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Local Well-Posedness for a Generalized Integrable Shallow Water Equation with Strong Dispersive Term

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Abstract: In this paper, we consider a kind of new nonlinear dispersive wave equation, which is generalized integrable shallow water equation with strong dispersive term. Applying Kato's semigroup approch, we obtained local well-posedness of Cauchy problem for the generalized integrable shallow water equation with strong dispersive term in Sobolev space $(H^s, s > (3/2))$.

Keywords: Integrable Shallow Water Equation, Strong Dispersive Term, Well-posedness, Kato's theory.

1 Introduction

Shallow water waves and their model equations are very important for mathematical and physical theory. Furthermore, water wave modeling is a complicated process and usually leads to models which are hard be analyzed mathematically and to be solved numerically [1].

Although the history of shallow water waves [2] goes back to the French and British mathematicians of the eighteenth and early nineteenth century, Stokes [3] is considered one of the pioneers of hydrodynamics [4]. He carefully derived the equations for the motion of incompressible, inviscid fluid, subject to a constant vertical gravitational force, where the fluid is bounded below by an impermeable bottom and above by a free surface [5]. Starting from these fundamental equations and by making further simplifying assumptions, various shallow water wave models can be derived. These shallow water models are widely used in oceanography and atmospheric science [5].

In this paper, we consider the Cauchy problem for the generalized integrable shallow water equation with strong dispersive term:

$$\begin{cases} u_t - \alpha^2 u_{xxt} + (g(u))_x + \gamma\left(u - \alpha^2 u_{xx}\right)_{xxx} = \\ \alpha^2 \left(\frac{h'(u)}{2} u_x^2 + h(u)\, u_{xx}\right)_x, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \end{cases} \tag{1}$$

where $g(u),\ h(u) : \mathbb{R} \to \mathbb{R}$ are given function, α and γ are constants. For $g(u) = 2\omega u + \frac{3}{2}u^2$ and $h(u) = u$, Eq.(1) takes the following form:

$$\begin{cases} u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma\left(u - \alpha^2 u_{xx}\right)_{xxx} = \\ \alpha^2 \left(2u_x u_{xx} + uu_{xxx}\right). \end{cases} \tag{2}$$

The strong dispersive term $\gamma\left(u - \alpha^2 u_{xx}\right)_{xxx}$ corresponds to the Lagrangian averaged Navier-Stokes alpha equations for turbulence and can provide analytical control over the solutions [6]. In [6], Tian et al. studied the well-posedness by applying Kato's semigroup approach. Moreover, they got the precise blow-up scenario and gave an explosion criterion of strong solutions of Eq.(2) with rather general initial data.

If $\alpha = 1$, $\gamma = 1$ and $\omega = 0$, Eq. (2) becomes the following fifth-order shallow water equation

$$u_t - u_{xxt} + u_{xxx} + 3uu_x - u_{xxxxx} = 2u_x u_{xx} + uu_{xxx} \tag{3}$$

which is a higher-order modification of the following Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

The well-posedness of the Cauchy problem of Eq. (3) in Sobolev spaces has been studied by several authors (see [7,8,9] and the references therein).

In Eq. (2), if the strong dispersive term $\gamma\left(u - \alpha^2 u_{xx}\right)_{xxx}$ is rewritten as the weak dispersive term

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γu_{xxx}, Eq. (2) becomes the following Dullin-Gottwald-Holm equation

$$\begin{cases} u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \\ \alpha^2 \left(2u_x u_{xx} + uu_{xxx}\right), \quad t > 0,\ x \in \mathbb{R} \end{cases} \tag{4}$$

which was derived by Dullin, Gottwald and Holm using asymptotic expansions directly in the Hamiltonian for Euler's equations in the shallow water regime in [10]. It is a model for unidirectional shallow water waves over a flat bottom. Here, the constants α^2 and $\frac{\gamma}{2\omega}$ are squares of length scales, and ω is the linear wave speed for undisturbed water at rest at spetial infinity. It has a bi-Hamiltonian structure and a Lax pair formulation (see [10]). Dullin-Gottwald-Holm equation (we call it DGH equation for short) is an integrable system via the inverse scattering transform (IST) method and contains both the Korteweg-de Vries (KdV) and Camassa-Holm (CH) equations [11] as limiting cases.

Recently, many papers have been conducted on the DGH equation. In [12], Tian et al. studied the well-posedness of the Cauchy problem and the scattering problem for the DGH equation. In [13], Hakkaev proved the orbital stability of the peaked solitary waves for the DGH equation using the method in [14]. It has been shown that DGH equation has global solutions and blow-up solutions in [15,16,17,18].

Recently, the Cauchy problem of the generalized DGH equation has been investigated in [19,20]. The well-posedness of problem for the following generalization of the DGH equation

$$\begin{cases} u_t - \alpha^2 u_{xxt} + h(u)_x + \gamma u_{xxx} = \\ \alpha^2 \left(\frac{g'(u)}{2} u_x^2 + g(u) u_{xx}\right)_x, \quad t > 0,\ x \in \mathbb{R} \end{cases} \tag{5}$$

has been studied in [20]. The difference between Eq.(1) and Eq.(5) is that Eq. (1) contains the strong dispersive term $\gamma\left(u - \alpha^2 u_{xx}\right)_{xxx}$. Thus, Eq.(1) involves higher-order derivative than Eq.(5). However, the Cauchy problem of the generalized integrable shallow water equation with strong dispersive term has not been discussed yet. The present paper aims to establish the local well-posedness of Eq. (1). The result in this paper generalizes the local well-posedness results in [6,20]. Moreover, using Lemma 2.4 and Lemma 2.5, we obtained the local well-posedness of Eq. (1) for $g,h \in C^{[s]+1}(\mathbb{R})$, $s > \frac{3}{2}$, but the local well-posedness of Eq. (5) was obtained for $g,h \in C^{m+3}(\mathbb{R})$, $m \geq 2$ in [20].

For convenience, we now reformulate our problem (1). Rescalling $x \mapsto \frac{x}{\alpha}$, we rewrite (1) as

$$\begin{cases} u_t - u_{xxt} + (g(u))_x + a(u - u_{xx})_{xxx} = \\ b\left(\frac{h'(u)}{2} u_x^2 + h(u) u_{xx}\right)_x, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{6}$$

where $a = \frac{\gamma}{\alpha^3}$ and $b = \frac{1}{\alpha}$.

We apply Kato's theory to establish the local well-posedness for the Cauchy problem of (1). We now provide the framework in which we shall reformulate (6). Note that if $p(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $\left(1-\partial_x^2\right)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$. Here, we denote by $*$ the convolution. If we denote $P(D)$ as the Fourier integral operator with the Fourier multiplier $-i\xi\left(1+\xi^2\right)^{-1}$ and $Q(D) = \left(1+\xi^2\right)^{-1}$, then we can rewrite (6), as follows:

$$\begin{cases} u_t + bh(u)u_x + au_{xxx} = P(D)\left(g(u) + \frac{b}{2}h'(u)u_x^2\right) \\ \quad +Q(D)\left(bh(u)u_x\right), \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{7}$$

Or in the equivalent form

$$\begin{cases} u_t + bh(u)u_x + au_{xxx} = -\partial_x \Lambda^{-2}\left(g(u) + \frac{b}{2}h'(u)u_x^2\right) \\ \quad +\Lambda^{-2}\left(bh(u)u_x\right), \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{8}$$

where $\Lambda = \left(1-\partial_x^2\right)^{1/2}$.

The main results:

Theorem 1.1. Assume that $g,h \in C^{[s]+1}(\mathbb{R})$, and $h(0) = g(0) = 0$. Given $u_0 \in H^s$, $s > \frac{3}{2}$, there exists a maximal $T = T(u_0) > 0$, and a unique solution u to (1) (or (8)) such that

$$u = u(.,u_0) \in C([0,T);H^s) \cap C^1\left([0,T);H^{s-1}\right).$$

Moreover, the solution depends continuously on the initial data, i.e the mapping

$$u_0 \to u(.,u_0) : H^s \to C([0,T);H^s) \cap C^1\left([0,T);H^{s-1}\right)$$

is continuous.

Theorem 1.2. Assume that $g,h \in C^{[s]+1}(\mathbb{R})$, and $h(0) = g(0) = 0$. Let $u_0 \in H^s$, $s > \frac{3}{2}$. Then, T in Theorem 1.1 may be chosen independent of s in the following sense. If $u = u(.,u_0) \in C([0,T);H^s) \cap C^1\left([0,T);H^{s-1}\right)$ to (1) (or (8)), and if $u_0 \in H^{s'}$ for some $s' \neq s$, $\frac{3}{2} < s'$, then $u \in C\left([0,T);H^{s'}\right) \cap C^1\left([0,T);H^{s'-1}\right)$ with the same T. In particular, if $g,h \in C^\infty(\mathbb{R})$ and $u_0 \in H^\infty = \cap_{s\geq 0} H^s$, then $u \in C([0,T);H^\infty)$.

The remainder of the paper is organized, as follows: In section 2, we present our basic notation and recall some required results. In section 3, by applying Kato's theory, the local well-posedness of the generalized integrable shallow water equation with strong dispersive term is investigated.

2 Preliminaries

First, we introduce some notations. $\Lambda^s = \left(1-\partial_x^2\right)^{s/2}$, $s \in \mathbb{R}$; $H^s = H^s(\mathbb{R})$ with norm

$$\|f\|_{H^s} = \|f\|_s = \left(\int_{\mathbb{R}} \left(1+|\xi|^2\right)^s \left|\widehat{f}(\xi)\right|^2 d\xi\right)^{1/2}$$

and $(.,.)$ for its inner product. $[A,B]=AB-BA$ denotes the commutator of the linear operators A and B. For the sake of simplicity, we will employ the same symbols c for different positive constants.

We will apply Kato's theory to establish the local well-posedness for the Cauchy problem of Eq. (1). Consider the abstract quasi-linear evolution equation:

$$\frac{dv}{dt}+A(v)v=f(v), \quad t\geq 0, \quad v(0)=v_0. \tag{9}$$

Let X and Y be two Hilbert spaces such that Y is continuously and densely embedded in X and let $Q:Y\to X$ be a topological isomorphism. $L(Y,X)$ denotes the space of all bounded linear operators from Y to X ($L(X)$, if $X=Y$). Assume that:

(I) $A(y)\in L(Y,X)$ for $y\in Y$ with

$$\|(A(y)-A(z))\,\omega\|_X\leq \kappa_1\|y-z\|_X\|\omega\|_Y, \quad y,z,\omega\in Y,$$

and $A(y)\in G(X,1,\beta)$, (i.e. $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(II) $QA(y)Q^{-1}=A(y)+B(y)$ where $B(y)\in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\|(B(y)-B(z))\,\omega\|_X\leq \kappa_2\|y-z\|_Y\|\omega\|_X, \quad y,z\in Y,\omega\in X.$$

(III) $f:Y\to Y$ and also extends to a map from X into X. f is bounded on bounded sets in Y, and satisfies

$$\|(f(y)-f(z))\|_Y\leq \kappa_3\|y-z\|_Y, \quad y,z\in Y,$$
$$\|(f(y)-f(z))\|_X\leq \kappa_4\|y-z\|_X, \quad y,z\in Y.$$

Here, $\kappa_1,\kappa_2,\kappa_3$ and κ_4 depend only on $\max\{\|y\|_Y,\|z\|_Y\}$.

Theorem 2.1. [21] Assume that (I), (II) and (III) hold. Given $v_0\in Y$, there is a $T>0$ depending only on $\|v_0\|_Y$ and a unique solution v to Eq. (9) such that

$$v=v(.,v_0)\in C([0,T);Y)\cap C^1([0,T);X).$$

Moreover, the map $v_0\to v(.,v_0)$ is continuous from Y to $C([0,T);Y)\cap C^1([0,T);X)$.

Some useful lemmas:

Lemma 2.1. [22] Let $f\in H^s$, $s>\frac{3}{2}$. Then,

$$\left\|\Lambda^{-r}\left[\Lambda^{r+t+1},M_f\right]\Lambda^{-t}\right\|_{L(L^2)}\leq c\|f\|_s, \quad |r|,|t|\leq s-1,$$

where M_f is the operator of multiplication by f and c is a positive constant depending only on r,t.

Lemma 2.2. [21] Let r,t be any real numbers such that $-r<t\leq r$. Then,

$$\|fg\|_t\leq c\|f\|_r\|g\|_t, \quad \text{if } r>\frac{1}{2},$$
$$\|fg\|_{r+t-\frac{1}{2}}\leq c\|f\|_r\|g\|_t, \quad \text{if } r<\frac{1}{2}$$

where c is a positive constant depending only on r,t.

Lemma 2.3. [23] Let X and Y be two Banach spaces and Y be continuously and densely embedded in X. Let $-A$ be the infinitesimal generator of the C_0-semigroup $T(t)$ on X and S be an isomorphism from Y onto X. Y is $-A$-admissible (i.e $T(t)Y\subset Y$, $\forall t\geq 0$, and the restriction of $T(t)$ to Y is a C_0-semigroup on Y) if and only if $-A_1=-SAS^{-1}$ is the infinitesimal generator of the C_0-semigroup $T_1(t)=-ST(t)S^{-1}$ on X. Moreover, if Y is $-A$-admissible, then the part of $-A$ in Y is the infinitesimal generator of the restriction of $T(t)$ to Y.

Lemma 2.4. [24] Let $f\in C^{[s]+1}(\mathbb{R})$, $s\geq 0$, with $f(0)=0$.Then, for any $R>0$, there is some constant $C_1(R)$ such that for all $u\in H^s\cap L^\infty$ with $\|u\|_{L^\infty}\leq R$, we have

$$\|f(u)\|_s\leq C_1(R)\|u\|_s.$$

Lemma 2.5. [24,25] Let $f\in C^{[s]+1}(\mathbb{R})$, $s\geq 0$. Then, for any $R>0$, there is some constant $C_2(R)$ such that for all $u,v\in H^s\cap L^\infty$ with $\|u\|_{L^\infty}\leq R$, $\|v\|_{L^\infty}\leq R$ and $\|u\|_s\leq R$, $\|v\|_s\leq R$, we have

$$\|f(u)-f(v)\|_s\leq C_2(R)\|u-v\|_s.$$

3 Local well-posedness

In this section, we prove Theorem 1.1 and Theorem 1.2.

To shorten our notation, we write $G(X,1,\beta)$ for the set of all linear operators A in X, such that $-A$ generates a C_0-semigroup $T(t)$ on X and that $\|T(t)\|_{L(X)}\leq e^{t\beta}$ for all $t\geq 0$. We will apply Theorem 2.1 with

$$A(u)=bh(u)\partial_x+a\partial_x^3,$$
$$f(u)=-\partial_x\Lambda^{-2}\left(g(u)+\frac{b}{2}h'(u)u_x^2\right)+\Lambda^{-2}(bh(u)u_x),$$

$Y=H^s$, $X=H^{s-1}$ and $Q=\Lambda=\left(1-\partial_x^2\right)^{\frac{1}{2}}$. We know that Q is an isomorphism of H^s onto H^{s-1}. To prove Theorem 1.1, by applying Theorem 2.1, we only need to check that $A(u)$ and $f(u)$ satisfy assumptions (I)–(III).

We break the argument into several lemmas.

Lemma 3.1. The operator $A(u)=bh(u)\partial_x+a\partial_x^3$, with $u\in H^s$, $s>\frac{3}{2}$ belongs to $G\left(L^2,1,\beta\right)$.

Proof. From [26], $A(u)\in G\left(L^2,1,\beta\right)$ if and only if there is a real number β such that

(i) $(A(u)y,y)_0\geq -\beta\|y\|_0^2$,

(ii) The range of $\lambda I+A$ is all of X, for some (or all) $\lambda>\beta$.

We first prove (i). Since $u\in H^s$, $s>\frac{3}{2}$, u and u_x belongs to L^∞. Note that $\|u_x\|_{L^\infty}\leq\|u\|_s$ and $\|u\|_{L^\infty}\leq\|u\|_s$. Thus,

$$\begin{aligned}|(A(u)y,y)_0| &= |((bh(u))\partial_x y,y)_0|+\left|\left(a\partial_x^3y,y\right)_0\right|\\ &\leq\left|\frac{b}{2}\|\partial_x h(u)\|_{L^\infty}\|y\|_0^2\right|\\ &\leq\left|\frac{b}{2}\|h(u)\|_s\|y\|_0^2\right|\\ &\leq cC_1(R)\|u\|_s\|y\|_0^2.\end{aligned}$$

Setting $\beta = cC_1(R)\|u\|_s$, we obtain $(A(u)y,y)_0 \geq -\beta\|y\|_0^2$. Next, we prove that $A(u)$ satisfies (ii). Because $A(u)$ is a closed operator and satisfies (i), $(\lambda I + A)$ has closed range in L^2 for all $\lambda > \beta$. Therefore, it suffices to show that $(\lambda I + A)$ has dense range in L^2 for all $\lambda > \beta$. Given $u \in H^s$, $s > \frac{3}{2}$ and $y \in L^2$. We have the generalized Leibniz formula

$$\partial_x\left(\left(bh(u) + a\partial_x^2\right)y\right) = \partial_x\left(bh(u) + a\partial_x^2\right)y + \left(bh(u) + a\partial_x^2\right)\partial_x y \quad \text{in } H^{-1}.$$

Due to u, $u_x \in L^\infty$, we have

$$\begin{aligned} D(A) &= D\left((bh(u))\partial_x + a\partial_x^3\right) \\ &= \left\{y \in L^2, (bh(u))\partial_x y + a\partial_x^3 y \in L^2\right\} \\ &= \left\{z \in L^2, -\partial_x\left(bh(u) + a\partial_x^2\right)z \in L^2\right\} \\ &= D\left(\left((bh(u))\partial_x + a\partial_x^3\right)^*\right) = D(A^*). \end{aligned}$$

Assume that the range of $(\lambda I + A)$ is not all of L^2, then there exists $z \in L^2$, $z \neq 0$, such that $((\lambda I + A)y, z)_0 = 0$, $\forall y \in D(A)$. Since $H^1 \subset D(A)$, we have that $D(A)$ is dense in L^2. Hence, it follows that $z \in D(A^*)$ and $\lambda z + A^* = 0$ in L^2. Since $D(A) = D(A^*)$, multiplying by z and integrating by parts, we obtain

$$\begin{aligned} 0 &= ((\lambda I + A^*)z, z)_0 \\ &= (\lambda z, z)_0 + (z, Az)_0 \geq (\lambda - \beta)\|z\|_0^2, \quad \forall \lambda > \beta, \end{aligned}$$

and so $z = 0$, which contradicts our assumption $z \neq 0$. This completes the proof of Lemma 3.1.

Lemma 3.2. The operator $A(u) = bh(u)\partial_x + a\partial_x^3$, with $u \in H^s$, $s > \frac{3}{2}$ belongs to $G\left(H^{s-1}, 1, \beta\right)$.

Proof. Since H^{s-1} is a Hilbert space, $A(u)$ belongs to $G(H^{s-1}, 1, \beta)$ for some real number β if and only if the following conditions hold [21]:

(i) $(A(u)y, y)_{s-1} \geq -\beta\|y\|_{s-1}^2$.

(ii) $-A(u)$ is the infinitesimal generator of a C_0-semigroup on H^{s-1} for some (or all) $\lambda > \beta$.

Using the equalities

$$\left(\Lambda^{s-1}a\partial_x^3 y, \Lambda^{s-1}y\right)_0 = 0, \tag{10}$$

$$\Lambda^{s-1}(bh(u)\partial_x y) = \left[\Lambda^{s-1}, bh(u)\right]\partial_x y + bh(u)\Lambda^{s-1}\partial_x y$$

we have

$$\begin{aligned} \left|(A(u)y,y)_{s-1}\right| &= \left|\left(\Lambda^{s-1}\left(\left(bh(u)\partial_x + a\partial_x^3\right)y\right), \Lambda^{s-1}y\right)_0\right| \\ &= \left|\left(\Lambda^{s-1}bh(u)\partial_x y, \Lambda^{s-1}y\right)_0\right| \\ &= \left|\left(\left[\Lambda^{s-1}, bh(u)\right]\partial_x y, \Lambda^{s-1}y\right)_0 + \left(bh(u)\Lambda^{s-1}\partial_x y, \Lambda^{s-1}y\right)_0\right| \\ &\leq \left|b\left(\left[\Lambda^{s-1}, h(u)\right]\partial_x\Lambda^{1-s}\Lambda^{s-1}y, \Lambda^{s-1}y\right)_0\right| + \left|\frac{b}{2}\left(\partial_x h(u), \left(\Lambda^{s-1}y\right)^2\right)_0\right| \\ &\leq \left\|b\left[\Lambda^{s-1}, h(u)\right]\Lambda^{2-s}\right\|_{L(L^2)}\left\|\Lambda^{s-1}y\right\|_0^2 + \left\|\frac{b}{2}\partial_x h(u)\right\|_{L^\infty}\left\|\left(\Lambda^{s-1}y\right)\right\|_0^2. \end{aligned}$$

From Lemma 2.1 with $r = 0$, $t = s-2$ and Lemma 2.4, we obtain

$$\begin{aligned} \left|(A(u)y,y)_{s-1}\right| &\leq c\left(\|h(u)\|_s + \|\partial_x h(u)\|_{L^\infty}\right)\|y\|_{s-1}^2 \\ &\leq c\left(\|h(u)\|_s + \|h(u)\|_s\right)\|y\|_{s-1}^2 \\ &\leq cC_1(R)\|u\|_s\|y\|_{s-1}^2. \end{aligned}$$

Choosing $\beta = cC_1(R)\|u\|_s$, we have $(A(u)y,y)_{s-1} \geq -\beta\|y\|_{s-1}^2$. Next, we prove (ii). Consider $S = \Lambda^{s-1}$. Note that S is an isomorphism of H^{s-1} onto L^2, and H^{s-1} is continuously and densely embedded in L^2 as $s > \frac{3}{2}$. Define

$$\begin{aligned} A_1(u) &= SA(u)S^{-1} = \Lambda^{s-1}A(u)\Lambda^{1-s}, \\ B_1(u) &= A_1(u) - A(u). \end{aligned}$$

Let $y \in L^2$ and $u \in H^s$. Moreover, we have $\left[\Lambda^{s-1}, (bh(u))\partial_x\right]\Lambda^{1-s} = \left[\Lambda^{s-1}, (bh(u))\right]\Lambda^{1-s}\partial_x$. Then, we obtain (note that (10)= 0)

$$\begin{aligned} \|B_1(u)y\|_0 &= \left\|\left[\Lambda^{s-1}, bh(u)\partial_x\right]\Lambda^{1-s}y\right\|_0 \\ &= \left\|\left[\Lambda^{s-1}, bh(u)\right]\Lambda^{1-s}\partial_x y\right\|_0 \\ &\leq \left\|b\left[\Lambda^{s-1}, h(u)\right]\Lambda^{2-s}\right\|_{L(L^2)}\left\|\Lambda^{-1}\partial_x y\right\|_0 \\ &\leq c\|h(u)\|_s\|y\|_0 \leq cC_1(R)\|u\|_s\|y\|_0, \end{aligned}$$

where we used Lemma 2.1 with $r = 0$, $t = s-2$ and Lemma 2.4. Hence, $B_1(u) \in L\left(L^2\right)$. Note that $A_1(u) = A(u) + B_1(u)$ and $A(u) \in G\left(L^2, 1, \beta\right)$ (see Lemma 3.1). By a perturbation theorem for a semigroup, we have that $A_1(u) \in G\left(L^2, 1, \beta_1\right)$. By Lemma 2.3 with $X = L^2, Y = H^{s-1}$, and $S = \Lambda^{s-1}$, we conclude that H^{s-1} is A-admissible. Therefore, $-A(u)$ is the infinitesimal generator of a C_0-semigroup on H^{s-1}. This completes the proof of Lemma 3.2.

Lemma 3.3. Let the operator $A(u) = bh(u)\partial_x + a\partial_x^3$ with $u \in H^s$, $s > \frac{3}{2}$. Then, $A(u) \in L\left(H^s, H^{s-1}\right)$. Moreover,

$$\|(A(u) - A(v))\omega\|_{s-1} \leq \kappa_1\|u - v\|_{s-1}\|\omega\|_s, \quad u, v, \omega \in H^s.$$

Proof. Let $u, v, \omega \in H^s, s > \frac{3}{2}$. Note that H^{s-1} is a Banach algebra. Then, we have

$$\begin{aligned} \|(A(u) - A(v))\omega\|_{s-1} &= \|b(h(u) - h(v))\partial_x\omega\|_{s-1} \\ &\leq \|b(h(u) - h(v))\|_{s-1}\|\partial_x\omega\|_{s-1} \\ &\leq cC_2(R)\|u - v\|_{s-1}\|\omega\|_s, \end{aligned}$$

where we used Lemma 2.5. Taking $v = 0$ in the above inequality, we obtain $A(u) \in L\left(H^s, H^{s-1}\right)$. Thus, proof of the Lemma 3.3 is completed.

Lemma 3.4. The operator

$$B(u) = \left[\Lambda, \left(bh(u) + a\partial_x^2\right)\partial_x\right]\Lambda^{-1} \in L\left(H^{s-1}\right),$$

with $u \in H^s$, $s > \frac{3}{2}$. Moreover,

$$\|(B(u) - B(v))\omega\|_{s-1} \leq \kappa_2\|u - v\|_s\|\omega\|_{s-1}.$$

Proof. Let $u,v\in H^s$ and $\omega\in H^{s-1}$, $s>\frac{3}{2}$. Moreover, we have $[\Lambda,(bh(u))\partial_x]\Lambda^{-1}=[\Lambda,(bh(u))]\Lambda^{-1}\partial_x$. Then, we obtain

$$
\begin{aligned}
&\|(B(u)-B(v))\omega\|_{s-1}\\
&=\left\|\Lambda^{s-1}[\Lambda,(bh(u)-bh(v))\partial_x]\Lambda^{-1}\omega\right\|_0\\
&=\left\|\Lambda^{s-1}[\Lambda,(bh(u)-bh(v))]\Lambda^{-1}\partial_x\omega\right\|_0\\
&\le\left\|b\Lambda^{s-1}[\Lambda,(h(u)-h(v))]\Lambda^{1-s}\right\|_{L(L^2)}\\
&\quad\times\left\|\Lambda^{s-2}\partial_x\omega\right\|_0.
\end{aligned}
$$

Applying Lemma 2.1 with $r=1-s$, $t=s-1$ and Lemma 2.5, we obtain

$$
\begin{aligned}
\|(B(u)-B(v))\omega\|_{s-1}&\le c\,\|(h(u)-h(v))\|_s\,\|\omega\|_{s-1}\\
&\le cC_2(R)\,\|u-v\|_s\,\|\omega\|_{s-1}.
\end{aligned}
$$

Taking $v=0$ in the above inequality, we obtain $B(u)\in L\left(H^{s-1}\right)$. Thus, proof of the Lemma 3.4 is completed.

Lemma 3.5. Let

$$
\begin{aligned}
f(u)&=-\partial_x\left(1-\partial_x^2\right)^{-1}\left(g(u)+\frac{b}{2}h'(u)u_x^2\right)\\
&\quad+\left(1-\partial_x^2\right)^{-1}(bh(u)u_x).
\end{aligned}
$$

Then, f is bounded on bounded sets in H^s, and satisfies

(i) $\|f(u)-f(v)\|_s\le\kappa_3\,\|u-v\|_s \qquad u,v\in H^s,$

(ii) $\|f(u)-f(v)\|_{s-1}\le\kappa_4\,\|u-v\|_{s-1} \qquad u,v\in H^s.$

Proof. Let $u,v\in H^s$, $s>\frac{3}{2}$. Setting

$$
\begin{aligned}
f_1(u)&=\left(1-\partial_x^2\right)^{-1}(bh(u)u_x)\\
&=b\partial_x\left(1-\partial_x^2\right)^{-1}(k(u)),\\
f_2(u)&=-\partial_x\left(1-\partial_x^2\right)^{-1}(g(u)),\\
f_3(u)&=-\partial_x\left(1-\partial_x^2\right)^{-1}\left(\frac{b}{2}h'(u)u_x^2\right),
\end{aligned}
$$

where $k'(u)=h(u)$, then

$$f(u)=f_1(u)+f_2(u)+f_3(u).$$

Applying Lemma 2.5, we get

$$
\begin{aligned}
\|f_1(u)-f_1(v)\|_s&\le c\,\|b(k(u)-k(v))\|_{s-1}\\
&\le cC_2(R)\,\|u-v\|_s
\end{aligned}
$$

and

$$
\begin{aligned}
\|f_2(u)-f_2(v)\|_s&\le c\,\|g(u)-g(v)\|_{s-1}\\
&\le cC_2(R)\,\|u-v\|_s.
\end{aligned}
$$

Since $s>\frac{3}{2}$ and H^{s-1} is a Banach algebra, then

$$
\begin{aligned}
\|f_3(u)-f_3(v)\|_s&\le c\left\|h'(u)u_x^2-h'(v)v_x^2\right\|_{s-1}\\
&\le c\Big(\left\|h'(u)\left(u_x^2-v_x^2\right)\right\|_{s-1}\\
&\qquad+\left\|v_x^2\left(h'(u)-h'(v)\right)\right\|_{s-1}\Big)\\
&\le c(\left\|h'(u)-h'(0)\right\|_{s-1}\|u_x-v_x\|_{s-1}\\
&\qquad\times\|u_x+v_x\|_{s-1}\\
&\qquad+\left|h'(0)\right|\|u_x-v_x\|_{s-1}\|u_x+v_x\|_{s-1}\\
&\qquad+\|v_x\|_{s-1}^2\left\|\left(h'(u)-h'(v)\right)\right\|_{s-1}).
\end{aligned}
$$

Again using Lemma 2.4 and Lemma 2.5, we obtain

$$
\begin{aligned}
\|f_3(u)-f_3(v)\|_s&\le c\left(C_1(R)+\left|h'(0)\right|\right)\|u-v\|_s\,\|u+v\|_s\\
&\qquad+cR^2C_2(R)\,\|u-v\|_s\\
&\le c\left(R\left(C_1(R)+\left|h'(0)\right|\right)+R^2C_2(R)\right)\\
&\qquad\times\|u-v\|_s\\
&\le c\,\|u-v\|_s.
\end{aligned}
$$

Taking $v=0$ in the above inequality, we obtain that f is bounded on bounded sets in H^s.
Next, we prove (ii). Let $u,v\in H^s$, $s>\frac{3}{2}$. Analogously of (i), we get

$$
\begin{aligned}
\|f_1(u)-f_1(v)\|_{s-1}&\le c\,\|b(k(u)-k(v))\|_{s-2}\\
&\le cC_2(R)\,\|u-v\|_{s-1},\\
\|f_2(u)-f_2(v)\|_{s-1}&\le c\,\|g(u)-g(v)\|_{s-2}\\
&\le cC_2(R)\,\|u-v\|_{s-1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\|f_3(u)-f_3(v)\|_{s-1}&\le c\left\|h'(u)u_x^2-h'(v)v_x^2\right\|_{s-2}\\
&\le c\Big(\left\|h'(u)\left(u_x^2-v_x^2\right)\right\|_{s-2}\\
&\qquad+\left\|v_x^2\left(h'(u)-h'(v)\right)\right\|_{s-2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le c\Big(\left\|h'(u)\partial_x(u+v)\right\|_{s-1}\|\partial_x(u-v)\|_{s-2}\Big)\\
&\quad+\|v_x\|_{s-1}^2\left\|\left(h'(u)-h'(v)\right)\right\|_{s-2}\\
&\le c(\left\|h'(u)-h'(0)\right\|_{s-1}\|u-v\|_{s-1}\|u+v\|_s\\
&\quad+\left|h'(0)\right|\|u-v\|_{s-1}\|u+v\|_s\\
&\quad+\|v\|_s^2\left\|\left(h'(u)-h'(v)\right)\right\|_{s-2})\\
&\le c\,\|u-v\|_{s-1}.
\end{aligned}
$$

This completes the proof of Lemma 3.5.

Proof of Theorem 1.1. Combining Theorem 2.1 and Lemmas 3.1-3.5, we can get the statement of Theorem 1.1.

Then, we will give the proof of Theorem 1.2.

Proof of Theorem 1.2. It suffices to consider the case $s'>s$, since the case $s'<s$ is obvious from uniqueness which is guaranteed by Theorem 1.1. To prove Theorem 1.2 for $s'>s$, let us return to (8), if we apply operator Λ^2 to

(8), we obtain the following equation for $y(t) = \Lambda^2 u(t) = u - u_{xx}$:

$$\frac{dy}{dt} + A(t)y + B(t)y = f(t), \quad y(0) = \Lambda^2 u(0),$$

where

$$A(t)y = \partial_x\left(\left(bh(u) + a\partial_x^2\right)y\right), \quad B(t)y = bh'(u)u_x y,$$

and

$$f(t) = u_x\left(\frac{b}{2}h''(u)u_x^2 - g'(u) + 2bh'(u)u + bh(u)\right).$$

Because $u \in C([0,T);H^s)$ and $u_0 \in H^{s'}$, we have

$$y \in C\left([0,T);H^{s-2}\right)$$

and

$$y(0) = \Lambda^2 u(0) \in C\left([0,T);H^{s'-2}\right).$$

It is our purpose to deduce $y \in C\left([0,T);H^{s'-2}\right)$, which implies $u \in C\left([0,T);H^{s'}\right)$. This will complete the proof of Theorem 1.2.

Note that $u \in C([0,T);H^s)$, $u_x \in H^{s-1}$, where H^{s-1} is a Banach algebra, and $g,h \in C^{[s]+1}(\mathbb{R})$. Then, we obtain

$$B(t) \in L\left(H^{s-1}\right) \text{ and } f(t) \in C\left([0,T);H^{s-1}\right).$$

To this end (see Lemmas 3.1-3.3 in [22]), we first need to prove that the family $A(t)$ has a unique evolution operator $\{U(t,\tau)\}$ associated with the spaces $X = H^\eta$ and $Y = H^k$, where $-s \leq \eta \leq s-2$, $1-s \leq k \leq s-1$, and $k \geq \eta+1$. Therefore, according to the proof of Lemma 3.1 in [22], we need to verify the following three conditions.

(i) $A(t) \in G(H^\eta, 1, \beta)$, $\forall y \in H^s$.

(ii) $\Lambda^\eta \partial_x\left[\Lambda^{k-\eta}, bh(u)\right]\Lambda^{-k}$ is uniformly bounded on L^2.

(iii) $A(t) \in L(H^k, H^\eta)$ is strongly continuous in t.

We begin by verifying (i). Since H^η is a Hilbert space, $A(t) \in G(H^\eta, 1, \beta)$ [21] if and only if there is a real number β such that

(a) $(A(t)y,y)_\eta \geq -\beta\|y\|_\eta^2$,

(b) $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^η for some (or all) $\lambda > \beta$.

First, we prove (a). Take $y \in H^\eta$ and note that

$$\begin{aligned}
&\Lambda^\eta\partial_x\left(bh(u)y + a\partial_x^2 y\right)\\
&= \Lambda^\eta\partial_x\left(-\left[\Lambda^{-\eta}, bh(u)\right]\Lambda^\eta y + \Lambda^{-\eta}\left(bh(u)\Lambda^\eta y\right)\right)\\
&\quad + a\Lambda^\eta\partial_x^3 y\\
&= -\Lambda^\eta\partial_x\left[\Lambda^{-\eta}, bh(u)\right]\Lambda^\eta y + \partial_x\left(bh(u)\Lambda^\eta y\right)\\
&\quad + a\Lambda^\eta\partial_x^3 y.
\end{aligned}$$

Then, we have

$$\begin{aligned}
(A(t)y,y)_\eta &= \left(-\Lambda^\eta\partial_x\left[\Lambda^{-\eta}, bh(u)\right]\Lambda^\eta y\right.\\
&\quad \left.+\partial_x\left(bh(u)\Lambda^\eta y\right) + a\Lambda^\eta\partial_x^3 y, \Lambda^\eta y\right)_0\\
&= \left(\Lambda^{\eta+1}\left[\Lambda^{-\eta}, bh(u)\right]\Lambda^\eta y, \partial_x\Lambda^{\eta-1}y\right)_0\\
&\quad + \frac{b}{2}\left(\partial_x\left(h(u)\Lambda^\eta y\right), \Lambda^\eta y\right)_0\\
&\leq \left\|\Lambda^{\eta+1}\left[\Lambda^{-\eta}, bh(u)\right]\right\|_{L(L^2)}\|\Lambda^\eta y\|_0^2\\
&\quad + \left\|\frac{b}{2}\partial_x h(u)\right\|_{L^\infty}\|\Lambda^\eta y\|_0^2.
\end{aligned}$$

From Lemma 2.1 with $r = -(\eta+1)$, $t = 0$ and Lemma 2.4, we obtain

$$\begin{aligned}
\left|(A(t)y,y)_\eta\right| &\leq c\left(\|h(u)\|_s + \|\partial_x h(u)\|_{L^\infty}\right)\|y\|_\eta^2\\
&\leq c\left(\|h(u)\|_s + \|h(u)\|_s\right)\|y\|_\eta^2\\
&\leq cC_1(R)\|u\|_s\|y\|_\eta^2.
\end{aligned}$$

Choosing $\beta = cC_1(R)\|u\|_s$, we have $(A(t)y,y)_\eta \geq -\beta\|y\|_\eta^2$.

Second, we prove (b). Let $S = \Lambda^{s-1-\eta}$. Note that S is an isomorphism of H^{s-1} onto H^η and H^{s-1} is continuously and densely embedded in H^η as $-s \leq \eta \leq s-2$. Define

$$\begin{aligned}
A_1(t) &= SA(t)S^{-1} = \Lambda^{s-1-\eta}A(t)\Lambda^{\eta+1-s},\\
B_1(t) &= A_1(t) - A(t) = [S, A(t)]S^{-1}.
\end{aligned}$$

Let $y \in H^\eta$ and $u \in H^s$, $s > \frac{3}{2}$. From Lemma 2.1 with $r = -(\eta+1)$, $t = s-1$ and Lemma 2.4, we have that

$$\begin{aligned}
\|B_1(t)y\|_\eta &= \left\|\Lambda^\eta\partial_x\left[\Lambda^{s-1-\eta}, bh(u)\right]\Lambda^{\eta+1-s}y\right\|_0\\
&\leq \left\|\Lambda^\eta\partial_x\left[\Lambda^{s-1-\eta}, bh(u)\right]\Lambda^{1-s}\right\|_{L(L^2)}\\
&\quad \times\|\Lambda^\eta y\|_0\\
&\leq c\|h(u)\|_s\|y\|_0\\
&\leq cC_1(R)\|u\|_s\|y\|_\eta.
\end{aligned}$$

Thus, we obtain $B_1(t) \in L(H^\eta)$. Note that

$$\begin{aligned}
A(t)y &= \partial_x\left(\left(bh(u) + a\partial_x^2\right)y\right)\\
&= \partial_x\left(bh(u)\right)y + bh(u)\partial_x y + a\partial_x^3 y
\end{aligned}$$

$$\text{and } u_x \in L\left(H^{s-1}\right).$$

Applying Lemma 3.2 and a perturbation theorem for semigroups, we have H^{s-1} is $A(t)$-admissible. Furthermore, applying Lemma 2.3 with $Y = H^{s-1}$, $X = H^\eta$ and $S = \Lambda^{s-1-\eta}$, we obtain that $-A_1(t)$ is the infinitesimal generator of a C_0-semigroup on H^η. Due to $A_1(t) = A(t) + B_1(t)$ and $B_1(t) \in L(H^\eta)$,by a perturbation theorem for a semigroups, we have that $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^η. This completes the proof of (b).

Next, we verify (ii). Take $y \in L^2$. Then, we have

$$\left\|\Lambda^\eta\partial_x\left[\Lambda^{k-\eta}, bh(u)\right]\Lambda^k y\right\|_0 \leq cC_1(R)\|u\|_s\|y\|_0,$$

where we applied Lemma 2.1 with $r = -(\eta+1)$, $t = k$ and Lemma 2.4.
Finally, we verify (iii). Take $y \in H^k$. Then,

$$\begin{aligned}
&\|(A(t+\tau) - A(t))y\|_\eta \\
&= \|\partial_x(bh(u(t+\tau)) - bh(u(t))y)\|_\eta \\
&\le \|bh(u(t+\tau)) - bh(u(t))y\|_{\eta+1} \\
&\le c\,\|b(h(u(t+\tau)) - h(u(t)))\|_{s-1}\|y\|_{\eta+1} \\
&\le cC_2(R)\|u(t+\tau) - u(t)\|_{s-1}\|y\|_{\eta+1} \\
&\le cC_2(R)\|(u(t+\tau) - u(t))\|_s\|y\|_k,
\end{aligned}$$

where we applied Lemma 2.2 with $r = s-1$, $t = \eta+1$ and Lemma 2.5. By the continuity of u, we prove (iii). Thus, the above three conditions imply the existence and uniqueness of evolution operator $U(t,\tau)$ for the family $A(t)$. In particular, $U(t,\tau)$ maps H^r into itself for $-s \le r \le s-1$.
Next, we choose $Y = H^{s-2}$ and $X = H^{s-3}$. Note that

$$y \in C\left([0,T);H^{s-2}\right) \cap C^1\left([0,T);H^{s-3}\right).$$

By the properties of evolution operator $U(t,\tau)$, we can obtain

$$\frac{d}{d\tau}(U(t,\tau)y(\tau)) = U(t,\tau)(-B(\tau)y(\tau) + f(\tau)).$$

An integrating over $\tau \in [0,t]$ gives

$$y(t) = U(t,0)y(0) + \int_0^t U(t,\tau)(-B(\tau)y(\tau) + f(\tau))\,d\tau. \tag{11}$$

If $s < s' \le s+1$, we have $f(t) \in C([0,T);H^{s-1}) \subset C\left([0,T);H^{s'-2}\right)$ and $B(t) = bh'(u)u_x \in L\left(H^{s'-2}\right)$ is strongly continuous in $[0,t)$, and $H^{s-1}H^{s'-2} \subset H^{s'-2}$ as $s-1 > \frac{1}{2}$. Due to $-s < s-2 < s'-2 \le s-1$, the family $\{U(t,\tau)\}$ is strongly continuous on $H^{s'-2}$ to itself. Note that $y(0) \in H^{s'-2}$. We regard Eq. (11) as an integral equation of Volterra type which can be solved for y by successive approximation. Then, the result of Theorem 1.2 is obtained.
If $s' > s+1$, we obtain the result of Theorem 1.2 by repeated application of the above argument. This completes the proof of Theorem 1.2.
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Conflict of Interest

The authors declare that they have no conflict of interest.

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