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A New Generalization of Garima Distribution with Application to Real Life Data

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Abstract: In this paper, we proposed a new distribution based on Garima distribution called as Alpha Power Transformed Garima distribution by using a technique developed by Mahdavi and Kundu. Some of the reliability and statistical properties of the distribution are obtained such as reliability function, hazard function, order statistics, Bonferroni and Lorenz curve. Model parameters are estimated by maximum likelihood and the least square method. Finally, the real-life data sets are investigated to know the performance and flexibility of this distribution.

Keywords: Alpha Power Transformation, Garima Distribution, Moments, Maximum Likelihood Estimation, Stress-Strength Reliability.

1 Introduction

In recent years, researchers proposed different methods of generating new continuous distributions in life-time data analysis to increase the ability to fit several lifetime data which have a high degree of skewness and kurtosis. These extensions to distributions provide better flexibility in modelling certain applications and data in practice. A detailed survey of methods for generating distributions has been studied by Lee [1] et al. and Jones [2]. Most of these distributions are special cases of the T-X class studied by Alzaatreh [3] et al. This class of distributions extends some recent families such as the beta-G pioneered by Eugene [4] et al., the gamma-G defined by Zografos and Balakrishnan [5], the Kw-G family proposed by Cordeiro and Castro [6] and the Weibull-G introduced by Bourguignon [7] et al. and so on.
Kus [8] introduced the two-parameter lifetime distribution with decreasing failure rate. The parameters of the distribution were obtained by EM algorithm using maximum likelihood estimate and the asymptotic variances and covariance of these estimates.

Some authors have discussed the situations where the data shows decreasing failure rate the upside-down bathtub (UBT) shape hazard rate. For example Proschan [9] found that the air-conditioning systems of planes follow decreasing failure rate. Efron [10] analyzed the data set related to head and neck cancer, in which the hazard rate initially increased, attain maximum and then decreased before it stabilized owing to a therapy. Bennette [11] analyzed lung cancer trial data which showed that failure rates were uni-modal in nature. Langlands [12] et al. have studied the breast carcinoma data and found that the mortality reached a peak after some finite period, and then declined gradually. It is interesting to know that the hazard rates of inverse versions of the probability distributions show the UBT shapes.

Dey [13] et al. discussed Alpha Power Lindley distribution its properties and application with earthquake data. Nassar et al. [14] also discussed the Alpha Power Weibull distribution, its properties and application to real-life data. Elbatal Ahmad [15] et al. discussed the newly Alpha Power transformed family of distribution, its properties and applications to the Weibull model. Alpha power Transformed Frechet Distribution was introduced by Suleman et al. [16] for modelling real-life data sets. The authors discussed some of the statistical properties of the distribution such as quantile function, moments, mean residual life, generating function, entropy, stochastic ordering, etc. The method of maximum likelihood estimation is used for estimating the parameters of the distribution. Dey [17] discuss the Alpha Power extended exponential distribution with application to ozone data. Ceren and Selen et al. [18] obtained the Alpha Power inverted Exponential distribution, its properties and application to real-life data.

Garima Distribution was introduced by Shankar [19] as a new one parameter lifetime model for behavioural and emotional science. Its usefulness and importance of life time model were better as compared to Lindley and exponential distribution. Shanker [20] obtained the discrete Poisson-Garima distribution. Shenbagaraja, Rather and Subramanian [21] obtained the length biased Garima distribution with properties and applications.

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Let F(x) be the cumulative distribution function (CDF) of random variable X, then the APT of CDF is defined as

$$F_{APT}(x)=\begin{cases}\dfrac{\alpha^{F(x)}-1}{\alpha-1}\ ; & if\ \alpha>0,\alpha\neq 1\\ F(x)\ ; & if\quad \alpha=1\end{cases} \tag{1}$$

and the corresponding Probability density function (PDF) is defined as:

$$f_{APT}(x)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}f(x)\alpha^{F(x)}\ ; & if\ \alpha>0,\alpha\neq 1\\ f(x)\ ; & if\quad \alpha=1\end{cases} \tag{2}$$

The APT survival function $S_{APT}(x)$ and $h_{APT}(x)$) are given by

$$S_{APT}(x)=\begin{cases}\dfrac{\alpha}{\alpha-1}(1-\alpha^{F(x)-1})\ ; & if\ \alpha\neq 1\\ 1-F(x)\ ; & if\ \alpha=1\end{cases} \tag{3}$$

$$h_{APT}(x)=\begin{cases}\dfrac{\alpha^{F(x)-1}}{1-\alpha^{F(x)-1}}f(x)\log\alpha\ ; & if\ \alpha\neq 1\\ \dfrac{f(x)}{S(x)}\ ; & if\ \alpha=1\end{cases} \tag{4}$$

The purpose of this article is to obtain a new distribution from the Garima distribution by α-power transformation as proposed by Mahdavi and Kundu [22]. The distribution is referred to as Alpha Power Transformed Garima (APTG) distribution. The additional parameter in the model can provide various properties and more flexibility in the form of the hazard and density functions.

In this paper, we will obtain the Alpha Power Transformed version of Garima distribution and discuss its various properties. Finally, the two lifetime data sets have been analysed, the results are compared to other distributions.

We are motivated to introduce the APTG distribution because:-

(i) it has an ability for modelling decreasing and upside-down bathtub shaped hazard rates.

(ii) it is the appropriate model for fitting the skewed data which may not be properly fitted by other common distributions and can also be used in a multiplicity of problems in various fields such as earthquake analysis, failure rate times, survival times of patients;

(iii) two real data applications show that it compares well with other competing lifetime distributions in modelling survival times of data.

Let X be a random variable with parameter, $\theta>0$. The PDF of one parameter Garima distribution is defined as follow

$$f(x;\theta)\ \frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\ ;\ x>0,\ \theta>0 \tag{5}$$

and its corresponding CDF of Garima distribution is given by

$$F(x;\theta)=1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\quad ;\ x>0,\theta>0 \tag{6}$$

2 Alpha Power Transformed Garima Distribution

By using APT Model, The probability density function (PDF) of the Alpha Power Transformed Garima (APTG) distribution with parameters θ > 0, α > 0 is given as

$$f_{APT}(x;\theta,\alpha)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}\left(\dfrac{\alpha\theta}{\theta+2}\right)(1+\theta+\theta x)\,e^{-\theta x}\ \alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}}\\ \qquad ;if\quad x>0\,;\ \theta>0,\alpha>0,\ \alpha\neq 1\\ \left(\dfrac{\theta}{\theta+2}\right)(1+\theta+\theta x)e^{-\theta x}\ ;\ if\quad x>0\,;\ \theta>0,\alpha=1\end{cases} \tag{7}$$

Where, θ, α are the scale and shape parameters.

and the CDF of the Alpha Power Transformed Garima distribution.

$$F_{APT}(x;\theta,\alpha)=\begin{cases}\dfrac{\alpha^{1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}}-1}{\alpha-1}\ ;if\ x>0;\theta,\alpha>0,\alpha\neq 1\\ 1-\left(1+\dfrac{\theta x}{\theta+2}\right)e^{-\theta x}\ ;if\ x>0;\theta>0,\alpha=1\end{cases}$$

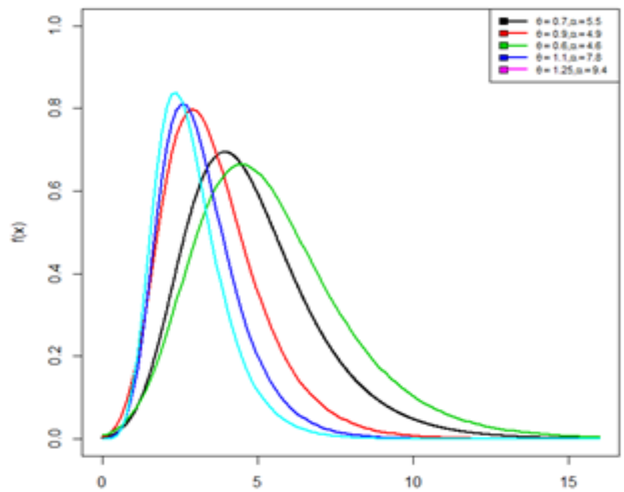

Fig. 1: cdf plot of APTGD distribution.

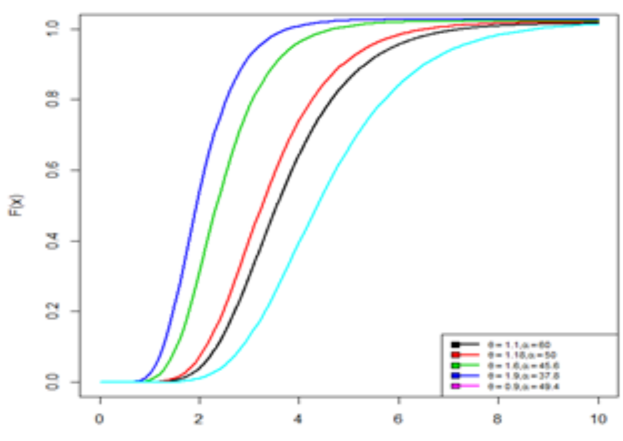

Fig. 2: cdf plot of APTG distribution.

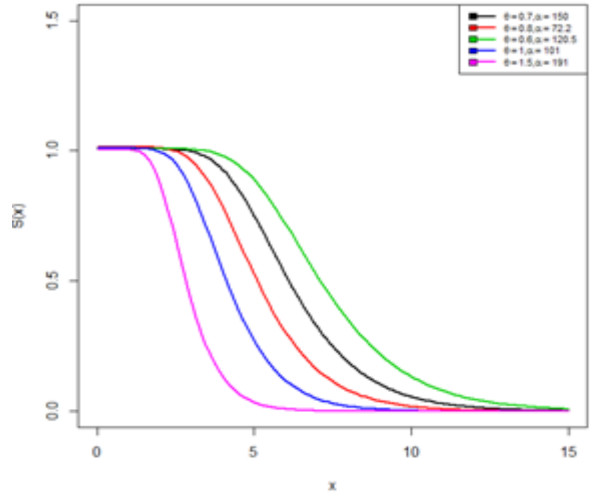

Fig. 3: Survival plot of APTG distribution.

3 Reliability Analysis

In this section we will discuss the Survival, hazard rate functions respectively, given by

3.1 Survival Function

$$S_{APT}(x;\theta,\alpha)=\begin{cases}\dfrac{\alpha}{\alpha-1}\left(1-\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}}\right); if\ \ x>0,\theta,\alpha>0,\alpha\neq 1\\[2ex] \left(1+\dfrac{\theta x}{\theta+2}\right)e^{-\theta x} \qquad ; if\ \ x>0,\theta>0,\alpha=1\end{cases} \tag{9}$$

3.2 Hazard Function

$$h_{APT}(x;\theta,\alpha)=$$

$$\begin{cases}\left(\dfrac{\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}}}{1-\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}}}\right)\log\alpha\left(\dfrac{\theta}{\theta+2}\right)\\ (1+\theta+\theta x) \qquad ;if\ x>0;\alpha>0,\theta>0,\ \alpha\neq 1\\[2ex] \left(\dfrac{\theta}{\theta x+\theta+2}\right); if\ \ x>0;\theta>0,\alpha=1\end{cases} \tag{10}$$

Hazard Plot

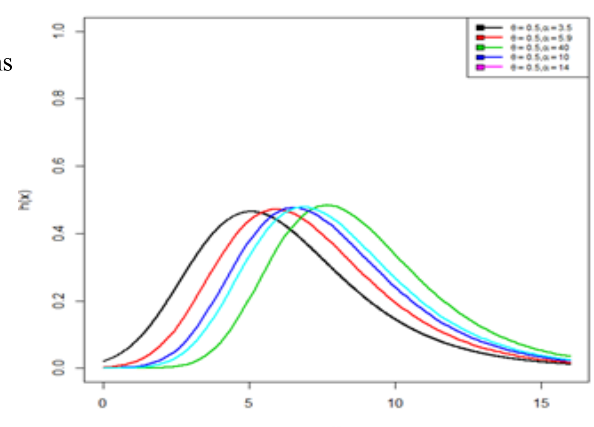

Fig. 4a: hazard plot of APTG distribution.

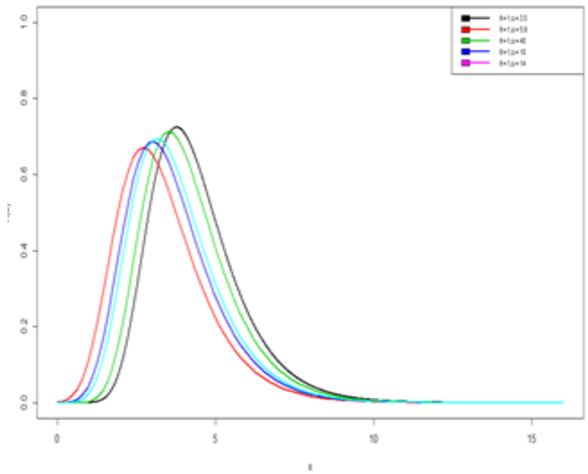

Fig. 4b: hazard plot of APTG distribution.

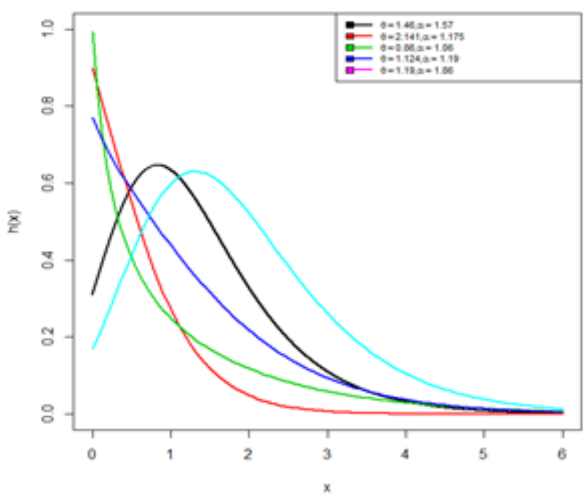

Fig.4c: hazard plot of APTG distribution.

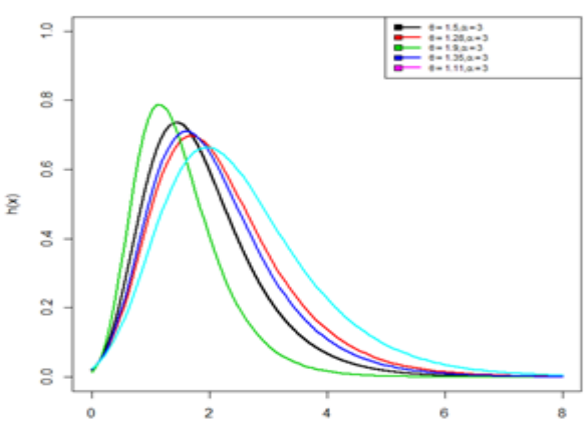

Fig. 4d: hazard plot of APTG distribution.

4 Statistical Properties

4.1 Moments

Let X denotes the random variable of Alpha Power Transformed Garima (APTG) distribution then the r^{th}order moment $E(X^r)$ of APTG distribution can be obtained as

$$E(X^r)=\mu_r'=\int_0^\infty x^r f_{APT}(x;\theta,\alpha)\,dx$$

$$E(X^r)=\int_0^\infty x^r\left(\frac{\log\alpha}{\alpha-1}\left(\frac{\theta}{\theta+2}\right)(1+\theta+\theta x)\,\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\theta x}\right)dx$$

$$E(X^r)=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)$$

$$\int_0^\infty x^r\left((1+\theta+\theta x)\,\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\theta x}\right)dx$$

By using the power series expansion to above equation,

$$\alpha^z=\sum_{k=0}^\infty\frac{(\log\alpha)^k}{k!}z^k \qquad (I)$$

$$(1-x)^{n-1}=\sum_{i=0}^\infty(-1)^i\binom{n-1}{i}x^i \qquad (II)$$

$$E(X^r)=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)$$

$$\left(\sum_{i=0}^\infty\frac{(\log\alpha)^i}{i!}\int_0^\infty x^r\left((1+\theta+\theta x)e^{-\theta x}\right)^i dx\right.$$

$$\left.\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)^i\right. \qquad (11)$$

Using binomial series expansion (I) and (II) to equation (11), we get

$$E\left(X^r\right)=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)$$

$$\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\theta x}{\theta+2}\right)^k\right)$$

$$\int_0^{\infty}x^r\left((1+\theta+\theta x)e^{-\theta x}e^{-\theta jx}\right)dx$$

$$E\left(X^r\right)=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{i=0}^{j}\binom{j}{k}\right)$$

$$\left(\frac{\theta^{k+1}}{(\theta+2)^{k+1}}\right)\int_0^{\infty}x^{r+k}\left((1+\theta+\theta x)e^{-\theta(j+1)x}\right)dx$$

$$\mu_r'=\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\theta^{k+1}}{(\theta+2)^{k+1}}\right)\right)$$

$$\left(\frac{\log\alpha}{\alpha-1}\right)\int_0^{\infty}x^{r+k+1-1}e^{-\theta(j+1)x}\,dx+$$

$$\theta\int_0^{\infty}x^{r+k+1-1}e^{-\theta(j+1)x}\,dx+\theta\int_0^{\infty}x^{r+k2-11}e^{-\theta(j+1)x}\,dx$$

$$\mu_r'=\left(\frac{\log\alpha}{\alpha-1}\right)$$

$$\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\theta^{k+1}}{(\theta+2)^{k+1}}\right)\right)$$

$$\left(\frac{\Gamma(r+k+1)}{(\theta(j+1))^{r+k+1}}+\frac{\theta\Gamma(r+k+1)}{(\theta(j+1))^{r+k+1}}+\frac{\theta\Gamma(r+k+2)}{(\theta(j+1))^{r+k+2}}\right) \tag{12}$$

Putting $r=1,2,3,\ldots$, so on in equation (12), we obtain the moments of APTG distribution.

$$\mu_1'=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\Gamma(k+2)}{(\theta(j+1))^{k+2}}+\frac{\theta\Gamma(k+21)}{(\theta(j+1))^{k+2}}+\frac{\theta\Gamma(k+3)}{(\theta(j+1))^{k+3}}\right)$$

$$\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\theta^{k+1}}{(\theta+2)^{k+1}}\right)\right)$$

$$\mu_2'=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\theta^{k+1}}{(\theta+2)^{k+1}}\right)\right)$$

$$\left(\frac{\Gamma(k+3)}{(\theta(j+1))^{k+31}}+\frac{\theta\Gamma(k+3)}{(\theta(j+1))^{k+3}}+\frac{\theta\Gamma(k+4)}{(\theta(j+1))^{k+4}}\right)$$

$$\mu_3'=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\theta^{k+1}}{(\theta+2)^{k+1}}\right)\right)$$

$$\left(\frac{\Gamma(k+4)}{(\theta(j+1))^{k+4}}+\frac{\theta\Gamma(k+4)}{(\theta(j+1))^{k+4}}+\frac{\theta\Gamma(k+5)}{(\theta(j+1))^{k+5}}\right)$$

4.2 Moment Generating Function

Let X have APTG distribution, the MGF of X is obtained as

$$M_X(t)=E(e^{tx})=\int_0^{\infty}e^{tx}f_{APT}(x;\theta,\alpha)\,dx$$

Using Taylor's series

$$E(e^{tx})=\int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+\frac{(tx)^3}{3!}+\ldots\ldots\right)f_{APT}(x;\theta,\alpha)\,dx$$

$$E(e^{tx})=\int_0^{\infty}\sum_{r=0}^{\infty}\frac{(tx)^r}{r!}f_{APT}(x;\theta,\alpha)\,dx$$

$$E(e^{tx})=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r f_{APT}(x;\theta,\alpha)\,dx$$

$$E(e^{tx})=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$$

$$E(e^{tx})=\left(\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\theta^{k+1}}{(\theta+2)^{k+1}}\right)\right)$$

$$\left(\frac{\Gamma(r+k+1)}{(\theta(j+1))^{r+k+1}}+\frac{\theta\Gamma(r+k+1)}{(\theta(j+1))^{r+k+1}}+\frac{\theta\Gamma(r+k+2)}{(\theta(j+1))^{r+k+2}}\right)$$

$$\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{t^r}{r!}\right)$$

Similarly, characteristic function of APTG distribution is given by

$$\varphi_x(t)=M_X(it)=\int_0^\infty e^{tix} f_{APT}(x;\theta,\alpha)\,dx$$

$$E(e^{itx})=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{(it)^r}{r!}\right)$$

$$\left(\sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\theta^{k+1}}{(\theta+2)^{k+1}}\right)\right)$$

$$\left(\frac{\Gamma(r+k+1)}{(\theta(j+1))^{r+k+1}}+\frac{\theta\,\Gamma(r+k+1)}{(\theta(j+1))^{r+k+1}}+\frac{\theta\,\Gamma(r+k+2)}{(\theta(j+1))^{r+k+2}}\right)$$

4.3 Stress Strength Reliability

Let X be the strength of the system which is subjected to a stress Y, and if $X\sim$ APTG (α_1,θ_1) and $Y\sim$ APTG (α_2,θ_2),provided X and Y are statistically independent random variables, then $R=P(Y<X)$.

$$R=\int_0^\infty f_{APT}(x;\theta_1,\alpha_1)\,F_{APT}(x;\theta_2,\alpha_2)\,dx$$

$$R=\left(\frac{\log\alpha_1}{\alpha_1-1}\right)\left(\frac{\theta_1}{\theta_1+2}\right)\int_0^\infty(1+\theta_1+\theta_1x)\alpha_1^{1-\left(1+\frac{\theta_1x}{\theta_1+2}\right)e^{-\theta_1x}}$$

$$\left(\frac{\alpha_2^{1-\left(1+\frac{\theta_2x}{\theta_2+2}\right)e^{-\theta_2x}}-1}{\alpha_2-1}\right)dx$$

$$R=\left(\frac{\theta_1\log\alpha_1}{(\alpha_1-1)(\theta_1+2)(\alpha_2-1)}\right)$$

$$\left(\begin{array}{l}\int_0^\infty(1+\theta_1+\theta_1x)\alpha_1^{1-\left(1+\frac{\theta_1x}{\theta_1+2}\right)e^{-\theta_1x}}\alpha_2^{1-\left(1+\frac{\theta_2x}{\theta_2+2}\right)e^{-\theta_2x}}dx\\ -\int_0^\infty(1+\theta_1+\theta_1x)\alpha_1^{1-\left(1+\frac{\theta_1x}{\theta_1+2}\right)e^{-\theta_1x}}dx\end{array}\right)\quad(13)$$

$$R=I_1+I_2$$

Using the power series expansion (I) and (II) to equation (13) as follows

$$I_1=\left(\frac{\theta_1\log\alpha_1}{(\alpha_1-1)(\theta_1+2)(\alpha_2-1)}\right)$$

$$\sum_{k=0}^{\infty}\frac{(\log\alpha_1)^k}{k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l}$$

$$\int_0^\infty(1+\theta_1+\theta_1x)\left(1+\frac{\theta_1x}{\theta_1+2}\right)^l e^{-l\theta_1}$$

$$\alpha_2^{1-\left(1+\frac{\theta_2x}{\theta_2+2}\right)e^{-x\theta_2}}dx$$

$$I_1=\left(\frac{\theta_1\log\alpha_1}{(\alpha_1-1)(\theta_1+2)(\alpha_2-1)}\right)\sum_{k=0}^{\infty}\frac{(\log\alpha_1)^k}{k!}$$

$$\int_0^\infty(1+\theta_1+\theta_1x)\alpha_2^{1-\left(1+\frac{\theta_2x}{\theta_2+2}\right)e^{-\theta_2x}}\left(1-\left(1+\frac{\theta_1x}{\theta_1+2}\right)e^{-\theta_1x}\right)^k dx$$

$$I_1=\left(\sum_{k=0}^{\infty}\frac{(\log\alpha_1)^k}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\sum_{m=0}^{l}\binom{l}{m}\sum_{p=0}^{\infty}\frac{(\log\alpha_2)^p}{p!}\right)$$

$$\left(\frac{\theta_1^{m+1}\log\alpha_1}{(\alpha_1-1)(\theta_1+2)^{m+1}(\alpha_2-1)}\right)$$

$$\int_0^\infty x^m\left((1+\theta_1+\theta_1x)e^{-\theta_1(l+1)x}\left(1-\left(1+\frac{\theta_2x}{\theta_2+2}\right)e^{-\theta_2x}\right)^p\right)dx$$

$$I_1=\left(\sum_{k=0}^{\infty}\frac{(\log\alpha_1)^k}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\sum_{m=0}^{l}\binom{l}{m}\sum_{p=0}^{\infty}\frac{(\log\alpha_2)^p}{p!}\right)$$

$$\sum_{q=0}^{p}\binom{p}{q}(-1)^q\left(\frac{\theta_1^{m+1}\log\alpha_1}{(\alpha_1-1)(\theta_1+2)^{m+1}(\alpha_2-1)}\right)$$

$$\int_0^\infty x^m\left((1+\theta_1+\theta_1x)e^{-\theta_1(l+1)x-\theta_2qx}\right)\left(1+\frac{\theta_2x}{\theta_2+2}\right)^q dx$$

$I_1 =$

$$\left(\sum_{k=0}^{\infty}\frac{(\log\alpha_1)^k}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\sum_{m=0}^{l}\binom{l}{m}\sum_{p=0}^{\infty}\frac{(\log\alpha_2)^p}{p!}\right)$$

$$\sum_{q=0}^{p}\binom{p}{q}(-1)^q\left(\frac{\theta_1^{m+1}\log\alpha_1}{(\alpha_1-1)(\theta_1+2)^{m+1}(\alpha_2-1)}\right)$$

$$\sum_{s=0}^{q}\binom{q}{s}\left(\frac{\theta_2}{\theta_2+2}\right)^s\int_0^{\infty}x^{m+s}\left((1+\theta_1+\theta_1 x)\right)$$

$$e^{-\theta_1(l+1)x-\theta_2 qx}dx$$

$$I_1 = \left(\sum_{k=0}^{\infty}\frac{(\log\alpha_1)^k}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\sum_{m=0}^{l}\binom{l}{m}\sum_{p=0}^{\infty}\frac{(\log\alpha_2)^p}{p!}\right)$$

$$\sum_{q=0}^{p}\binom{p}{q}(-1)^q\left(\frac{\theta_1^{m+1}\log\alpha_1}{(\alpha_1-1)(\theta_1+2)^{m+1}(\alpha_2-1)}\right)$$

$$\left(\frac{\theta_2}{\theta_2+2}\right)^q\sum_{s=0}^{q}\binom{q}{s}\int_0^{\infty}x^{m+s}\left((1+\theta_1+\theta_1 x)e^{-\theta_1(l+1)x-\theta_2 qx}\right)dx$$

On simplification we get,

$$I_1 = \left(\sum_{k=0}^{\infty}\frac{(\log\alpha_1)^k}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\sum_{m=0}^{l}\binom{l}{m}\sum_{p=0}^{\infty}\frac{(\log\alpha_2)^p}{p!}\right)$$

$$\sum_{q=0}^{pr}\binom{p}{q}(-1)^q\left(\frac{\theta_1^{m+1}\log\alpha_1}{(\alpha_1-1)(\theta_1+2)^{m+1}(\alpha_2-1)}\right)$$

$$\left(\frac{\theta_2}{\theta_2+2}\right)^q\sum_{s=0}^{q}\binom{q}{s}$$

$$\left(\begin{array}{l}\frac{\Gamma m+s+1}{(\theta_1(l+1)-q\theta_2)^{m+s+1}}+\frac{\theta_1\Gamma m+s+1}{(\theta_1(l+1)-q\theta_2)^{m+s+1}}\\ +\frac{\theta_1\Gamma m+s+2}{(\theta_1(l+1)-q\theta_2)^{m+s+2}}\end{array}\right)$$

Similarly, we can do for I_2 and we will get,

$$I_2 = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{z=0}^{j}(-1)^j\frac{(\log\alpha_1)^i}{i!}\binom{i}{j}\binom{j}{z}\left(\frac{\theta_1}{\theta_1+2}\right)^{z+1}$$

$$\left(\frac{\Gamma(s+1)}{(\theta_1(j+1))^{s+1}}+\frac{\theta_1\Gamma(s+1)}{(\theta_1(j+1))^{s+1}}+\frac{\theta_1\Gamma(s+2)}{(\theta_1(j+1))^{s+2}}\right)$$

$$\left(\frac{\log\alpha_1}{\alpha_1-1}\right)$$

4.4 Mean Waiting Time and Mean Residual Time

4.4.1 Mean waiting time

The mean waiting time represents the waiting time elapsed since the failure on an object on condition that this failure have occurred in the interval [0, t]. The mean waiting time of X , say $\bar{\boldsymbol{\mu}}(\boldsymbol{t})$

$$\bar{\mu}(t) = t - \frac{1}{F(t)}\int_0^t xf_{APT}(x;\theta,\alpha)dx$$

$$\bar{\mu}(t) = t - \frac{1}{F(t)}\int_0^t x\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)(1+\theta+\theta x)$$

$$\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\theta x}\,dx$$

$$\bar{\mu}(t) = t - \frac{1}{F(t)}\left(\begin{array}{l}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)\\ \int_0^t x(1+\theta+\theta x)\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\theta x}dx\end{array}\right) \quad (14)$$

By using power series representation, to

equation, (14),

$$\alpha^z = \sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}z^k \qquad (I)$$

$$\bar{\mu}(t) = t - \frac{1}{F(t)}$$

$$\left(\begin{array}{l}\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\int_0^t x\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)\\ (1+\theta+\theta x)\left(1-\left(1+\frac{\theta x}{\theta+2}\right)\right)e^{-\theta x}\end{array}\right)^i dx$$

$$\bar{\mu}(t)=t-\frac{1}{F(t)}\left(\begin{array}{l}\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i}}{i!}\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}\left(\frac{\log\alpha}{\alpha-1}\right)\\ \left(\frac{\theta}{\theta+2}\right)\int_{0}^{t}x(1+\theta+\theta x)\\ \left(1+\frac{\theta x}{\theta+2}\right)^{j}e^{-\theta(j+1)x}\end{array}\right)dx \quad (15)$$

By using series expansion to (15),

$$(1-x)^{n-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{n-1}{i}x^{i} \quad \text{(II)}$$

On simplification, we get

$$\bar{\mu}(t)=\int_{0}^{t}x^{k+1}(1+\theta+\theta x)e^{-\theta(j+1)x}\,dx$$

$$t-\frac{1}{F(t)}\left(\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i}}{i!}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^{j}\binom{i}{j}\binom{j}{k}\left(\frac{\theta}{\theta+2}\right)^{k+1}\right)$$

Put $x\theta(j+1)=z,\theta(j+1)dz=dx,\ dx=\frac{dz}{\theta(j+1)}$

as $x\to 0,\ z\to 0;$ as $x\to t,\ z\to\theta t(j+1)$

$$\bar{\mu}(t)=t-\frac{1}{F(t)}\left(\begin{array}{l}\left(\frac{\log\alpha}{\alpha-1}\right)\\ \sum_{i=0}^{\infty}\frac{(\log\alpha)^{i}}{i!}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^{j}\binom{i}{j}\binom{j}{k}\end{array}\right)$$

$$\frac{1}{\theta(j+1)}\left(\frac{\theta}{\theta+2}\right)^{k+1}\left(\begin{array}{l}\int_{0}^{t\theta(j+1)}\left(\frac{z}{\theta(j+1)}\right)^{(k+2)-1}e^{-z}dz+\\ \int_{0}^{t\theta(j+1)}\left(\frac{z}{\theta(j+1)}\right)^{(k+2)-1}e^{-z}dz+\\ \int_{0}^{t\theta(j+1)}\left(\frac{z}{\theta(j+1)}\right)^{(k+3-1)}e^{-z}dz\end{array}\right)$$

Substituting equation (3) into equation (15) we get,

$$\bar{\mu}(t)=t-\frac{\alpha-1}{\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}}$$

$$\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{(\alpha-1)i!}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^{j}\binom{i}{j}\binom{j}{k}\left(\frac{\theta}{\theta+2}\right)^{k+1}\right)$$

$$\left(\frac{1}{\theta(j+1)}\right)\left(\begin{array}{l}\left(\frac{1}{\theta(j+1)}\right)^{k+1}\gamma((k+2);t\theta(j+1))+\\ \left(\frac{1}{\theta(j+1)}\right)^{k+1}\gamma((k+2);t\theta(j+1))+\\ \left(\frac{1}{\theta(j+1)}\right)^{k+2}\gamma((k+3);t\theta(j+1))\end{array}\right)$$

4.4.2 Mean Residual Life

Assuming that X is a continuous random variable with survival function (6).then the mean residual life is the expected additional lifetime that the component has survived after a fixed time point t. the mean residual life function is given by

$$\mu(t)=\frac{1}{S(t)}\left(E(t)-\int_{0}^{t}xf(x;\alpha,\theta)dx\right)-t \quad (16)$$

Where,

$$\int_{0}^{t}xf(x;\alpha,\theta)dx=$$

$$\left(\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i}}{i!}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^{j}\binom{i}{j}\binom{j}{k}\left(\frac{\theta}{\theta+2}\right)^{k+1}\right)$$

$$\left(\frac{1}{\theta(j+1)}\right)\left(\begin{array}{l}\left(\frac{1}{\theta(j+1)}\right)^{k+1}\gamma((k+2);t\theta(j+1))+\\ \left(\frac{1}{\theta(j+1)}\right)^{k+1}\gamma((k+2);t\theta(j+1))+\\ \left(\frac{1}{\theta(j+1)}\right)^{k+2}\gamma((k+3);t\theta(j+1))\end{array}\right) \quad (17)$$

Substituting the equation (10), equation (17) and equation (8) in equation (16) we get,

$$\mu(t)=\frac{\alpha-1}{\alpha\left(1-\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}}\right)}$$

$$\left(\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\frac{(\log\alpha)^{i+1}}{(\alpha-1)i!}(-1)^{j}\binom{i}{j}\binom{j}{k}\left(\frac{\theta}{(\theta+2)}\right)^{k+1}\right.$$

$$\left.\left(\begin{array}{l}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{k}{l}\binom{l}{m}\frac{\Gamma(k+m+l+2)}{(\theta(j+1))^{(k+m+l+2)}}-\left(\frac{1}{\theta(j+1)}\right)^{k+2}\times\\ \left(\begin{array}{l}\gamma((k+2);t\theta(j+1))+\gamma((k+2);t\theta(j+1))\\ +\left(\frac{1}{\theta(j+1)}\right)^{k+1}\gamma((k+3);t\theta(j+1))\end{array}\right)\end{array}\right)-t\right)$$

5 Entropies

The concept of Entropy is used to measure the randomness of systems and is widely used in areas like physics, molecular imaging of tumours and sparse kernel density estimation.

5.1 Renyi Entropy

Renyi entropy was introduced by Alfred Renyi [23]. Some recent applications of Renyi entropy have been considered such as sparse kernel density estimations, high-resolution scalar quantization, estimation of the number of components of multi-component non-stationary signal. Renyi entropy is given by

$$e(\beta)=\frac{1}{1-\beta}\log\int_{0}^{\infty}f^{\beta}(x)\,dx\ ,\ \beta>0,\beta\neq1$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^{\beta}\left(\frac{\theta}{\theta+2}\right)^{\beta}$$

$$\int_{0}^{\infty}(1+\theta+\theta x)^{\beta}\alpha^{\beta\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\beta\theta x}\,dx \qquad (18)$$

Using the power series expansion (I) and (II) to

equation (18),

$$e(\beta)=\frac{1}{1-\beta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^{\beta}\left(\frac{\theta}{\theta+2}\right)^{\beta}\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i}}{i!}$$

$$\int_{0}^{\infty}(1+\theta+\theta x)^{\beta}\left(\beta\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)\right)^{i}e^{-\beta\theta x}\,dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^{\beta}\left(\frac{\theta}{\theta+2}\right)^{\beta}$$

$$\sum_{i=0}^{\infty}\frac{(\beta\log\alpha)^{i}}{i!}\sum_{j=0}^{\infty}\binom{i}{j}(-1)^{j}$$

$$\int_{0}^{\infty}(1+\theta+\theta x)^{\beta}\left(1+\frac{\theta x}{\theta+2}\right)^{j}e^{-\theta jx-\beta\theta x}\,dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^{\beta}\left(\frac{\theta}{\theta+2}\right)^{\beta}$$

$$\sum_{i=0}^{\infty}\frac{(\beta\log\alpha)^{i}}{i!}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{j}{k}\binom{i}{j}(-1)^{j}\left(\frac{\theta}{\theta+2}\right)^{k}$$

$$\int_{0}^{\infty}(1+\theta+\theta x)^{\beta}x^{k}e^{-\theta(j+\beta)x}\,dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^{\beta}\left(\frac{\theta}{\theta+2}\right)^{\beta+l}$$

$$\sum_{i=0}^{\infty}\frac{(\beta\log\alpha)^{i}}{i!}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\beta}{l}\binom{j}{k}\binom{i}{j}(-1)^{j}\left(\frac{\theta}{\theta+2}\right)^{k}$$

$$\int_{0}^{\infty}(1+x)^{l}x^{k}e^{-\theta(j+\beta)x}\,dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^{\beta}\left(\frac{\theta}{\theta+2}\right)^{\beta+l}$$

$$\sum_{i=0}^{\infty}\frac{(\beta\log\alpha)^{i}}{i!}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{l}{m}\binom{\beta}{l}\binom{j}{k}\binom{i}{j}(-1)^{j}$$

$$\left(\frac{\theta}{\theta+2}\right)^{k}\int_{0}^{\infty}(x)^{(k+l+1)-1}e^{-\theta(j+\beta)x}dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^{\beta}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\beta\log\alpha)^{i}}{i!}\binom{i}{j}\binom{j}{k}\binom{\beta}{l}\binom{l}{m}$$

$$(-1)^{j}\left(\frac{\theta}{\theta+2}\right)^{\beta+l+k}\left(\frac{\Gamma(k+l+1)}{(\theta(j+\beta))^{(k+l+1)}}\right)$$

5.2 Tsallis Entropy:

A generalization of Boltzmann-Gibbs a statistical mechanics initiated by Tsallis has focused a great deal of attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy [24] for a continuous random variable is defined as follows

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\int_0^{\infty} f^{\lambda}(x)dx\right)$$

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\int_0^{\infty}\left(\frac{\log\alpha}{\alpha-1}\left(\frac{\alpha\theta}{\theta+2}\right)(1+\theta+\theta x)\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\theta x}\right)^{\lambda}\right)$$

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\left(\left(\frac{\log\alpha}{\alpha-1}\right)^{\lambda}\left(\frac{\theta}{\theta+2}\right)^{\lambda}\int_0^{\infty}(1+\theta+\theta x)^{\lambda}\alpha^{\lambda\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\lambda\theta x}\right)\right)dx \quad (19)$$

Using the power series expansion to (I) & (II) to

equation (19),

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\left(\left(\frac{\log\alpha}{\alpha-1}\right)^{\lambda}\left(\frac{\theta}{\theta+2}\right)^{\lambda}\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}\int_0^{\infty}(1+\theta+\theta x)^{\lambda}\left(\lambda\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)\right)^{i}e^{-\lambda\theta x}\right)\right)dx$$

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\left(\left(\frac{\log\alpha}{\alpha-1}\right)^{\lambda}\left(\frac{\theta}{\theta+2}\right)^{\lambda}\sum_{i=0}^{\infty}\frac{(\lambda\log\alpha)^i}{i!}\sum_{j=0}^{\infty}\binom{i}{j}(-1)^j\int_0^{\infty}(1+\theta+\theta x)^{\lambda}\left(1+\frac{\theta x}{\theta+2}\right)^{j}e^{-\theta jx-\lambda\theta x}\right)\right)dx$$

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\left(\left(\frac{\log\alpha}{\alpha-1}\right)^{\lambda}\left(\frac{\theta}{\theta+2}\right)^{\lambda}\left(\frac{\theta}{\theta+2}\right)\sum_{i=0}^{\infty}\frac{(\lambda\log\alpha)^i}{i!}\sum_{j=0}^{\infty}\binom{i}{j}(-1)^j\, k\int_0^{\infty}(1+\theta+\theta x)^{\lambda}\left(1+\frac{\theta x}{\theta+2}\right)^{j}e^{-\theta jx-\lambda\theta x}\right)\right)dx$$

$$S(\lambda)=\frac{1}{\lambda-1}$$

$$\left(1-\left(\left(\frac{\log\alpha}{\alpha-1}\right)^{\lambda}\frac{\Gamma(k+l+1)}{(\theta(j+\lambda))^{(k+l+1)}}\binom{\lambda}{l}\binom{l}{m}(-1)^j\left(\frac{\theta}{\theta+2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\lambda\log\alpha)^i}{i!}\binom{i}{j}\binom{j}{k}^{\lambda+l+k}\right)\right)$$

6 Bonferroni And Lorenz Curve

The Bonferroni [25] and Lorenz [26] curve have applications not only in economics to study income and poverty, but also in other fields such as reliability, demography, insurance, and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{\mu' p}\int_0^{q} x\, f_{APT}(x)dx$$

and

$$L(p)=\frac{1}{\mu'}\int_0^{q} x\, f_{APT}(x)dx$$

$$B(p)=\frac{1}{\mu' p}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)\int_0^{q} x\,(1+\theta+\theta x)\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\theta x}dx \quad (20)$$

Using the power series expansion to equation (20),

$$\alpha^{z}=\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}z^k$$

$$B(p)=\frac{1}{\mu' p}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!} \quad (21)$$

$$\int_0^{q} x\,(1+\theta+\theta x)\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)^{i}e^{-\theta x}dx$$

By using series expansion, to equation (21),

$$(1-x)^{n-1}=\sum_{i=0}^{\infty}(-1)^i\binom{n-1}{i}x^i$$

$$B(p)=\frac{1}{\mu'p}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta}{\theta+2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\log\alpha)^i}{i!}(-1)^j\binom{i}{j}$$

$$\int_0^q x\,(1+\theta+\theta x)\left(1+\frac{\theta x}{\theta+2}\right)^j e^{-\theta jx}e^{-\theta x}dx$$

On simplification we get,

$$B(p)=\frac{1}{\mu'p}\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\log\alpha)^i}{i!}(-1)^j\binom{i}{j}\binom{j}{k}$$

$$\left(\frac{\theta}{\theta+2}\right)^{k+1}\int_0^q x^{k+1}(1+\theta+\theta x)\,e^{-\theta(j+1)x}dx$$

$$B(p)=\frac{1}{\mu'p}\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\log\alpha)^i}{i!}(-1)^j\binom{i}{j}\binom{j}{k}$$

$$\left(\frac{\theta}{\theta+2}\right)^{k+1}\int_0^q x^{k+1}\,e^{-\theta(j+1)x}dx$$

$$+\theta\int_0^q x^{k+1}\,e^{-\theta(j+1)x}dx+\theta\int_0^q x^{k+2}\,e^{-\theta(j+1)x}dx$$

Put $x\theta(j+1)=z,\ \theta(j+1)dx=dz, dx=\frac{dz}{\theta(j+1)}$

as $x\to 0$, $z\to 0$; as $x\to q,\quad z\to\theta q(j+1)$

$$B(p)=\frac{1}{\mu'p}\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\log\alpha)^i}{i!}$$

$$(-1)^j\binom{i}{j}\binom{j}{k}\left(\frac{\theta}{\theta+2}\right)^{k+1}\left(\frac{1}{\theta(j+1)}\right)$$

$$\int_0^{q\theta(j+1)}\left(\frac{z}{\theta(j+1)}\right)^{(k+1)}e^{-z}dz+\theta$$

$$\int_0^{q\theta(j+1)}\left(\frac{z}{\theta(j+1)}\right)^{(k+1)}e^{-z}dz\ +$$

$$\theta\int_0^{q\theta(j+1)}\left(\frac{z}{\theta(j+1)}\right)^{(k+2)}e^{-z}dz$$

$$B(p)=\frac{1}{\mu'p}\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\log\alpha)^i}{i!}(-1)^j\binom{i}{j}\binom{j}{k}\left(\frac{\theta}{\theta+2}\right)^{k+1}$$

$$\left(\frac{1}{\theta(j+1)}\right)^{k+2}\int_0^{q\theta(j+1)}z^{k+2-1}e^{-z}dz\ +\theta\int_0^{q\theta(j+1)}z^{k+2-1}e^{-z}dz\ +$$

$$\frac{1}{(j+1)}\int_0^{q\theta(j+1)}z^{k+3-1}e^{-z}dz$$

$$B(p)=\frac{1}{\mu'p}\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\log\alpha)^i}{i!}(-1)^j\binom{i}{j}\binom{j}{k}$$

$$\left(\frac{\theta}{\theta+2}\right)^{k+1}\left(\frac{1}{\theta(j+1)}\right)^{k+2}\left(\begin{array}{l}\gamma(k+2,\theta q(j+1))+\theta\,\gamma(k+2,\theta q(j+1))\\+\left(\frac{1}{(j+1)}\right)\gamma(k+3,\theta q(j+1))\end{array}\right)$$

$$L(p)=pB(p)$$

$$L(p)=\frac{1}{\mu'}\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\log\alpha)^i}{i!}(-1)^j\binom{i}{j}\binom{j}{k}\left(\frac{\theta}{\theta+2}\right)^{k+1}$$

$$\left(\frac{1}{\theta(j+1)}\right)^{k+2}\left(\begin{array}{l}\gamma(k+2,\theta q(j+1))+\theta\,\gamma(k+2,\theta q(j+1))+\\\left(\frac{1}{(j+1)}\right)\gamma(k+3,\theta q(j+1))\end{array}\right)$$

7 Order Statistics

Suppose $X_1,X_2,X_3,\dots,X_n$ is a random sample from $f_{APT}(x;\theta,\alpha)$.

Let $X_{(1)},X_{(2)},X_{(3)},\dots,X_{(n)}$ be the order statistics of a random sample $X_1,X_2,X_3,\dots,X_n$ drawn from the continuous population with probability density function $f_{APT}(x)$ and cumulative density function $F_{APT}(x)$, then the PDF of r^{th} order statistics $X_{(r)}$ is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!\,(n-r)!}f_X(x)\ [F_X(x)]^{r-1}[1-F_X(x)]^{n-r}\quad(22)$$

Using the equation (7) and equation (8) in equation (22), the probability density function of r^{th} order statistics $X_{(r)}$ of APTG distribution is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!}$$

$$\left(\frac{\log\alpha}{\alpha-1} \left(\frac{\alpha\theta}{\theta+2}\right) (1+\theta+\theta x)\,\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}} \right)$$

$$\left(\frac{\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)}-1}{\alpha-1} \right)^{r-1} \left(\frac{\alpha-\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)}}{\alpha-1} \right)^{n-r}$$

Therefore, the probability density function of higher order statistics $X_{(n)}$ can be obtained as

$$f_{X(n)}(x) = n\,f(x)\left(F(x)\right)^{n-1}$$

$$f_{X(n)}(x) = n \left(\frac{\log\alpha}{\alpha-1} \left(\frac{\alpha\theta}{\theta+2}\right) (1+\theta+\theta x)\,\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}} \right)$$

$$\left(\frac{\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)}-1}{\alpha-1} \right)^{n-1}$$

and the PDF of first order statistic $X_{(1)}$ can be obtained as

$$f_{X(1)}(x) = n\,f(x)\left(1-F(x)\right)^{n-1}$$

$$f_{X(1)}(x) = n \left(\frac{\log\alpha}{\alpha-1} \left(\frac{\alpha\theta}{\theta+2}\right) (1+\theta+\theta x)\,\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}} \right)$$

$$\left(\frac{\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)}}{\alpha-1} \right)^{n-1}$$

8 Parameter Estimation

8.1 Least Square Estimation

Suppose that $X_1, X_2, X_3, \ldots, X_n$ is a random sample of n size from APTG distribution and let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ is the corresponding ordered sample then Least Square estimators can be obtained by minimizing the sum of squares errors

$$Z(\alpha,\theta) = \sum_{i=1}^{n} \left(F\left(X_{(i)}\right) - \left(\frac{i}{n+1}\right) \right)^2$$

Where $F\left(X_{(i)}\right)$ is the CDF of APTG Distribution with $X_{(i)}$ being the i^{th} Order statistic.

With respect to unknown parameters. So the LSE [27] of the population parameters (α,θ) can be obtained by the simultaneous solutions of the normal equations.

$$\frac{\partial Z(\alpha,\theta)}{\partial\alpha} = \sum_{i=1}^{n} F'\left(X_{(i)},\alpha,\theta\right)\left(F\left(X_{(i)},\alpha,\theta\right) - \left(\frac{i}{n+1}\right) \right) = 0 \quad (23)$$

$$\frac{\partial Z(\alpha,\theta)}{\partial\theta} = \sum_{i=1}^{n} F'\left(X_{(i)},\alpha,\theta\right)\left(F\left(X_{(i)},\alpha,\theta\right) - \left(\frac{i}{n+1}\right) \right) = 0 \quad (24)$$

By using the equations (23) and equation (24), the LSE can be obtained as follows,

$$\frac{\partial Z(\alpha,\theta)}{\partial\alpha} = \sum_{i=1}^{n} \left(\frac{\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)}-1}{\alpha-1} - \left(\frac{i}{n+1}\right) \right) \left(\alpha^{-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)} \frac{(\alpha-1)\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)-\alpha}{(\alpha-1)^2} \right) = 0 \quad (25)$$

$$\frac{\partial Z(\alpha,\theta)}{\partial\theta} = \sum_{i=1}^{n} \left(\frac{\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)}-1}{\alpha-1} - \left(\frac{i}{n+1}\right) \right) \left(\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)} \right) \log\alpha = 0 \quad (26)$$

From the above said equation (25) and equation (26) are difficult to obtain the estimators (α, θ), so they can be computed by using R software.

8.2 Maximum Likelihood Estimator

8.2.1 Complete Data

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample from APTG (α,θ), distribution then

$$L(x_i;\alpha,\theta)=\prod_{i=1}^{n}\left(f(x_i;\alpha,\theta)\right)_i$$

The log-likelihood function becomes

$$L=L(\theta,\alpha)=n\left\{\log(\log\alpha)-\log(\alpha-1)+\log(\theta)-\log(\theta+2)\right\} \tag{27}$$

$$+\sum_{i=1}^{n}(1+\theta+\theta x_i)-\theta\sum_{i=1}^{n}x_i+\log\alpha\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta x_i}{\theta+2}\right)e^{-\theta x_i}\right)$$

Therefore, the maximum likelihood equations are given by

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha\log\alpha}-\frac{n}{\alpha-1}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta x_i}{\theta+2}\right)e^{-\theta x_i}\right) \tag{28}$$

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}-\frac{n}{(\theta+2)}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\left(\frac{1+x_i}{1+\theta+\theta x_i}\right)$$

$$+\log\alpha\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta x_i}{\theta+2}\right)e^{-\theta x_i}\right) \tag{29}$$

The MLEs of α and θ can be obtained by solving (28) and (29) equations simultaneously, and they will be denoted by $\hat{\alpha}$, $\hat{\theta}$.

Because of the complicated form of likelihood equations (26) and (27) algebraically it is very difficult to solve the system of nonlinear equations. Therefore, we use R and wolfram mathematics for estimating the required parameters.

8.2.2 Censored Data

Consider a data set $D=(x:r)$, where $x=x_1,x_2,x_3,\ldots,x_n$ the observed failure are times and $r_i=r_1,r_2,r_3,\ldots,r_n$ are the censored failure times where r_i is equal to 1 if a failure is observed and 0 otherwise. Suppose that the data are independently and identically distributed follows a distribution with probability density $f(x,\theta)$ and survival functions $S(x,\theta)$ respectively.

$$f_{APT}(x;\theta,\alpha)=\left(\frac{\log\alpha}{\alpha-1}\left(\frac{\theta}{\theta+2}\right)(1+\theta+\theta x)\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\theta x}\right)$$

Survival function:

$$S_{APT}(x;\alpha,\theta)=\frac{\alpha}{\alpha-1}\left(1-\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}}\right)$$

The likelihood function for the observations is given by

$$L(x;\alpha,\theta)=\prod_{i=1}^{n}\left(\frac{\log\alpha}{\alpha-1}\left(\frac{\theta}{\theta+2}\right)(1+\theta+\theta x)\;\alpha^{\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)}e^{-\theta x}\right)^{r_i}$$

$$\left(\frac{\alpha}{\alpha-1}\left(1-\alpha^{-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}}\right)\right)^{1-r_i}$$

$$\log L(x;\alpha,\theta)=\log\left(\frac{\log\alpha}{\alpha-1}\right)\sum_{i=1}^{n}r_i+\log\left(\frac{\theta}{\theta+2}\right)\sum_{i=1}^{n}r_i+$$

$$\sum_{i=1}^{n}r_i\log(1+\theta+\theta x_i)-\theta\sum_{i=1}^{n}r_i x_i+\log\left(\frac{\alpha}{\alpha-1}\right)^{i}$$

$$\sum_{i=1}^{n}(1-r_i)+\sum_{i=1}^{n}(1-r_i)\log\left(1-\alpha^{-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+2}\right)}\right)$$

$$\log L=\left(\log(\log(\alpha))-\log(\alpha-1)\right)\sum_{i=1}^{n}r_i+\left(\log(\theta)-\log(\theta+2)\right)$$

$$\sum_{i=1}^{n}r_i+\sum_{i=1}^{n}r_i\log(1+\theta+\theta x_i)-\theta\sum_{i=1}^{n}r_i x_i+\left(\log(\alpha)-\log(\alpha-1)\right)^{i}$$

$$\sum_{i=1}^{n}(1-r_i)+\sum_{i=1}^{n}(1-r_i)\log\left(1-\alpha^{-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+2}\right)}\right)$$

The log likelihood function in can be maximized numerically in order to obtain the ML estimates.

9 Data Analysis

In this section, we work with an application based on a real data set and found that APTG distribution gives the best fit for the considered data.

9.1 Data Set 1

Gross and Clark [28] reported a set of data relating relief in minutes receiving of 20 patients.

The data is given below:

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3.0	1.7	2.3	1.6	2.0

In order to compare the distributions, we consider the criteria like AIC (Akaike Information Criterion), AICC (Corrected Akaike Information Criterion) And BIC (Bayesian Information Criterion). The better distribution corresponds to lesser AIC, AICC and BIC values.

$$AIC = 2k - 2\log L$$

$$BIC = k\log n - 2\log L$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

Where, k is the number of parameters in the statistical model, n is the sample size and $-2\,log\,L$ the maximized value of the log-likelihood function and are shown in Table1and Table 2.

9.2 Data Set 2

The data set represents the life time of 50 devices from Aarset [29] et al. it consists of the following set of observations.

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Table 1: MLE's and Criteria for Comparison for data set 1.

Distribution	MLE	S.D	-2logL	AIC	BIC	AICC
PLD	$\hat{\theta}$=0.34448830 $\hat{\beta}$=2.25294202	$\hat{\theta}$=0.0996896 $\hat{\beta}$=0.3067661	40.86395	44.86396	46.85543	46.36396
LBWNQLD	$\hat{\alpha}$ =1.578852 $\hat{\beta}$ =8.378050	$\hat{\alpha}$=2.038191 $\hat{\beta}$=3.355450	45.77610	49.7761	51.76757	51.2761
APTGD	$\hat{\theta}$=2.044349 $\hat{\alpha}$ = 1.608602	$\hat{\theta}$ =1.531384 $\hat{\alpha}$= 3.355495	34.84120	38.84121	40.83267	40.34121
LBG	$\hat{\theta}$=1.306236	$\hat{\theta}$= 0.183792	50.6912	52.6912	53.68693	52.857629
LBETE	$\hat{\alpha}$ =1.162584 $\hat{\beta}$ =1.531495	$\hat{\alpha}$=167.86310 $\hat{\beta}$ =116.95185	52.32636	56.32636	58.31782	57.82636
ODOMA	$\hat{\theta}$=1.7229491	$\hat{\theta}$= 0.1326213	62.05256	64.05256	65.0483	65.55256
GARIMA	$\hat{\theta}$=0.7395722	$\hat{\theta}$ = 0.1405294	63.2116	65.21155	66.20728	65.43377
Exponential	$\hat{\theta}$=0.5263157	$\hat{\theta}$= 0.1176873	65.67416	67.67416	68.66989	67.89638

Table 2: MLE's and Criteria for Comparison for data set 2.

Distribution	MLE	S.D	-2logL	AIC	BIC	AICC
APTGD	$\hat{\theta}$=0.03624992 $\hat{\alpha}$ =1.7764411	$\hat{\theta}$ =0.0061332 $\hat{\alpha}$ =1.3098916	479.8391	483.8394	487.6634	484.3611
PLD	$\hat{\theta}$=0.16120455 $\hat{\beta}$=0.66379655	$\hat{\theta}$=0.04475142 $\hat{\beta}$=0.06848726	484.1747	488.1746	491.9987	488.6964
LBWTPAD	$\hat{\alpha}$=1.6632881 $\hat{\theta}$=6.421260	$\hat{\alpha}$=1.0755112 θ = 6.972958	512.6821	516.6821	520.5061	517.2038
APTLD	$\hat{\alpha}$ =1.2593115 $\hat{\theta}$=0.04439636	$\hat{\alpha}$=0.90296576 $\hat{\theta}$=0.00631427	502.7629	506.7619	510.586	507.2837
LBG	$\hat{\theta}$=0.05648725	$\hat{\theta}$=0.00512144	524.5255	526.5255	528.4375	526.6088
WAD	$\hat{\alpha}$=0.00100000 $\hat{\theta}$=0.08690334	$\hat{\alpha}$=0.74490857 $\hat{\theta}$=0.01747111	590.0163	594.0163	597.8403	594.5380
ODOMA	$\hat{\theta}$=0.11063692	$\hat{\theta}$=0.00696766	668.2526	670.2527	672.1647	670.7744

It has been observed from Table1 and Table 2 that the APTG distribution have the lesser AIC, BIC, AICC values as compared to other distributions. Therefore, we conclude that the Alpha Power Transformed Garima Distribution leads to a better fit than the other distributions.

10 Conclusions

In this paper, we have introduced a new generalization of Garima distribution called Alpha Power Transformed Garima Distribution. We have studied various mathematical and statistical properties of distribution. The moments, survival function, hazard function and the maximum likelihood estimates, LSE of the parameters, have been investigated. The application of the new distribution has also been demonstrated with real-life data over one and two parameter Like APTLD, LBETE, LBG, WAD etc. The results are compared with APTG distribution, revealed that the APTG distribution provides a better fit than other distributions.

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Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] Lee, F. Famoye, A. Alzaatreh, Methods for generating families of continuous distribution in the recent decades, Wiley Interdiscip, *Rev. Comput Stat*., **5**, 219–238 (2013).

[2] M. C. Jones, On families of distribution with shape parameters, *Int. Stat Rev,* **83(2)**, 175-192 (2015).

[3] A. Alzaatreh, C. Lee, F. Famoye, A new method for generating families of continuous distributions, *Metron*, **71**, 63–79 (2013).

[4] M. Eugene, C. Lee, Famoye, F. Beta-normal distribution and its applications, *Commun Stat Theory Methods,* **31**, 497–512 (2002).

[5] K. Zografos, N. Balakrishnan, On families of beta- and generalized gamma-generated distributions and associated inference, *Stat Methodoly*, **6**, 344-362 (2009).

[6] G.M. Cordeiro, M. Castro, A new family of generalized distributions, *J. Stat Comput. Simul.*, **81**, 883–898 (2011).

[7] M. Bourguignon, R. B. Silva, G. M. Cordeiro, the Weibull-G family of probability distribution, *J. Data Sci.,* **12**, 53-68 (2014).

[8] C. Kus, A new lifetime distribution*, Comput. Stat. Data Annals,* **51**, 4497–4509 (2007).

[9] Proschan, Theoretical explanation of observed decreasing failure rate, *Technometrics,* **5**, 375–383

(1963).

[10] Efron, Logistic regression, survival analysis and the Kaplan-Meier curve*, J. Am. Stat. Assoc.,* **83**, 414–425 (1988).

[11] Langlands, S. Pocock, G. Kerr, S. Gore, Long-term survival of patients with breast cancer: a study of the curability of the disease, *Br Med J*., **2**, 1247–1251 (1997).

[12] S. Bennette, Log-logistic regression models for survival data, *Appl. Stat.,* **32**, 165-171 (1963).

[13] S. Dey, D. Kumar, Alpha power transformed Lindley distribution, properties and associated inference with application to earthquake data, *Annals data Sci.,* **6(4)**, 625-650 (2019).

[14] M. Nassar, A. Alzaatreh, M. Mead, O. Abo-Kasem, Alpha power Weibull distribution, Properties and applications*, Communications in Statistics—Theory and Methods,* **46(20)**, 10236–10252 (2017).

[15] I. Elbatal, A. Zubair, et.al A new alpha power transformed family of distributions, properties and application to Weibull model, *J. Non-linear Sci. Appl*., **12**, 1-20 (2019).

[16] S. Nasira, P.N. Mwitha, Alpha Power Transformed Frechet Distribution, *Applied Mathematics and Information Sciences,* **13(1)**, 129-141 (2019).

[17] S. Dey, A. Alzaatreh, C. Zhang, D. Kumar, An New extension of generalized exponential distribution with application to ozone data, *Ozone Sci. and Eng*., **39(4),** 273-285 (2017a).

[18] U. Ceren, and C. Salen, Alpha Power Inverted Exponential Distribution*, Gazi University Journal of Science,* **31(3)**, 954-965 (2018).

[19] R. Shanker, Garima distribution and its application to model behavioral science data, *Biometrics and Biostatistics International Journal,* **4(7)**, (2016).

[20] R. Shanker, The discrete Poisson Garima distribution, *Biometrics and Biostatistics International Journal,* **5(2)** (2017).

[21] R. Shenbagaraja, C. Subramanian, On Some Aspects of Length Biased Technique with Real Life Data, *Science, Technology, and Development*, **8(9)**, 326-335 (2019).

[22] A. Mahdavi, D. Kundu, A new method for generating distributions with an application to exponential distribution, *Commun. Stat Theory Methods*, **46(13)**, 6543–6557 (2017).

[23] A. Rényi, *On measures of entropy and information, Proceedings of fourth Berkeley symposium mathematics statistics and probability*, University of California Press, Berkeley, **1**, 547–561 (1961).

[24] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, *J. Stat Phys*., **52**, 479-487 (1988).

[25] C. E. Bonferroni, Elmenti di statistic a generale, *Libreria Seber, Firen*, (1930).

[26] M. O. Lorenz, Methods of measuring the concentration of wealth, *Publ. Am. Stat. Assoc*., **9**, 209–219 (1997).

[27] J. Swain, S. Venkatraman, and J. Wilson, Least square estimation of distribution function in Johnson's translation system, *J. Statist. Comput. Simul.*, **29**, 271-297 (1988).

[28] J. Gross, V. A. Clark, *Survival Distributions Reliability Applications in the Biometrical Sciences,* John Wiley, New York, (1975).

[29] M.V. Aarset, How to identify bathtub hazard rate, *IEEE Trans. Rel. R.,* **36**, 106-108 (1987).

[30] S. Dey, V. K. Sharma, M. Mesfioui, A new extension of Weibull distribution with application to lifetime data, *Ann. Data Sci,.***4(1),** 31-62 (2017b).

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