^{29}-2.01692\times 10^{-20}\xi^{31}. \quad (44)$$

5.3 ADM

According to the recurrence relation (36), the initial value fractional order problem (17)-(11) gives:

$$h_\theta(\xi)=a\xi+\frac{b\xi^3}{6},$$

$$h_1=-\frac{6abR\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{5abR\beta\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{abR\beta^2\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{b^2R\xi^{3+\beta}}{\Gamma(4+\beta)},$$

$$h_2(\xi)=\frac{-3\times 2^{2-2\beta}a^2b\sqrt{pi}R^2\beta\xi^{-1+2\beta}}{\Gamma(\frac{1}{2}+\beta)\Gamma(4+\beta)}+\frac{3\times 2^{2-2\beta}ab^2\sqrt{pi}R^2\beta\xi^{1+2\beta}}{(1+2\beta)\Gamma(\frac{1}{2}+\beta)\Gamma(4+\beta)}+\cdots,$$

$$h_3(\xi)=\frac{9\times 2^{3-2\beta}a^2b^2\sqrt{\pi}R^3\xi^{-1-3\beta}\Gamma(2\beta)\Gamma(2+\beta}{(1+2\beta)(3+(2\beta)\Gamma(\beta)\Gamma(3\beta)\Gamma(\frac{1}{2}+\beta)\Gamma(4+\beta)}-2^{2-2\beta}b^4\sqrt{\pi}R^3\xi^{3+3\beta}\Gamma(2\beta)\Gamma(5+\beta)/\Big[(1+2\beta)\cdot(3+2\beta)(1+3\beta)(2+3\beta)\Gamma(3\beta)\Gamma\left(\frac{1}{2}+\beta\right)\cdot\Gamma(4+\beta)^2\Big]+\cdots.$$

Considering the NIM 4^{th} solution, we have the following:

$$h(\xi)=\sum_{i=0}^{3}h_i=a\xi+\frac{b\xi^3}{6}-\frac{6abR\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{5abR\beta\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{abR\beta^2\xi^{1+\beta}}{\Gamma(4+\beta)}-\frac{b^2R\xi^{3+\beta}}{\Gamma(4+\beta)}-\frac{3\times 2^{2-2\beta}a^2b\sqrt{\pi}R^2\beta\xi^{-1+2\beta}}{\Gamma(\frac{1}{2}+\beta)\Gamma(4+\beta)}+\frac{3\times 2^{2-2\beta}ab^2\sqrt{\pi}R^2\beta\xi^{1+2\beta}}{(1+2\beta)\Gamma(\frac{1}{2}+\beta)\Gamma(4+\beta)}+\frac{9\times 2^{3-2\beta}a^2b^2\sqrt{\pi}R^3\Gamma(2\beta)\Gamma(2+\beta)\xi^{-1+3\beta}}{(1+2\beta)(3+2\beta)\Gamma(\beta)\Gamma(3\beta)\Gamma(\frac{1}{2}\beta)\Gamma(4+\beta)}+\cdots. \quad (45)$$

In the special case, $\beta=4$, (45) becomes

$$h(\xi)=a\xi+\frac{b\xi^3}{6}-8.33333\times 10^{-3}abR\xi^5-1.98413\times 10^{-4}b^2R\xi^7+1.57470\times 10^{-7}ab^2R^2\xi^9+2.16901\times 10^{-10}a^2b^2R^3\xi^{11}+\cdots.$$

On using (40) the 4-order term solution for $R=1$ is

$$h(\xi)=7.1897\times 10^{-1}\xi+2.91518\times 10^{-1}\xi^3-1.04796\times 10^{-2}\xi^5-6.88387\times 10^{-5}\xi^7+7.01138\times 10^{-5}\xi^9-3.49258\times 10^{-6}\xi^{11}-4.22954\times 10^{-7}\xi^{13}+1.04129\times 10^{-7}\xi^{16}+1.32402\times 10^{-8}\xi^{-17}+4.22479\times 10^{-10}\xi^{19}. \quad (46)$$

From the previous results for (17)-(11), obtained by the three considered methods, it is clear that the approximate solutions obtained by NIM in (39), PM in (43) and ADM in (44) are approximately the same and these methods are suitable for solving this kind of problems.

The residual error of the problem is:

$$\mathrm{Re}h(\xi)=\text{Residual Error}=\frac{d^\beta\bar{h}}{d\xi^\beta}+R\bar{h}\frac{d^3\bar{h}}{d\xi^3}, \quad (47)$$

where $\bar{h}$ is the 4-term approximate solutions in (39) or (43) and (45).

If $\mathrm{Re}=0$, then $\bar{h}$ will be the exact solution. However, this usually does not occur in nonlinear problems.

It is clear from the obtained results that the above-considered methods minimize the limitations of the ordinary perturbation methods. In the same time, these methods take full advantages of the traditional perturbation methods. Therefore, these methods are powerful methods for solving the nonlinear fractional order differential equations.

6 Numerical Results and Discussion

In this work, an axisymmetric flow between two infinite plates is considered. The resulting nonlinear fractional order boundary value problem is solved analytically in case of slip boundary conditions using NIM, PM, and ADM.

Table 1 present the solutions for different values of the fractional order β along with absolute residual errors $|\mathrm{Re}|$ at $\beta=4$ for fixed value of the Reynolds number R.

Figures 1 and 2 indicate the approximate solutions for different values of $\beta = 3.7, 3.8, 3.9, 4.0, 4.1$ at $R = \gamma = 1$ in cases of NIM, PM and ADM. In addition to the above figures, figures 3, 4 represent the r and ξ component for the velocity profile at $R = 1$ given by NIM and PM.

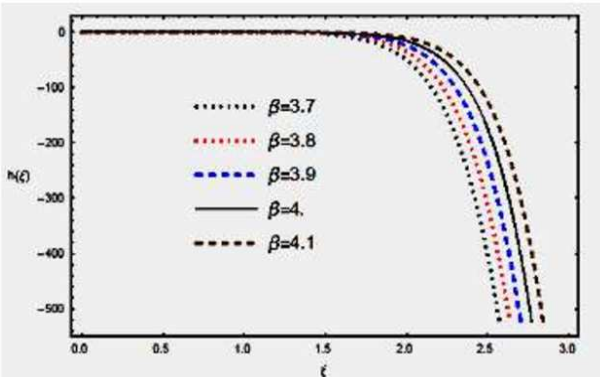


Fig. 1: Solutions for $\beta = 3.7, 3.8, 3.9, 4.0, 4.1$ at $R = \gamma = 1$ using NIM, PM.

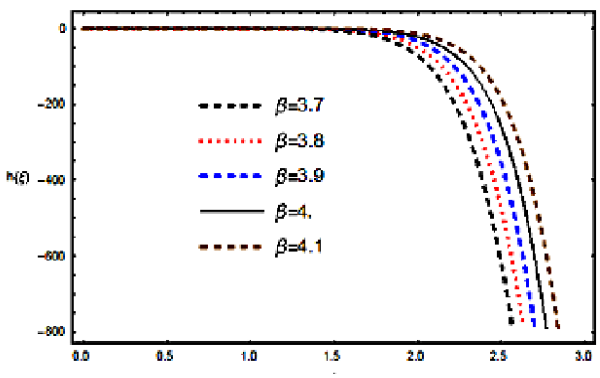


Fig. 2: Solutions for $\beta = 3.7, 3.8, 3.9, 4.0, 4.1$ at $R = \gamma = 1$ using ADM.

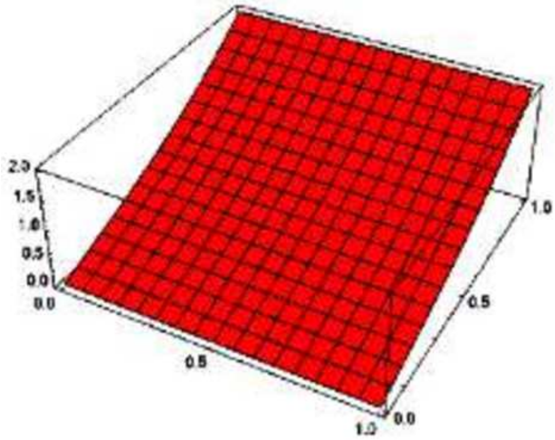


Fig. 3: ξ-component $V_{\xi}(r, \theta, \xi) = -\frac{1}{r}\frac{\partial \phi}{\partial \xi}$ of the velocity profile given by NIM and PM at $R = \gamma = 1$.

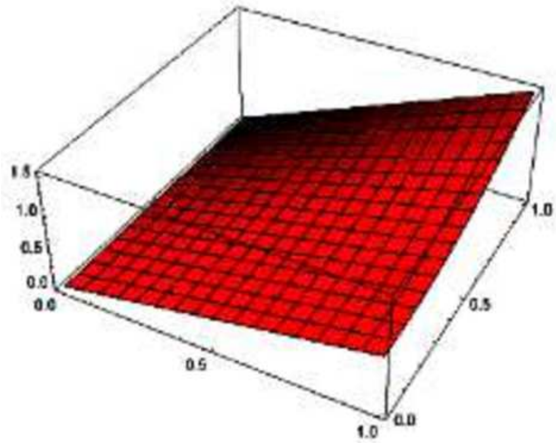


Fig. 4: r-component $V_{\xi}(r, \theta, \xi) = \frac{1}{r}\frac{\partial \phi}{\partial \xi}$ of the velocity profile given by NIM and PM at $R = \gamma = 1$.

7 Conclusion

In this work, an analytical solution for an axisymmetric squeezing flow between to infinite parallel plates in fractional form is obtained using the NIM, PM and ADM in cases of slip boundary condition. Analysis of the residual errors confirms that the NIM and PM are identical and efficient schemes. The convergence of the considered methods is confirmed by absolute residual errors. Therefore, we concluded that the considered methods can be effectively used in various fields of science and engineering as they give better results in terms of accuracy.

Acknowledgement

The authors gratefully thank to Editor-in-Chief and the anonymous referees for their helpful and suggestive comments.

Table 1: Solutions for different values of ξ and β at $R = 1$.

ξ	$\beta = 3.7$		$\beta = 3.8$		$\beta = 3.9$	
	NIM & PM	ADM	NIM & PM	ADM	NIM & PM	ADM
0.0	0.00	0.00	0.00	0.00	0.00	0.00
0.1	7.21882×10^{-2}	7.21882×10^{-2}	7.1883×10^{-2}	7.21883×10^{-2}	7.21884×10^{-2}	7.21884×10^{-2}
0.2	1.46117×10^{-1}	1.46117×10^{-1}	1.46120×10^{-1}	1.46120×10^{-1}	1.46121×10^{-1}	1.46121×10^{-1}
0.3	2.23502×10^{-1}	2.23502×10^{-1}	2.23517×10^{-1}	2.23517×10^{-1}	2.23528×10^{-1}	2.23528×10^{-1}
0.4	3.06014×10^{-1}	3.06014×10^{-1}	3.06065×10^{-1}	3.06065×10^{-1}	3.06106×10^{-1}	3.06106×10^{-1}
0.5	3.95266×10^{-1}	3.95266×10^{-1}	3.92401×10^{-1}	3.92401×10^{-1}	3.95510×10^{-1}	3.95510×10^{-1}
0.6	4.92804×10^{-1}	4.92804×10^{-1}	4.93096×10^{-1}	4.93096×10^{-1}	4.93337×10^{-1}	4.93337×10^{-1}
0.7	6.00092×10^{-1}	6.00093×10^{-1}	6.00649×10^{-1}	6.00649×10^{-1}	6.01116×10^{-1}	6.01116×10^{-1}
0.8	7.18508×10^{-1}	7.18510×10^{-1}	7.19473×10^{-1}	7.19474×10^{-1}	7.20296×10^{-1}	7.20296×10^{-1}
0.9	8.49330×10^{-1}	8.49338×10^{-1}	8.50886×10^{-1}	8.50890×10^{-1}	8.52234×10^{-1}	8.52235×10^{-1}
1.0	9.93737×10^{-1}	9.93761×10^{-1}	9.93761×10^{-1}	9.96118×10^{-1}	9.98188×10^{-1}	9.98192×10^{-1}

Table 1: Continue

ξ	$\beta = 4.0$		$\beta = 4.1$		$Res_{\beta=4}$	
	NIM & PM	ADM	NIM & PM	ADM	NIM & PM	ADM
0.0	0.00	0.00	0.00	0.00	0.00	0.00
0.1	7.21884×10^{-2}	7.21884×10^{-2}	7.21884×10^{-2}	7.21884×10^{-2}	-9.8392×10^{-20}	8.89760×10^{-13}
0.2	1.46123×10^{-1}	1.46123×10^{-1}	1.46124×10^{-1}	1.46124×10^{-1}	-1.29926×10^{-7}	4.78321×10^{-10}
0.3	2.23537×10^{-1}	2.23537×10^{-1}	2.23543×10^{-1}	2.23543×10^{-1}	-2.33646×10^{-6}	21.99410×10^{-8}
0.4	3.06138×10^{-1}	3.06138×10^{-1}	3.06162×10^{-1}	3.06162×10^{-1}	-1.87760×10^{-5}	2.961550×10^{-7}
0.5	3.95597×10^{-1}	3.95597×10^{-1}	3.95666×10^{-1}	3.95666×10^{-1}	-9.77465×10^{-5}	2.523930×10^{-6}
0.6	4.93534×10^{-1}	4.93534×10^{-1}	4.93694×10^{-1}	4.93694×10^{-1}	-3.88561×10^{-4}	1.522890×10^{-5}
0.7	6.01506×10^{-1}	6.01506×10^{-1}	6.01829×10^{-1}	6.01829×10^{-1}	-1.28640×10^{-3}	7.265870×10^{-5}
0.8	7.20994×10^{-1}	7.20994×10^{-1}	7.21583×10^{-1}	7.21584×10^{-1}	-3.73261×10^{-3}	2.924660×10^{-4}
0.9	8.53394×10^{-1}	8.53395×10^{-1}	8.54389×10^{-1}	8.54389×10^{-1}	-9.8035×10^{-3}	1.034700×10^{-3}
1.0	1.00000	1.00001	1.00158	1.00158	-2.38132×10^{-2}	3.306640×10^{-3}

Conflict of Interest

The authors declare that they have no conflict of interest.

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