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Construct Extended Cubic B-Splines in n-Dimensional for Solving n-Dimensional Partial Differential Equations

K. R. Raslan[1], *Khalid K. Ali*[1,*] *and Hilal M. Y. Al-Bayatti* [2]

[1]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt
[2]Department of Computer Science, College of Arts and Science, Applied Science University, P.O. Box 5055, East Al-Ekir, Kingdom of Bahrain

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Abstract: In this work, we present a solution to a major problem that most researchers meet, which is the solution of differential equations of different dimensional by presenting a new structure to n-dimensional for the Extended cubic B-spline collocation algorithm. The Extended cubic B-spline collocation forms are displayed in one, two and three dimensional. These constructs are of prime importance in solving mathematical models that have applications in various sciences. The efficiency and accuracy of these algorithms through a few test problems in two and three dimensional. Also, comparing our solutions and with the results obtained by using other numerical methods available in the literature as much as possible.

Keywords: Collocation method; n-dimensional; The Extended cubic B-splines; Error estimation.

1 Introduction

We all know that obtaining solutions to partial differential equations is of great importance in various fields. The solutions to these equations are divided into analytical solutions and numerical solutions [1–5]. Recently, researchers have tended to use different methods to find these solutions, whether analytical or numerical. With the existence of models for these equations that have a degree of difficulty in finding solutions to them, especially if they are in two dimensional or in three dimensional or more than that, serious work continued on how to find these solutions. Over time, some researchers in this field found it difficult to find analytical solutions with different dimensional for these models, so some of them went to find numerical solutions to them. Several researchers have used different numerical methods to find solutions to these equations with different dimensional [6–16]. Now, we are continuing to work on developing a different basis B-spline collocation method to find numerical solutions to partial differential equations in two and three dimensional and so on. This work is a continuation of the works in [17, 18]. The collocation strategy began by Frazer et al. [19] in 1937. Afterward, the collocation strategy at the side the least-squares strategy and Galerkin strategy was utilized by Bickley [20] to ponder shaky heat condition issues. Afterward on, in 1975, the collocation strategy beside B-splines was connected to shaky warm conduction and boundary layer streams [21] and it was found that the comes about gotten were superior when compared with comes about gotten with limited contrast strategies. Since at that point, the collocation strategy is being utilized over a wide extend of issues [22–29]. In combination with the collocation strategy, there has been serious utilization of polynomial B-splines for understanding halfway differential conditions. Cubic B-splines, quasi B-splines, quartic B-splines, quintic B-splines, and so on are utilized in combination with the collocation strategy in [22–27] for managing with different straight and nonlinear boundary esteem issues. Strategies like Haar wavelet collocation strategy [30], a slope replicating part collocation strategy [31] and Newton premise capacities collocation strategy [32] are moreover picking up ubiquity to illuminate differential conditions.

In this work, we presente the Extended cubic B-spline collocation algorithm forms in n-dimensional. In addition, some numerical examples are proposed to study the effectiveness and accuracy of this method.

* Corresponding author e-mail: khalidkaram2012@azhar.edu.eg

This article is organized as follows: The second section introduces n-dimensional Extended cubic B-spline formulations. The third section introduces numerical examples. The error estimates are present in section four. Finally, show the conclusion part.

2 Construct Extended cubic B-spline Formulas

The forms for n-dimensional Extended cubic B-splines were introduced in this part.

2.1 One dimension Extended cubic B-spline [21]

Let $x \in [a,b]$ and $\Lambda_i(x)$ are those Extended cubic B-spline with knots at the points x_i. Then the set of Extended cubic B-splines $\Lambda_{-1}(x), \Lambda_0(x), ..., \Lambda_{N-1}(x)$, $\Lambda_N(x)$, Λ_{N+1}, forms a basis for functions defined over the interval. The approximation $\mathscr{H}^N(x)$ to $\mathscr{H}(x)$ which uses these splines as :

$$\mathscr{H}^N(x) = \sum_{i=-1}^{N+1} \mathscr{L}_i \; \Lambda_i(x), \tag{1}$$

where $\mathscr{L}_i$ unknown term. The formulations of $\mathscr{H}_i$, $\frac{dU_i}{dx}$, $\frac{d^2\mathscr{H}_i}{dx^2}$ are given by:

$$\begin{aligned}
\mathscr{H}_i &= \frac{1}{24}\left(-(\rho-4)\mathscr{L}_{i-1} + 2(\rho+8)\mathscr{L}_i - (\rho-4)\mathscr{L}_{i+1}\right),\\
\frac{d\ \mathscr{H}_i}{d\ x} &= \frac{\mathscr{L}_{i+1} - \mathscr{L}_{i-1}}{2h},\\
\frac{d^2\mathscr{H}_i}{dx^2} &= \frac{(\rho+2)\left(\mathscr{L}_{i-1} - 2\mathscr{L}_i + \mathscr{L}_{i+1}\right)}{2h^2}.
\end{aligned} \tag{2}$$

2.2 Two dimensional Extended cubic B-spline

In this subsection, we show the formula of Extended cubic B-spline in two dimensional on a rectangular grid divided into regular rectangular finite elements on both sides. $h = \Delta x, k = \Delta y$ by the knots (x_m, y_n) where $m = 0, 1, ..., M, n = 0, 1, ..., N$. The approximation $\mathscr{H}^N(x,y)$ to $\mathscr{H}(x,y)$ given by:

$$\mathscr{H}^N(x,\,y) = \sum_{m=-1}^{M+1}\sum_{n=-1}^{N+1} \mathscr{L}_{m,n}\mho_{m,n}(x,y), \tag{3}$$

where $\mathscr{L}_{m,n}$ are the amplitudes of the Extended cubic B-splines $\mho_{m,n}(x,y)$ given by

$$\mho_{m,n}(x,y) = \Lambda_m(x)\Lambda_n(y).$$

Which peaks on the knot (x_m, y_n) and $\Lambda_m(x)$, $\Lambda_n(y)$ are identical in form to the one dimension Extended cubic B-splines. Then the formulations of $\mathscr{H}_{m,n}$, $\frac{\partial\mathscr{H}_{m,n}}{\partial x}$, $\frac{\partial\mathscr{H}_{m,n}}{\partial y}$, $\frac{\partial^2\mathscr{H}_{m,n}}{\partial x^2}$, $\frac{\partial^2\mathscr{H}_{m,n}}{\partial y^2}$, $\frac{\partial^2\mathscr{H}_{m,n}}{\partial x\partial y}$, ... are given by:

$$\begin{aligned}
\mathscr{H}_{m,\,n} = \frac{1}{576}
&\Big(-2\left(\rho^2+4\rho-32\right)\mathscr{L}_{m-1,n} + \rho^2\mathscr{L}_{m-1,n+1}\\
&-2\rho^2\mathscr{L}_{m,n-1} + 4\rho^2\mathscr{L}_{m,n} - 2\rho^2\mathscr{L}_{m,n+1} + \rho^2\mathscr{L}_{m+1,n-1}\\
&-2\rho^2\mathscr{L}_{m+1,n} + \rho^2\mathscr{L}_{m+1,n+1} + (\rho-4)^2\mathscr{L}_{m-1,n-1}\\
&-8\rho\mathscr{L}_{m-1,n+1} - 8\rho\mathscr{L}_{m,n-1} + 64\rho\mathscr{L}_{m,n} - 8\rho\mathscr{L}_{m,n+1}\\
&-8\rho\mathscr{L}_{m+1,n-1} - 8\rho\mathscr{L}_{m+1,n} - 8\rho\mathscr{L}_{m+1,n+1}\\
&+16\mathscr{L}_{m-1,n+1} + 64\mathscr{L}_{m,n-1} + 256\mathscr{L}_{m,n} + 64\mathscr{L}_{m,n+1}\\
&+16\mathscr{L}_{m+1,n-1} + 64\mathscr{L}_{m+1,n} + 16\mathscr{L}_{m+1,n+1}\Big).
\end{aligned} \tag{4}$$

$$\begin{aligned}
\frac{\partial\mathscr{H}_{m,n}}{\partial x} = \frac{1}{48h}&\Big((\rho-4)\mathscr{L}_{m-1,n-1} - 2(\rho+8)\mathscr{L}_{m-1,n}\\
&+\rho\mathscr{L}_{m-1,n+1} - \rho\mathscr{L}_{m+1,n-1} + 2\rho\mathscr{L}_{m+1,n}\\
&-\rho\mathscr{L}_{m+1,n+1} - 4\mathscr{L}_{m-1,n+1} + 4\mathscr{L}_{m+1,n-1}\\
&+16\mathscr{L}_{m+1,n} + 4\mathscr{L}_{m+1,n+1}\Big),\\
\frac{\partial\mathscr{H}_{m,n}}{\partial y} = \frac{1}{48k}&\Big((\rho-4)\mathscr{L}_{m-1,n-1}\\
&(\rho-4)\mathscr{L}_{m-1,n+1} - 2\rho\mathscr{L}_{m,n-1} + 2\rho\mathscr{L}_{m,n+1}\\
&+\rho\mathscr{L}_{m+1,n-1} - \rho\mathscr{L}_{m+1,n+1} - 16\mathscr{L}_{m,n-1} + 16\mathscr{L}_{m,n+1}\\
&-4\mathscr{L}_{m+1,n-1} + 4\mathscr{L}_{m+1,n+1}\Big).
\end{aligned} \tag{5}$$

$$\begin{aligned}
\frac{\partial^2\mathscr{H}_{m,n}}{\partial x^2} = -\frac{\rho+2}{48h^2}&\\
\Big((\rho-4)\mathscr{L}_{m-1,n-1} &- 2(\rho+8)\mathscr{L}_{m-1,n} + \rho\mathscr{L}_{m-1,n+1}\\
-2\rho\mathscr{L}_{m,n-1} &+ 4\rho\mathscr{L}_{m,n} - 2\rho\mathscr{L}_{m,n+1} + \rho\mathscr{L}_{m+1,n-1}\\
-2\rho\mathscr{L}_{m+1,n} &+ \rho\mathscr{L}_{m+1,n+1} - 4\mathscr{L}_{m-1,n+1}\\
+8\mathscr{L}_{m,n-1} &+ 32\mathscr{L}_{m,n} + 8\mathscr{L}_{m,n+1} - 4\mathscr{L}_{m+1,n-1}\\
-16\mathscr{L}_{m+1,n} &- 4\mathscr{L}_{m+1,n+1}\Big),\\
\frac{\partial^2\mathscr{H}_{m,n}}{\partial y^2} = -\frac{\rho+2}{48k^2}&\Big((\rho-4)\mathscr{L}_{m-1,n-1} - 2(\rho-4)\mathscr{L}_{m-1,n}\\
+\rho\mathscr{L}_{m-1,n+1} &- 2\rho\mathscr{L}_{m,n-1} + 4\rho\mathscr{L}_{m,n} - 2\rho\mathscr{L}_{m,n+1}\\
+\rho\mathscr{L}_{m+1,n-1} &- 2\rho\mathscr{L}_{m+1,n} + \rho\mathscr{L}_{m+1,n+1} - 4\mathscr{L}_{m-1,n+1}\\
-16\mathscr{L}_{m,n-1} &+ 32\mathscr{L}_{m,n} - 16\mathscr{L}_{m,n+1} - 4\mathscr{L}_{m+1,n-1}\\
+8\mathscr{L}_{m+1,n} &- 4\mathscr{L}_{m+1,n+1}\Big).
\end{aligned} \tag{6}$$

$$\frac{\partial^2 \mathscr{H}_{m,n}}{\partial x \partial y} = \frac{\mathscr{L}_{m-1,n-1} - \mathscr{L}_{m-1,n+1} - \mathscr{L}_{m+1,n-1} + \mathscr{L}_{m+1,n+1}}{4hk},$$
$$\frac{\partial^3 \mathscr{H}_{m,n}}{\partial x^2 \partial y} = -\frac{(\rho+2)}{4h^2k}$$
$$(\mathscr{L}_{m-1,n-1} - \mathscr{L}_{m-1,n+1} - 2\mathscr{L}_{m,n-1} + 2\mathscr{L}_{m,n+1} + \mathscr{L}_{m+1,n-1} - \mathscr{L}_{m+1,n+1}),$$
$$\frac{\partial^3 \mathscr{H}_{m,n}}{\partial x \partial y^2} = -\frac{(\rho+2)}{4hk^2}$$
$$(\mathscr{L}_{m-1,n-1} - 2\mathscr{L}_{m-1,n} + \mathscr{L}_{m-1,n+1} - \mathscr{L}_{m+1,n-1} + 2\mathscr{L}_{m+1,n} - \mathscr{L}_{m+1,n+1}),$$
$$\vdots \tag{7}$$

2.3 The three dimensional Extended cubic B-spline

Now, we obtain the Extended cubic B-spline in three measurements approximates on a framework subdivided into limited components of sides $h = \Delta x, k = \Delta y\ , q = \Delta z$ by the knots (x_m, y_n, z_r) where $m = 0,1,...,M, n = 0,1,...,N, r = 0,1,...,R$ can be interpolated in terms of piecewise Extended cubic B-splines. If $\mathscr{H}(x,y,z)$ is a function of x, y and z, it can be shown there exists a unique approximation $\mathscr{H}^N(x,y,z)$ as

$$\mathscr{H}^N(x,y,z) = \sum_{m=-1}^{M+1} \sum_{n=-1}^{N+1} \sum_{r=-1}^{R+1} \mathscr{L}_{m,n,r} \mho_{m,n,r}(x,y,z), \tag{8}$$

where $\mathscr{L}_{m,n,r}$ are the amplitudes of the Extended cubic B-splines $\mho_{m,n,r}(x,y,z)$ given by

$$\mho_{m,n,r}(x,y,z) = \Lambda_m(x)\Lambda_n(y)\Lambda_r(z).$$

Also, $\Lambda_m(x), \Lambda_n(y)$ and $\Lambda_r(z)$ have the same form as the one dimension Extended cubic B-splines. The formulations of $\mathscr{H}_{m,n,r}$, $\frac{\partial \mathscr{H}_{m,n,r}}{\partial x}$, $\frac{\partial \mathscr{H}_{m,n,r}}{\partial y}$, $\frac{\partial \mathscr{H}_{m,n,r}}{\partial z}$, $\frac{\partial^2 \mathscr{H}_{m,n,r}}{\partial x^2}$, $\frac{\partial^2 \mathscr{H}_{m,n,r}}{\partial y^2}$, $\frac{\partial^2 \mathscr{H}_{m,n,r}}{\partial z^2}$, $\frac{\partial^2 \mathscr{H}_{m,n,r}}{\partial x \partial y}$, $\frac{\partial^2 \mathscr{H}_{m,n,r}}{\partial x \partial z}$, $\ldots$, are given in terms of the $\mathscr{L}_{m,n,r}$ by:

$$\begin{aligned}
\mathscr{H}_{m,n,r} = \frac{1}{13824}\Big(& 2(\rho+8)\mathscr{L}_{m-1,n-1,r}(\rho-4)^2 \\
& - (\rho-4)^3 \mathscr{L}_{m-1,n-1,r-1} - \rho^3 \mathscr{L}_{m-1,n-1,r+1} \\
& + 12\rho^2 \mathscr{L}_{m-1,n-1,r+1} - 48\rho \mathscr{L}_{m-1,n-1,r+1} \\
& + 64\mathscr{L}_{m-1,n-1,r+1} + 2\rho^3 \mathscr{L}_{m-1,n,r-1} \\
& - 96\rho \mathscr{L}_{m-1,n,r-1} + 256\mathscr{L}_{m-1,n,r-1} - 4\rho^3 \mathscr{L}_{m-1,n,r} \\
& - 48\rho^2 \mathscr{L}_{m-1,n,r} + 1024\mathscr{L}_{m-1,n,r} + 2\rho^3 \mathscr{L}_{m-1,n,r+1} \\
& - 96\rho \mathscr{L}_{m-1,n,r+1} + 256\mathscr{L}_{m-1,n,r+1} \\
& - \rho^3 \mathscr{L}_{m-1,n+1,r-1} + 12\rho^2 \mathscr{L}_{m-1,n+1,r-1} \\
& - 48\rho \mathscr{L}_{m-1,n+1,r-1} + 64\mathscr{L}_{m-1,n+1,r-1} \\
& + 2\rho^3 \mathscr{L}_{m-1,n+1,r} - 96\rho \mathscr{L}_{m-1,n+1,r} \\
& + 256\mathscr{L}_{m-1,n+1,r} - \rho^3 \mathscr{L}_{m-1,n+1,r+1} \\
& + 12\rho^2 \mathscr{L}_{m-1,n+1,r+1} - 48\rho \mathscr{L}_{m-1,n+1,r+1} \\
& + 64\mathscr{L}_{m-1,n+1,r+1} + 2\rho^3 \mathscr{L}_{m,n-1,r-1} \\
& - 96\rho \mathscr{L}_{m,n-1,r-1} + 256\mathscr{L}_{m,n-1,r-1} \\
& - 4\rho^3 \mathscr{L}_{m,n-1,r} - 48\rho^2 \mathscr{L}_{m,n-1,r} \\
& + 1024\mathscr{L}_{m,n-1,r} + 2\rho^3 \mathscr{L}_{m,n-1,r+1} \\
& - 96\rho \mathscr{L}_{m,n-1,r+1} + 256\mathscr{L}_{m,n-1,r+1} \\
& - 4\rho^3 \mathscr{L}_{m,n,r-1} - 48\rho^2 \mathscr{L}_{m,n,r-1} + 1024\mathscr{L}_{m,n,r-1} \\
& + 8\rho^3 \mathscr{L}_{m,n,r} + 192\rho^2 \mathscr{L}_{m,n,r} + 1536\rho \mathscr{L}_{m,n,r} \\
& + 4096\mathscr{L}_{m,n,r} - 4\rho^3 \mathscr{L}_{m,n,r+1} \\
& + 256\mathscr{L}_{m,n+1,r-1} - 4\rho^3 \mathscr{L}_{m,n+1,r} - 48\rho^2 \mathscr{L}_{m,n+1,r} \\
& + 1024\mathscr{L}_{m,n+1,r} + 2\rho^3 \mathscr{L}_{m,n+1,r+1} - 96\rho \mathscr{L}_{m,n+1,r+1} \\
& + 256\mathscr{L}_{m,n+1,r+1} - \rho^3 \mathscr{L}_{m+1,n-1,r-1} \\
& + 12\rho^2 \mathscr{L}_{m+1,n-1,r-1} - 48\rho \mathscr{L}_{m+1,n-1,r-1} \\
& + 64\mathscr{L}_{m+1,n-1,r-1} + 2\rho^3 \mathscr{L}_{m+1,n-1,r} \\
& - 96\rho \mathscr{L}_{m+1,n-1,r} + 256\mathscr{L}_{m+1,n-1,r} \\
& - \rho^3 \mathscr{L}_{m+1,n-1,r+1} + 12\rho^2 \mathscr{L}_{m+1,n-1,r+1} \\
& - 48\rho \mathscr{L}_{m+1,n-1,r+1} + 64\mathscr{L}_{m+1,n-1,r+1} \\
& + 2\rho^3 \mathscr{L}_{m+1,n,r-1} - 96\rho \mathscr{L}_{m+1,n,r-1} + 256\mathscr{L}_{m+1,n,r-1} \\
& - 4\rho^3 \mathscr{L}_{m+1,n,r} - 48\rho^2 \mathscr{L}_{m+1,n,r} + 1024\mathscr{L}_{m+1,n,r} \\
& + 2\rho^3 \mathscr{L}_{m+1,n,r+1} - 96\rho \mathscr{L}_{m+1,n,r+1} \\
& + 256\mathscr{L}_{m+1,n,r+1} - \rho^3 \mathscr{L}_{m+1,n+1,r-1} \\
& + 12\rho^2 \mathscr{L}_{m+1,n+1,r-1} - 48\rho \mathscr{L}_{m+1,n+1,r-1} \\
& + 64\mathscr{L}_{m+1,n+1,r-1} + 2\rho^3 \mathscr{L}_{m+1,n+1,r} \\
& - 96\rho \mathscr{L}_{m+1,n+1,r} + 256\mathscr{L}_{m+1,n+1,r} \\
& - \rho^3 \mathscr{L}_{m+1,n+1,r+1} + 12\rho^2 \mathscr{L}_{m+1,n+1,r+1} \\
& - 48\rho \mathscr{L}_{m+1,n+1,r+1} + 64\mathscr{L}_{m+1,n+1,r+1}\Big).
\end{aligned} \tag{9}$$

$$
\begin{aligned}
\frac{\partial \mathscr{H}_{m,n,r}}{\partial x} = \frac{1}{1152h}\Big(& 2\left(\rho^2+4\rho-32\right)\mathscr{L}_{m-1,n-1,r} \\
& -\rho^2\mathscr{L}_{m-1,n-1,r+1}+2\rho^2\mathscr{L}_{m-1,n,r-1}-4\rho^2\mathscr{L}_{m-1,n,r} \\
& +2\rho^2\mathscr{L}_{m-1,n,r+1}-\rho^2\mathscr{L}_{m-1,n+1,r-1}+2\rho^2\mathscr{L}_{m-1,n+1,r} \\
& -\rho^2\mathscr{L}_{m-1,n+1,r+1}+\rho^2\mathscr{L}_{m+1,n-1,r-1}-2\rho^2\mathscr{L}_{m+1,n-1,r} \\
& +\rho^2\mathscr{L}_{m+1,n-1,r+1}-2\rho^2\mathscr{L}_{m+1,n,r-1}+4\rho^2\mathscr{L}_{m+1,n,r} \\
& -2\rho^2\mathscr{L}_{m+1,n,r+1}+\rho^2\mathscr{L}_{m+1,n+1,r-1}-2\rho^2\mathscr{L}_{m+1,n+1,r} \\
& +\rho^2\mathscr{L}_{m+1,n+1,r+1}-(\rho-4)^2\mathscr{L}_{m-1,n-1,r-1} \\
& +8\rho\mathscr{L}_{m-1,n-1,r+1}+8\rho\mathscr{L}_{m-1,n,r-1} \\
& -64\rho\mathscr{L}_{m-1,n,r}+8\rho\mathscr{L}_{m-1,n,r+1}+8\rho\mathscr{L}_{m-1,n+1,r-1} \\
& +8\rho\mathscr{L}_{m-1,n+1,r}+8\rho\mathscr{L}_{m-1,n+1,r+1}-8\rho\mathscr{L}_{m+1,n-1,r-1} \\
& -8\rho\mathscr{L}_{m+1,n-1,r}-8\rho\mathscr{L}_{m+1,n-1,r+1}-8\rho\mathscr{L}_{m+1,n,r-1} \\
& +64\rho\mathscr{L}_{m+1,n,r}-8\rho\mathscr{L}_{m+1,n,r+1}-8\rho\mathscr{L}_{m+1,n+1,r-1} \\
& -8\rho\mathscr{L}_{m+1,n+1,r}-8\rho\mathscr{L}_{m+1,n+1,r+1}-16\mathscr{L}_{m-1,n-1,r+1} \\
& -64\mathscr{L}_{m-1,n,r-1}-256\mathscr{L}_{m-1,n,r}-64\mathscr{L}_{m-1,n,r+1} \\
& -16\mathscr{L}_{m-1,n+1,r-1}-64\mathscr{L}_{m-1,n+1,r}-16\mathscr{L}_{m-1,n+1,r+1} \\
& +16\mathscr{L}_{m+1,n-1,r-1}+64\mathscr{L}_{m+1,n-1,r}+16\mathscr{L}_{m+1,n-1,r+1} \\
& +64\mathscr{L}_{m+1,n,r-1}+256\mathscr{L}_{m+1,n,r} \\
& +64\mathscr{L}_{m+1,n,r+1}+16\mathscr{L}_{m+1,n+1,r-1} \\
& +64\mathscr{L}_{m+1,n+1,r}+16\mathscr{L}_{m+1,n+1,r+1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathscr{H}_{m,n,r}}{\partial y} = \frac{1}{1152k}\Big(& 2\left(\rho^2+4\rho-32\right)\mathscr{L}_{m-1,n-1,r} \\
& -\rho^2\mathscr{L}_{m-1,n-1,r+1}+\rho^2\mathscr{L}_{m-1,n+1,r-1}-2\rho^2\mathscr{L}_{m-1,n+1,r} \\
& +\rho^2\mathscr{L}_{m-1,n+1,r+1}+2\rho^2\mathscr{L}_{m,n-1,r-1}-4\rho^2\mathscr{L}_{m,n-1,r} \\
& +2\rho^2\mathscr{L}_{m,n-1,r+1}-2\rho^2\mathscr{L}_{m,n+1,r-1}+4\rho^2\mathscr{L}_{m,n+1,r} \\
& -2\rho^2\mathscr{L}_{m,n+1,r+1}-\rho^2\mathscr{L}_{m+1,n-1,r-1}+2\rho^2\mathscr{L}_{m+1,n-1,r} \\
& -\rho^2\mathscr{L}_{m+1,n-1,r+1}+\rho^2\mathscr{L}_{m+1,n+1,r-1}-2\rho^2\mathscr{L}_{m+1,n+1,r} \\
& +\rho^2\mathscr{L}_{m+1,n+1,r+1}-(\rho-4)^2\mathscr{L}_{m-1,n-1,r-1} \\
& +8\rho\mathscr{L}_{m-1,n-1,r+1}-8\rho\mathscr{L}_{m-1,n+1,r-1} \\
& -8\rho\mathscr{L}_{m-1,n+1,r}-8\rho\mathscr{L}_{m-1,n+1,r+1}+8\rho\mathscr{L}_{m,n-1,r-1} \\
& -64\rho\mathscr{L}_{m,n-1,r}+8\rho\mathscr{L}_{m,n-1,r+1}-8\rho\mathscr{L}_{m,n+1,r-1} \\
& +64\rho\mathscr{L}_{m,n+1,r}-8\rho\mathscr{L}_{m,n+1,r+1} \\
& +8\rho\mathscr{L}_{m+1,n-1,r-1}+8\rho\mathscr{L}_{m+1,n-1,r} \\
& +8\rho\mathscr{L}_{m+1,n-1,r+1}-8\rho\mathscr{L}_{m+1,n+1,r-1} \\
& -8\rho\mathscr{L}_{m+1,n+1,r}-8\rho\mathscr{L}_{m+1,n+1,r+1}-16\mathscr{L}_{m-1,n-1,r+1} \\
& +16\mathscr{L}_{m-1,n+1,r-1}+64\mathscr{L}_{m-1,n+1,r} \\
& +16\mathscr{L}_{m-1,n+1,r+1}-64\mathscr{L}_{m,n-1,r-1} \\
& -256\mathscr{L}_{m,n-1,r}-64\mathscr{L}_{m,n-1,r+1} \\
& +64\mathscr{L}_{m,n+1,r-1}+256\mathscr{L}_{m,n+1,r}+64\mathscr{L}_{m,n+1,r+1} \\
& -16\mathscr{L}_{m+1,n-1,r-1}-64\mathscr{L}_{m+1,n-1,r} \\
& -16\mathscr{L}_{m+1,n-1,r+1}+16\mathscr{L}_{m+1,n+1,r-1} \\
& +64\mathscr{L}_{m+1,n+1,r}+16\mathscr{L}_{m+1,n+1,r+1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathscr{H}_{m,n,r}}{\partial z} = \frac{1}{1152q}\Big(& 2\rho^2\mathscr{L}_{m-1,n,r-1}-2\rho^2\mathscr{L}_{m-1,n,r+1} \\
& -\rho^2\mathscr{L}_{m-1,n+1,r-1}+\rho^2\mathscr{L}_{m-1,n+1,r+1}+2\rho^2\mathscr{L}_{m,n-1,r-1} \\
& -2\rho^2\mathscr{L}_{m,n-1,r+1}-4\rho^2\mathscr{L}_{m,n,r-1} \\
& +4\rho^2\mathscr{L}_{m,n,r+1}+2\rho^2\mathscr{L}_{m,n+1,r-1}-2\rho^2\mathscr{L}_{m,n+1,r+1} \\
& -\rho^2\mathscr{L}_{m+1,n-1,r-1}+\rho^2\mathscr{L}_{m+1,n-1,r+1}+2\rho^2\mathscr{L}_{m+1,n,r-1} \\
& -2\rho^2\mathscr{L}_{m+1,n,r+1}-\rho^2\mathscr{L}_{m+1,n+1,r-1}+\rho^2\mathscr{L}_{m+1,n+1,r+1} \\
& +(\rho-4)^2\mathscr{L}_{m-1,n-1,r+1}-(\rho-4)^2\mathscr{L}_{m-1,n-1,r-1} \\
& +8\rho\mathscr{L}_{m-1,n,r-1}-8\rho\mathscr{L}_{m-1,n,r+1} \\
& +8\rho\mathscr{L}_{m-1,n+1,r-1}-8\rho\mathscr{L}_{m-1,n+1,r+1} \\
& +8\rho\mathscr{L}_{m,n-1,r-1}-8\rho\mathscr{L}_{m,n-1,r+1} \\
& -64\rho\mathscr{L}_{m,n,r-1}+64\rho\mathscr{L}_{m,n,r+1} \\
& +8\rho\mathscr{L}_{m,n+1,r-1}-8\rho\mathscr{L}_{m,n+1,r+1} \\
& +8\rho\mathscr{L}_{m+1,n-1,r-1}-8\rho\mathscr{L}_{m+1,n-1,r+1} \\
& +8\rho\mathscr{L}_{m+1,n,r-1}-8\rho\mathscr{L}_{m+1,n,r+1} \\
& +8\rho\mathscr{L}_{m+1,n+1,r-1}-8\rho\mathscr{L}_{m+1,n+1,r+1} \\
& -64\mathscr{L}_{m-1,n,r-1}+64\mathscr{L}_{m-1,n,r+1}-16\mathscr{L}_{m-1,n+1,r-1} \\
& +16\mathscr{L}_{m-1,n+1,r+1}-64\mathscr{L}_{m,n-1,r-1} \\
& +64\mathscr{L}_{m,n-1,r+1}-256\mathscr{L}_{m,n,r-1}+256\mathscr{L}_{m,n,r+1} \\
& -64\mathscr{L}_{m,n+1,r-1}+64\mathscr{L}_{m,n+1,r+1} \\
& -16\mathscr{L}_{m+1,n-1,r-1}+16\mathscr{L}_{m+1,n-1,r+1} \\
& -64\mathscr{L}_{m+1,n,r-1}+64\mathscr{L}_{m+1,n,r+1} \\
& -16\mathscr{L}_{m+1,n+1,r-1}+16\mathscr{L}_{m+1,n+1,r+1}\Big).
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathscr{H}_{m,n,r}}{\partial x\partial y} = -\frac{1}{96hk}\Big(& (\rho-4)\mathscr{L}_{m-1,n-1,r-1} \\
& -2(\rho+8)\mathscr{L}_{m-1,n-1,r}+\rho\mathscr{L}_{m-1,n-1,r+1} \\
& -\rho\mathscr{L}_{m-1,n+1,r-1}+2\rho\mathscr{L}_{m-1,n+1,r} \\
& -\rho\mathscr{L}_{m-1,n+1,r+1}-\rho\mathscr{L}_{m+1,n-1,r-1} \\
& +2\rho\mathscr{L}_{m+1,n-1,r}-\rho\mathscr{L}_{m+1,n-1,r+1}+\rho\mathscr{L}_{m+1,n+1,r-1} \\
& -2\rho\mathscr{L}_{m+1,n+1,r}+\rho\mathscr{L}_{m+1,n+1,r+1}-4\mathscr{L}_{m-1,n-1,r+1} \\
& +4\mathscr{L}_{m-1,n+1,r-1}+16\mathscr{L}_{m-1,n+1,r}+4\mathscr{L}_{m-1,n+1,r+1} \\
& +4\mathscr{L}_{m+1,n-1,r-1}+16\mathscr{L}_{m+1,n-1,r}+4\mathscr{L}_{m+1,n-1,r+1} \\
& -4\mathscr{L}_{m+1,n+1,r-1}-16\mathscr{L}_{m+1,n+1,r}-4\mathscr{L}_{m+1,n+1,r+1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathscr{H}_{m,n,r}}{\partial x\partial z} = \frac{1}{96hs}\Big(& -(\rho-4)\mathscr{L}_{m-1,n-1,r-1} \\
& +(\rho-4)\mathscr{L}_{m-1,n-1,r+1}+2\rho\mathscr{L}_{m-1,n,r-1} \\
& -2\rho\mathscr{L}_{m-1,n,r+1}-\rho\mathscr{L}_{m-1,n+1,r-1}+\rho\mathscr{L}_{m-1,n+1,r+1} \\
& +\rho\mathscr{L}_{m+1,n-1,r-1}-\rho\mathscr{L}_{m+1,n-1,r+1}-2\rho\mathscr{L}_{m+1,n,r-1} \\
& +2\rho\mathscr{L}_{m+1,n,r+1}+\rho\mathscr{L}_{m+1,n+1,r-1}-\rho\mathscr{L}_{m+1,n+1,r+1} \\
& +16\mathscr{L}_{m-1,n,r-1}-16\mathscr{L}_{m-1,n,r+1}+4\mathscr{L}_{m-1,n+1,r-1}
\end{aligned}
$$

$$
\begin{aligned}
&-4\mathscr{L}_{m-1,n+1,r+1}-4\mathscr{L}_{m+1,n-1,r-1}+4\mathscr{L}_{m+1,n-1,r+1}\\
&-16\mathscr{L}_{m+1,n,r-1}+16\mathscr{L}_{m+1,n,r+1}\\
&-4\mathscr{L}_{m+1,n+1,r-1}+4\mathscr{L}_{m+1,n+1,r+1}\Big),\\
&\frac{\partial^2\mathscr{H}_{m,n,r}}{\partial y\partial z}=\frac{1}{96kq}\Big(-(\rho-4)\mathscr{L}_{m-1,n-1,r-1}\\
&+(\rho-4)\mathscr{L}_{m-1,n-1,r+1}+\rho\mathscr{L}_{m-1,n+1,r-1}\\
&-\rho\mathscr{L}_{m-1,n+1,r+1}+2\rho\mathscr{L}_{m,n-1,r-1}-2\rho\mathscr{L}_{m,n-1,r+1}\\
&-2\rho\mathscr{L}_{m,n+1,r-1}+2\rho\mathscr{L}_{m,n+1,r+1}-\rho\mathscr{L}_{m+1,n-1,r-1}\\
&+\rho\mathscr{L}_{m+1,n-1,r+1}+\rho\mathscr{L}_{m+1,n+1,r-1}-\rho\mathscr{L}_{m+1,n+1,r+1}\\
&-4\mathscr{L}_{m-1,n+1,r-1}+4\mathscr{L}_{m-1,n+1,r+1}+16\mathscr{L}_{m,n-1,r-1}\\
&-16\mathscr{L}_{m,n-1,r+1}-16\mathscr{L}_{m,n+1,r-1}+16\mathscr{L}_{m,n+1,r+1}\\
&+4\mathscr{L}_{m+1,n-1,r-1}-4\mathscr{L}_{m+1,n-1,r+1}\\
&-4\mathscr{L}_{m+1,n+1,r-1}+4\mathscr{L}_{m+1,n+1,r+1}\Big),\\
&\frac{\partial^3\mathscr{H}_{m,n,r}}{\partial x\partial y\partial z}=\frac{1}{8hkq}\Big(-\mathscr{L}_{m-1,n-1,r-1}+\mathscr{L}_{m-1,n-1,r+1}\\
&+\mathscr{L}_{m-1,n+1,r-1}-\mathscr{L}_{m-1,n+1,r+1}\\
&+\mathscr{L}_{m+1,n-1,r-1}-\mathscr{L}_{m+1,n-1,r+1}\\
&-\mathscr{L}_{m+1,n+1,r-1}+\mathscr{L}_{m+1,n+1,r+1}\Big),\\
&\vdots
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\frac{\partial^2\mathscr{H}_{m,n,r}}{\partial x^2}=\frac{\rho+2}{1152h^2}\Big(\mathscr{L}_{m-1,n-1,r-1}(\rho-4)^2\\
&-2\left(\rho^2+4\rho-32\right)\mathscr{L}_{m-1,n-1,r}+\rho^2\mathscr{L}_{m-1,n-1,r+1}\\
&-8\rho\mathscr{L}_{m-1,n-1,r+1}+16\mathscr{L}_{m-1,n-1,r+1}\\
&-2\rho^2\mathscr{L}_{m-1,n,r-1}-8\rho\mathscr{L}_{m-1,n,r-1}+64\mathscr{L}_{m-1,n,r-1}\\
&+4\rho^2\mathscr{L}_{m-1,n,r}+64\rho\mathscr{L}_{m-1,n,r}+256\mathscr{L}_{m-1,n,r}\\
&-2\rho^2\mathscr{L}_{m-1,n,r+1}-8\rho\mathscr{L}_{m-1,n,r+1}+64\mathscr{L}_{m-1,n,r+1}\\
&+\rho^2\mathscr{L}_{m-1,n+1,r-1}-8\rho\mathscr{L}_{m-1,n+1,r-1}+16\mathscr{L}_{m-1,n+1,r-1}\\
&-2\rho^2\mathscr{L}_{m-1,n+1,r}-8\rho\mathscr{L}_{m-1,n+1,r}+64\mathscr{L}_{m-1,n+1,r}\\
&+\rho^2\mathscr{L}_{m-1,n+1,r+1}-8\rho\mathscr{L}_{m-1,n+1,r+1}+16\mathscr{L}_{m-1,n+1,r+1}\\
&-2\rho^2\mathscr{L}_{m,n-1,r-1}+16\rho\mathscr{L}_{m,n-1,r-1}-32\mathscr{L}_{m,n-1,r-1}\\
&+4\rho^2\mathscr{L}_{m,n-1,r}+16\rho\mathscr{L}_{m,n-1,r}-128\mathscr{L}_{m,n-1,r}\\
&-2\rho^2\mathscr{L}_{m,n-1,r+1}+16\rho\mathscr{L}_{m,n-1,r+1}-32\mathscr{L}_{m,n-1,r+1}\\
&+4\rho^2\mathscr{L}_{m,n,r-1}+16\rho\mathscr{L}_{m,n,r-1}-128\mathscr{L}_{m,n,r-1}\\
&-8\rho^2\mathscr{L}_{m,n,r}-128\rho\mathscr{L}_{m,n,r}-512\mathscr{L}_{m,n,r}\\
&+4\rho^2\mathscr{L}_{m,n,r+1}+16\rho\mathscr{L}_{m,n,r+1}-128\mathscr{L}_{m,n,r+1}\\
&-2\rho^2\mathscr{L}_{m,n+1,r-1}+16\rho\mathscr{L}_{m,n+1,r-1}-32\mathscr{L}_{m,n+1,r-1}\\
&+4\rho^2\mathscr{L}_{m,n+1,r}+16\rho\mathscr{L}_{m,n+1,r}-128\mathscr{L}_{m,n+1,r}\\
&-2\rho^2\mathscr{L}_{m,n+1,r+1}+16\rho\mathscr{L}_{m,n+1,r+1}-32\mathscr{L}_{m,n+1,r+1}\\
&+\rho^2\mathscr{L}_{m+1,n-1,r-1}-8\rho\mathscr{L}_{m+1,n-1,r-1}+16\mathscr{L}_{m+1,n-1,r-1}\\
&-2\rho^2\mathscr{L}_{m+1,n-1,r}-8\rho\mathscr{L}_{m+1,n-1,r}+64\mathscr{L}_{m+1,n-1,r}\\
&+\rho^2\mathscr{L}_{m+1,n-1,r+1}-8\rho\mathscr{L}_{m+1,n-1,r+1}+16\mathscr{L}_{m+1,n-1,r+1}\\
&-2\rho^2\mathscr{L}_{m+1,n,r-1}-8\rho\mathscr{L}_{m+1,n,r-1}+64\mathscr{L}_{m+1,n,r-1}\\
&+4\rho^2\mathscr{L}_{m+1,n,r}+64\rho\mathscr{L}_{m+1,n,r}+256\mathscr{L}_{m+1,n,r}\\
&-2\rho^2\mathscr{L}_{m+1,n,r+1}-8\rho\mathscr{L}_{m+1,n,r+1}+64\mathscr{L}_{m+1,n,r+1}\\
&+\rho^2\mathscr{L}_{m+1,n+1,r-1}-8\rho\mathscr{L}_{m+1,n+1,r-1}+16\mathscr{L}_{m+1,n+1,r-1}\\
&-2\rho^2\mathscr{L}_{m+1,n+1,r}-8\rho\mathscr{L}_{m+1,n+1,r}+64\mathscr{L}_{m+1,n+1,r}\\
&+\rho^2\mathscr{L}_{m+1,n+1,r+1}-8\rho\mathscr{L}_{m+1,n+1,r+1}+16\mathscr{L}_{m+1,n+1,r+1}\Big),\\
&\frac{\partial^2\mathscr{H}_{m,n,r}}{\partial y^2}=\frac{\rho+2}{1152k^2}\Big(\mathscr{L}_{m-1,n-1,r-1}(\rho-4)^2\\
&-2\left(\rho^2+4\rho-32\right)\mathscr{L}_{m-1,n-1,r}+\rho^2\mathscr{L}_{m-1,n-1,r+1}\\
&-8\rho\mathscr{L}_{m-1,n-1,r+1}+16\mathscr{L}_{m-1,n-1,r+1}-2\rho^2\mathscr{L}_{m-1,n,r-1}\\
&+16\rho\mathscr{L}_{m-1,n,r-1}-32\mathscr{L}_{m-1,n,r-1}+4\rho^2\mathscr{L}_{m-1,n,r}\\
&+16\rho\mathscr{L}_{m-1,n,r}-128\mathscr{L}_{m-1,n,r}-2\rho^2\mathscr{L}_{m-1,n,r+1}\\
&+16\rho\mathscr{L}_{m-1,n,r+1}-32\mathscr{L}_{m-1,n,r+1}+\rho^2\mathscr{L}_{m-1,n+1,r-1}\\
&-8\rho\mathscr{L}_{m-1,n+1,r-1}+16\mathscr{L}_{m-1,n+1,r-1}-2\rho^2\mathscr{L}_{m-1,n+1,r}\\
&-8\rho\mathscr{L}_{m-1,n+1,r}+64\mathscr{L}_{m-1,n+1,r}+\rho^2\mathscr{L}_{m-1,n+1,r+1}\\
&-8\rho\mathscr{L}_{m-1,n+1,r+1}+16\mathscr{L}_{m-1,n+1,r+1}-2\rho^2\mathscr{L}_{m,n-1,r-1}\\
&-8\rho\mathscr{L}_{m,n-1,r-1}+64\mathscr{L}_{m,n-1,r-1}+4\rho^2\mathscr{L}_{m,n-1,r}\\
&+64\rho\mathscr{L}_{m,n-1,r}+256\mathscr{L}_{m,n-1,r}-2\rho^2\mathscr{L}_{m,n-1,r+1}\\
&-8\rho\mathscr{L}_{m,n-1,r+1}+64\mathscr{L}_{m,n-1,r+1}+4\rho^2\mathscr{L}_{m,n,r-1}\\
&+16\rho\mathscr{L}_{m,n,r-1}-128\mathscr{L}_{m,n,r-1}\\
&-8\rho^2\mathscr{L}_{m,n,r}-128\rho\mathscr{L}_{m,n,r}-512\mathscr{L}_{m,n,r}\\
&+4\rho^2\mathscr{L}_{m,n,r+1}+16\rho\mathscr{L}_{m,n,r+1}-128\mathscr{L}_{m,n,r+1}\\
&-2\rho^2\mathscr{L}_{m,n+1,r-1}-8\rho\mathscr{L}_{m,n+1,r-1}+64\mathscr{L}_{m,n+1,r-1}\\
&+4\rho^2\mathscr{L}_{m,n+1,r}+64\rho\mathscr{L}_{m,n+1,r}+256\mathscr{L}_{m,n+1,r}\\
&-2\rho^2\mathscr{L}_{m,n+1,r+1}-8\rho\mathscr{L}_{m,n+1,r+1}+64\mathscr{L}_{m,n+1,r+1}\\
&+\rho^2\mathscr{L}_{m+1,n-1,r-1}-8\rho\mathscr{L}_{m+1,n-1,r-1}+16\mathscr{L}_{m+1,n-1,r-1}\\
&-2\rho^2\mathscr{L}_{m+1,n-1,r}-8\rho\mathscr{L}_{m+1,n-1,r}+64\mathscr{L}_{m+1,n-1,r}\\
&+\rho^2\mathscr{L}_{m+1,n-1,r+1}-8\rho\mathscr{L}_{m+1,n-1,r+1}+16\mathscr{L}_{m+1,n-1,r+1}\\
&-2\rho^2\mathscr{L}_{m+1,n,r-1}+16\rho\mathscr{L}_{m+1,n,r-1}-32\mathscr{L}_{m+1,n,r-1}\\
&+4\rho^2\mathscr{L}_{m+1,n,r}+16\rho\mathscr{L}_{m+1,n,r}-128\mathscr{L}_{m+1,n,r}\\
&-2\rho^2\mathscr{L}_{m+1,n,r+1}+16\rho\mathscr{L}_{m+1,n,r+1}-32\mathscr{L}_{m+1,n,r+1}\\
&+\rho^2\mathscr{L}_{m+1,n+1,r-1}-8\rho\mathscr{L}_{m+1,n+1,r-1}+16\mathscr{L}_{m+1,n+1,r-1}\\
&-2\rho^2\mathscr{L}_{m+1,n+1,r}-8\rho\mathscr{L}_{m+1,n+1,r}+64\mathscr{L}_{m+1,n+1,r}\\
&+\rho^2\mathscr{L}_{m+1,n+1,r+1}-8\rho\mathscr{L}_{m+1,n+1,r+1}+16\mathscr{L}_{m+1,n+1,r+1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathscr{H}_{m,n,r}}{\partial z^2} = \frac{\rho+2}{1152q^2}\Big(&\mathscr{L}_{m-1,n-1,r-1}(\rho-4)^2\\
&-2\mathscr{L}_{m-1,n-1,r}(\rho-4)^2+\rho^2\mathscr{L}_{m-1,n-1,r+1}\\
&-8\rho\mathscr{L}_{m-1,n-1,r+1}+16\mathscr{L}_{m-1,n-1,r+1}-2\rho^2\mathscr{L}_{m-1,n,r-1}\\
&-8\rho\mathscr{L}_{m-1,n,r-1}+64\mathscr{L}_{m-1,n,r-1}+4\rho^2\mathscr{L}_{m-1,n,r}\\
&+16\rho\mathscr{L}_{m-1,n,r}-128\mathscr{L}_{m-1,n,r}-2\rho^2\mathscr{L}_{m-1,n,r+1}\\
&-8\rho\mathscr{L}_{m-1,n,r+1}+64\mathscr{L}_{m-1,n,r+1}+\rho^2\mathscr{L}_{m-1,n+1,r-1}\\
&-8\rho\mathscr{L}_{m-1,n+1,r-1}+16\mathscr{L}_{m-1,n+1,r-1}-2\rho^2\mathscr{L}_{m-1,n+1,r}\\
&+16\rho\mathscr{L}_{m-1,n+1,r}-32\mathscr{L}_{m-1,n+1,r}+\rho^2\mathscr{L}_{m-1,n+1,r+1}\\
&-8\rho\mathscr{L}_{m-1,n+1,r+1}+16\mathscr{L}_{m-1,n+1,r+1}-2\rho^2\mathscr{L}_{m,n-1,r-1}\\
&-8\rho\mathscr{L}_{m,n-1,r-1}+64\mathscr{L}_{m,n-1,r-1}+4\rho^2\mathscr{L}_{m,n-1,r}\\
&+16\rho\mathscr{L}_{m,n-1,r}-128\mathscr{L}_{m,n-1,r}-2\rho^2\mathscr{L}_{m,n-1,r+1}\\
&-8\rho\mathscr{L}_{m,n-1,r+1}+64\mathscr{L}_{m,n-1,r+1}+4\rho^2\mathscr{L}_{m,n,r-1}\\
&+64\rho\mathscr{L}_{m,n,r-1}+256\mathscr{L}_{m,n,r-1}-8\rho^2\mathscr{L}_{m,n,r}\\
&-128\rho\mathscr{L}_{m,n,r}-512\mathscr{L}_{m,n,r}+4\rho^2\mathscr{L}_{m,n,r+1}\\
&+64\rho\mathscr{L}_{m,n,r+1}+256\mathscr{L}_{m,n,r+1}-2\rho^2\mathscr{L}_{m,n+1,r-1}\\
&-8\rho\mathscr{L}_{m,n+1,r-1}+64\mathscr{L}_{m,n+1,r-1}+4\rho^2\mathscr{L}_{m,n+1,r}\\
&+16\rho\mathscr{L}_{m,n+1,r}-128\mathscr{L}_{m,n+1,r}-2\rho^2\mathscr{L}_{m,n+1,r+1}\\
&-8\rho\mathscr{L}_{m,n+1,r+1}+64\mathscr{L}_{m,n+1,r+1}+\rho^2\mathscr{L}_{m+1,n-1,r-1}\\
&-8\rho\mathscr{L}_{m+1,n-1,r-1}+16\mathscr{L}_{m+1,n-1,r-1}-2\rho^2\mathscr{L}_{m+1,n-1,r}\\
&+16\rho\mathscr{L}_{m+1,n-1,r}-32\mathscr{L}_{m+1,n-1,r}+\rho^2\mathscr{L}_{m+1,n-1,r+1}\\
&-8\rho\mathscr{L}_{m+1,n-1,r+1}+16\mathscr{L}_{m+1,n-1,r+1}-2\rho^2\mathscr{L}_{m+1,n,r-1}\\
&-8\rho\mathscr{L}_{m+1,n,r-1}+64\mathscr{L}_{m+1,n,r-1}+4\rho^2\mathscr{L}_{m+1,n,r}\\
&+16\rho\mathscr{L}_{m+1,n,r}-128\mathscr{L}_{m+1,n,r}-2\rho^2\mathscr{L}_{m+1,n,r+1}\\
&-8\rho\mathscr{L}_{m+1,n,r+1}+64\mathscr{L}_{m+1,n,r+1}+\rho^2\mathscr{L}_{m+1,n+1,r-1}\\
&-8\rho\mathscr{L}_{m+1,n+1,r-1}+16\mathscr{L}_{m+1,n+1,r-1}-2\rho^2\mathscr{L}_{m+1,n+1,r}\\
&+16\rho\mathscr{L}_{m+1,n+1,r}-32\mathscr{L}_{m+1,n+1,r}+\rho^2\mathscr{L}_{m+1,n+1,r+1}\\
&-8\rho\mathscr{L}_{m+1,n+1,r+1}+16\mathscr{L}_{m+1,n+1,r+1}\Big).
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathscr{H}_{m,n,r}}{\partial x^2\partial y} = \frac{\rho+2}{96h^2k}\Big(&(\rho-4)\mathscr{L}_{m-1,n-1,r-1}\\
&-2(\rho+8)\mathscr{L}_{m-1,n-1,r}+\rho\mathscr{L}_{m-1,n-1,r+1}\\
&-\rho\mathscr{L}_{m-1,n+1,r-1}+2\rho\mathscr{L}_{m-1,n+1,r}-\rho\mathscr{L}_{m-1,n+1,r+1}\\
&-2\rho\mathscr{L}_{m,n-1,r-1}+4\rho\mathscr{L}_{m,n-1,r}-2\rho\mathscr{L}_{m,n-1,r+1}\\
&+2\rho\mathscr{L}_{m,n+1,r-1}-4\rho\mathscr{L}_{m,n+1,r}+2\rho\mathscr{L}_{m,n+1,r+1}\\
&+\rho\mathscr{L}_{m+1,n-1,r-1}-2\rho\mathscr{L}_{m+1,n-1,r}+\rho\mathscr{L}_{m+1,n-1,r+1}\\
&-\rho\mathscr{L}_{m+1,n+1,r-1}+2\rho\mathscr{L}_{m+1,n+1,r}-\rho\mathscr{L}_{m+1,n+1,r+1}\\
&-4\mathscr{L}_{m-1,n-1,r+1}+4\mathscr{L}_{m-1,n+1,r-1}+16\mathscr{L}_{m-1,n+1,r}\\
&+4\mathscr{L}_{m-1,n+1,r+1}+8\mathscr{L}_{m,n-1,r-1}+32\mathscr{L}_{m,n-1,r}\\
&+8\mathscr{L}_{m,n-1,r+1}-8\mathscr{L}_{m,n+1,r-1}-32\mathscr{L}_{m,n+1,r}\\
&-8\mathscr{L}_{m,n+1,r+1}-4\mathscr{L}_{m+1,n-1,r-1}-16\mathscr{L}_{m+1,n-1,r}\\
&-4\mathscr{L}_{m+1,n-1,r+1}+4\mathscr{L}_{m+1,n+1,r-1}\\
&+16\mathscr{L}_{m+1,n+1,r}+4\mathscr{L}_{m+1,n+1,r+1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathscr{H}_{m,n,r}}{\partial x^2\partial z} = \frac{\rho+2}{96h^2q}\Big(&(\rho-4)\mathscr{L}_{m-1,n-1,r-1}\\
&-(\rho-4)\mathscr{L}_{m-1,n-1,r+1}-2\rho\mathscr{L}_{m-1,n,r-1}\\
&+2\rho\mathscr{L}_{m-1,n,r+1}+\rho\mathscr{L}_{m-1,n+1,r-1}-\rho\mathscr{L}_{m-1,n+1,r+1}\\
&-2\rho\mathscr{L}_{m,n-1,r-1}+2\rho\mathscr{L}_{m,n-1,r+1}+4\rho\mathscr{L}_{m,n,r-1}\\
&-4\rho\mathscr{L}_{m,n,r+1}-2\rho\mathscr{L}_{m,n+1,r-1}+2\rho\mathscr{L}_{m,n+1,r+1}\\
&+\rho\mathscr{L}_{m+1,n-1,r-1}-\rho\mathscr{L}_{m+1,n-1,r+1}-2\rho\mathscr{L}_{m+1,n,r-1}\\
&+2\rho\mathscr{L}_{m+1,n,r+1}+\rho\mathscr{L}_{m+1,n+1,r-1}-\rho\mathscr{L}_{m+1,n+1,r+1}\\
&-16\mathscr{L}_{m-1,n,r-1}+16\mathscr{L}_{m-1,n,r+1}-4\mathscr{L}_{m-1,n+1,r-1}\\
&+4\mathscr{L}_{m-1,n+1,r+1}+8\mathscr{L}_{m,n-1,r-1}-8\mathscr{L}_{m,n-1,r+1}\\
&+32\mathscr{L}_{m,n,r-1}-32\mathscr{L}_{m,n,r+1}+8\mathscr{L}_{m,n+1,r-1}\\
&-8\mathscr{L}_{m,n+1,r+1}-4\mathscr{L}_{m+1,n-1,r-1}+4\mathscr{L}_{m+1,n-1,r+1}\\
&-16\mathscr{L}_{m+1,n,r-1}+16\mathscr{L}_{m+1,n,r+1}\\
&-4\mathscr{L}_{m+1,n+1,r-1}+4\mathscr{L}_{m+1,n+1,r+1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathscr{H}_{m,n,r}}{\partial y^2\partial z} = \frac{\rho+2}{96k^2q}\Big(&(\rho-4)\mathscr{L}_{m-1,n-1,r-1}\\
&-(\rho-4)\mathscr{L}_{m-1,n-1,r+1}-2\rho\mathscr{L}_{m-1,n,r-1}\\
&+2\rho\mathscr{L}_{m-1,n,r+1}+\rho\mathscr{L}_{m-1,n+1,r-1}-\rho\mathscr{L}_{m-1,n+1,r+1}\\
&-2\rho\mathscr{L}_{m,n-1,r-1}+2\rho\mathscr{L}_{m,n-1,r+1}+4\rho\mathscr{L}_{m,n,r-1}\\
&-4\rho\mathscr{L}_{m,n,r+1}-2\rho\mathscr{L}_{m,n+1,r-1}+2\rho\mathscr{L}_{m,n+1,r+1}\\
&+\rho\mathscr{L}_{m+1,n-1,r-1}-\rho\mathscr{L}_{m+1,n-1,r+1}-2\rho\mathscr{L}_{m+1,n,r-1}\\
&+2\rho\mathscr{L}_{m+1,n,r+1}+\rho\mathscr{L}_{m+1,n+1,r-1}-\rho\mathscr{L}_{m+1,n+1,r+1}\\
&+8\mathscr{L}_{m-1,n,r-1}-8\mathscr{L}_{m-1,n,r+1}-4\mathscr{L}_{m-1,n+1,r-1}\\
&+4\mathscr{L}_{m-1,n+1,r+1}-16\mathscr{L}_{m,n-1,r-1}+16\mathscr{L}_{m,n-1,r+1}\\
&+32\mathscr{L}_{m,n,r-1}-32\mathscr{L}_{m,n,r+1}-16\mathscr{L}_{m,n+1,r-1}\\
&+16\mathscr{L}_{m,n+1,r+1}-4\mathscr{L}_{m+1,n-1,r-1}+4\mathscr{L}_{m+1,n-1,r+1}\\
&+8\mathscr{L}_{m+1,n,r-1}-8\mathscr{L}_{m+1,n,r+1}\\
&-4\mathscr{L}_{m+1,n+1,r-1}+4\mathscr{L}_{m+1,n+1,r+1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathscr{H}_{m,n,r}}{\partial y\partial z^2} = \frac{\rho+2}{96kq^2}\Big(&(\rho-4)\mathscr{L}_{m-1,n-1,r-1}\\
&-2(\rho-4)\mathscr{L}_{m-1,n-1,r}+\rho\mathscr{L}_{m-1,n-1,r+1}\\
&-\rho\mathscr{L}_{m-1,n+1,r-1}+2\rho\mathscr{L}_{m-1,n+1,r}-\rho\mathscr{L}_{m-1,n+1,r+1}\\
&-2\rho\mathscr{L}_{m,n-1,r-1}+4\rho\mathscr{L}_{m,n-1,r}-2\rho\mathscr{L}_{m,n-1,r+1}\\
&+2\rho\mathscr{L}_{m,n+1,r-1}-4\rho\mathscr{L}_{m,n+1,r}+2\rho\mathscr{L}_{m,n+1,r+1}\\
&+\rho\mathscr{L}_{m+1,n-1,r-1}-2\rho\mathscr{L}_{m+1,n-1,r}+\rho\mathscr{L}_{m+1,n-1,r+1}\\
&-\rho\mathscr{L}_{m+1,n+1,r-1}+2\rho\mathscr{L}_{m+1,n+1,r}-\rho\mathscr{L}_{m+1,n+1,r+1}\\
&-4\mathscr{L}_{m-1,n-1,r+1}+4\mathscr{L}_{m-1,n+1,r-1}-8\mathscr{L}_{m-1,n+1,r}\\
&+4\mathscr{L}_{m-1,n+1,r+1}-16\mathscr{L}_{m,n-1,r-1}+32\mathscr{L}_{m,n-1,r}\\
&-16\mathscr{L}_{m,n-1,r+1}+16\mathscr{L}_{m,n+1,r-1}-32\mathscr{L}_{m,n+1,r}
\end{aligned}
$$

$$
\begin{aligned}
&+16\mathscr{L}_{m,n+1,r+1}-4\mathscr{L}_{m+1,n-1,r-1}+8\mathscr{L}_{m+1,n-1,r}\\
&-4\mathscr{L}_{m+1,n-1,r+1}+4\mathscr{L}_{m+1,n+1,r-1}\\
&-8\mathscr{L}_{m+1,n+1,r}+4\mathscr{L}_{m+1,n+1,r+1}\Big),\\
&\frac{\partial^3\mathscr{H}_{m,n,r}}{\partial x\partial z^2}=\frac{\rho+2}{96hq^2}\Big((\rho-4)\mathscr{L}_{m-1,n-1,r-1}\\
&-2(\rho-4)\mathscr{L}_{m-1,n-1,r}+\rho\mathscr{L}_{m-1,n-1,r+1}\\
&-2\rho\mathscr{L}_{m-1,n,r-1}+4\rho\mathscr{L}_{m-1,n,r}-2\rho\mathscr{L}_{m-1,n,r+1}\\
&+\rho\mathscr{L}_{m-1,n+1,r-1}-2\rho\mathscr{L}_{m-1,n+1,r}+\rho\mathscr{L}_{m-1,n+1,r+1}\\
&-\rho\mathscr{L}_{m+1,n-1,r-1}+2\rho\mathscr{L}_{m+1,n-1,r}-\rho\mathscr{L}_{m+1,n-1,r+1}\\
&+2\rho\mathscr{L}_{m+1,n,r-1}-4\rho\mathscr{L}_{m+1,n,r}+2\rho\mathscr{L}_{m+1,n,r+1}\\
&-\rho\mathscr{L}_{m+1,n+1,r-1}+2\rho\mathscr{L}_{m+1,n+1,r}-\rho\mathscr{L}_{m+1,n+1,r+1}\\
&-4\mathscr{L}_{m-1,n-1,r+1}-16\mathscr{L}_{m-1,n,r-1}+32\mathscr{L}_{m-1,n,r}\\
&-16\mathscr{L}_{m-1,n,r+1}-4\mathscr{L}_{m-1,n+1,r-1}+8\mathscr{L}_{m-1,n+1,r}\\
&-4\mathscr{L}_{m-1,n+1,r+1}+4\mathscr{L}_{m+1,n-1,r-1}-8\mathscr{L}_{m+1,n-1,r}\\
&+4\mathscr{L}_{m+1,n-1,r+1}+16\mathscr{L}_{m+1,n,r-1}-32\mathscr{L}_{m+1,n,r}\\
&+16\mathscr{L}_{m+1,n,r+1}+4\mathscr{L}_{m+1,n+1,r-1}\\
&-8\mathscr{L}_{m+1,n+1,r}+4\mathscr{L}_{m+1,n+1,r+1}\Big),\\
&\frac{\partial^3\mathscr{H}_{m,n,r}}{\partial x\partial y^2}=\frac{\rho+2}{96hk^2}\Big((\rho-4)\mathscr{L}_{m-1,n-1,r-1}\\
&-2(\rho+8)\mathscr{L}_{m-1,n-1,r}+\rho\mathscr{L}_{m-1,n-1,r+1}\\
&-2\rho\mathscr{L}_{m-1,n,r-1}+4\rho\mathscr{L}_{m-1,n,r}-2\rho\mathscr{L}_{m-1,n,r+1}\\
&+\rho\mathscr{L}_{m-1,n+1,r-1}-2\rho\mathscr{L}_{m-1,n+1,r}+\rho\mathscr{L}_{m-1,n+1,r+1}\\
&-\rho\mathscr{L}_{m+1,n-1,r-1}+2\rho\mathscr{L}_{m+1,n-1,r}-\rho\mathscr{L}_{m+1,n-1,r+1}\\
&+2\rho\mathscr{L}_{m+1,n,r-1}-4\rho\mathscr{L}_{m+1,n,r}+2\rho\mathscr{L}_{m+1,n,r+1}\\
&-\rho\mathscr{L}_{m+1,n+1,r-1}+2\rho\mathscr{L}_{m+1,n+1,r}-\rho\mathscr{L}_{m+1,n+1,r+1}\\
&-4\mathscr{L}_{m-1,n-1,r+1}+8\mathscr{L}_{m-1,n,r-1}+32\mathscr{L}_{m-1,n,r}\\
&+8\mathscr{L}_{m-1,n,r+1}-4\mathscr{L}_{m-1,n+1,r-1}-16\mathscr{L}_{m-1,n+1,r}\\
&-4\mathscr{L}_{m-1,n+1,r+1}+4\mathscr{L}_{m+1,n-1,r-1}+16\mathscr{L}_{m+1,n-1,r}\\
&+4\mathscr{L}_{m+1,n-1,r+1}-8\mathscr{L}_{m+1,n,r-1}-32\mathscr{L}_{m+1,n,r}\\
&-8\mathscr{L}_{m+1,n,r+1}+4\mathscr{L}_{m+1,n+1,r-1}\\
&+16\mathscr{L}_{m+1,n+1,r}+4\mathscr{L}_{m+1,n+1,r+1}\Big),\\
&\vdots
\end{aligned}
\tag{13}
$$

In all n-dimensional PDE's with collocation method we get a system of algebraic equations in this form

$$A\underset{\sim}{\mathscr{L}}=\underset{\sim}{b}, \tag{14}$$

We solve the above system using newton's method to find the unknown values of $\mathscr{L}$.

3 The error estimates

Lemma 1. *Suppose that $\hat{\mathscr{H}}$ is an estimation of smoothness class C^2. At that point error gauges of the insertion on a square work of side h are*

$$
\begin{gathered}
\|\mathscr{H}-\hat{\mathscr{H}}\|\le\beta_0h^4,\ \ \|\frac{\partial\mathscr{H}}{\partial x}-\frac{\partial\hat{\mathscr{H}}}{\partial x}\|\le\beta_1h^3,\\
|\frac{\partial\mathscr{H}}{\partial z}-\frac{\partial\hat{\mathscr{H}}}{\partial z}\|\le\beta_2h^3,\ \ \|\frac{\partial\mathscr{H}}{\partial y}-\frac{\partial\hat{\mathscr{H}}}{\partial y}\|\le\beta_3h^3,\\
\|\frac{\partial^2\mathscr{H}}{\partial x^2}-\frac{\partial^2\hat{\mathscr{H}}}{\partial x^2}\|\le\beta_4h^2,\ \ \|\frac{\partial^2\mathscr{H}}{\partial y^2}-\frac{\partial^2\hat{\mathscr{H}}}{\partial y^2}\|\le\beta_5h^2,\\
|\frac{\partial^2\mathscr{H}}{\partial z^2}-\frac{\partial^2\hat{\mathscr{H}}}{\partial z^2}\|\le\beta_6h^2,\\
\|\frac{\partial^2\mathscr{H}}{\partial x\partial y}-\frac{\partial^2\hat{\mathscr{H}}}{\partial x\partial y}\|\le\beta_7h^2,\ \ \|\frac{\partial^2\mathscr{H}}{\partial x\partial z}-\frac{\partial^2\hat{\mathscr{H}}}{\partial x\partial z}\|\le\beta_8h^2,\\
|\frac{\partial^2\mathscr{H}}{\partial y\partial z}-\frac{\partial^2\hat{\mathscr{H}}}{\partial y\partial z}\|\le\beta_9h^2,
\end{gathered}
$$

where the β_i are constants.

The proof of above lemma see [9].

4 The numerical results

Presently, we must know whether this method, which was developed by presenting its constructions in different dimensional, is accurate and effective or not. To prove that this method is of high accuracy, we present in this section various numerical examples in different dimensional. We also show some figures of the results obtained. In addition to providing comparisons of our results with pre-existing results.

The first test problem: [18]
We take the first test problem in the 2-dimensional in this form:

$$
\begin{aligned}
u_{xx}(x,y)+u_{yy}(x,y)+u_x(x,y)+u_y(x,y)\\
-3e^{2x+3y}(x^2(18y^2-4y-5)\\
+x(5-8y^2-6y))-3y^2+3y)=0,\quad x,y\in[a,b]
\end{aligned}
\tag{15}
$$

The exact solution to that problem given as follows:

$$u(x,y)=3e^{2x+3y}(x-x^2)(y-y^2). \tag{16}$$

We take the boundary conditions to the first problem in this form:

$$u(a,y)=u(x,a)=\alpha,\quad u(b,y)=u(x,b)=\beta. \tag{17}$$

By substitution from (4)-(6) into (15) with (17) we obtain the numerical results as in the next table:

Table 1: The computational results to the first problem at $y = 0.5, x, y \in [0,1]$

x	Numerical results	Exact results	Absolute error	Quadratic B-Spline [18]
0.1	0.36856	0.36949	9.228 E-4	1.069 E-3
0.2	0.80015	0.80230	2.150 E-3	2.323 E-3
0.3	1.28295	1.28617	3.217 E-3	3.409 E-3
0.4	1.79122	1.79535	4.129 E-3	4.316 E-3
0.5	2.27931	2.28422	4.908 E-3	5.042 E-3
0.6	2.67274	2.67835	5.612 E-3	5.606 E-3
0.7	2.85609	2.86243	6.341 E-3	6.054 E-3
0.8	2.65652	2.66375	7.237 E-3	6.468 E-3
0.9	1.82167	1.83010	8.433 E-3	6.931 E-3

In Table 1, we compared the results about the 2-dimensional Extended cubic B-spline method employing a work of 50×50 and the exact results together. We show that our results are accepted with regard to the exact results. In Fig. 1, we introduce the numerical arrangements with the exact solution at $y = 0.5$. In Fig. 2, we show the numerical results and the exact solution at $x = 0.5$.

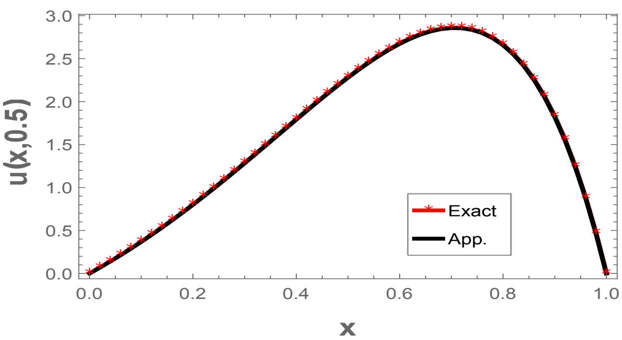

Fig. 1: The numerical results with the exact results at $y = 0.5$.

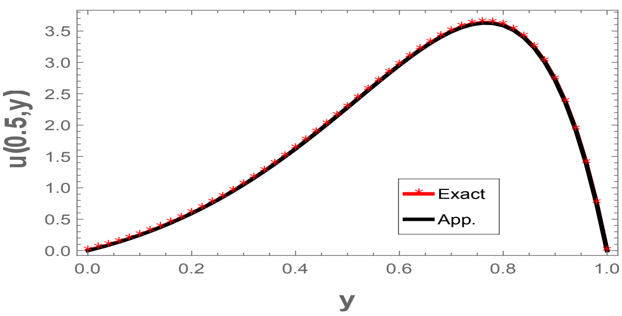

Fig. 2: The numerical results with the exact results at $x = 0.5$.

The second test problem: MHD duct flow [7–9, 16]

The cross-section of an infinitely long rectangular duct is oriented with its sides parallel to the $x-$ and $y-$axes and the origin of coordinates at its center. The duct width is $2a$ and height $2b$ so that the sides of the duct have equations $x = \pm a$ and $y = \pm b$. A conducting fluid flows in the z direction along the duct and is subjected to a constant applied magnetic field M acting in a direction lying in the xy-plane and making an angle Λ with the y-axis. The equations governing the flow may be expressed in the normalized form [7, 15].

$$\frac{\partial P}{\partial z} = \mu \nu \nabla^2 \nabla_z + \frac{A_{0x}}{\mu_0}\frac{\partial P_z}{\partial x'} + \frac{A_{0y}}{\mu_0}\frac{\partial P_z}{\partial y'}, \tag{18}$$

and the z-component of the curl of Ohm's law,

$$\nabla^2 A_z + \xi \mu_0 (A_{0x}\frac{\partial U_z}{\partial x'} + A_{0y}\frac{\partial U_z}{\partial y'}) = 0, \tag{19}$$

with the boundary conditions: $U = A = 0$ at $x' = \pm\alpha, y' = \pm b$,
where ν, μ and ξ are, respectively, the kinematic viscosity, density and electric conductivity of the fluid; μ_0 is the magnetic permeability in vacuum; dP/dz is the constant axial pressure gradient; $\mho_{0x}$ and $\mho_{0y}$ are the x' and y' components of the applied magnetic field; and U_z and A_z are the z components of velocity and induced magnetic field, respectively. Following the notation of P. C. Lu [15], who solved this problem using the Kantorovieh method, Eqs. (18) and (19) become in non-dimensionalized form,

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})U + M_x\frac{\partial A}{\partial x} + M_y\frac{\partial A}{\partial y} = -1, \tag{20}$$

and

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})A + M_x\frac{\partial U}{\partial x} + M_y\frac{\partial U}{\partial y} = -1, \tag{21}$$

with boundary conditions $U = A = 0, x = \pm\alpha, y = \pm 1$. Distance has been scaled to the duct semi-height b so that $x = x'/b$, $y = y'/b$, and $\alpha = a/b$. The following normalisations have also been used.

$$U = \frac{U_z}{\frac{-b^2}{\nu\mu}\frac{dP}{dz}},$$
$$A = \frac{A_z}{\frac{-b^2}{\nu\mu}\frac{dP}{dz}\mu_0(\nu\mu\xi)^{\frac{1}{2}}},$$
$$M_x = A_{0x'}b(\frac{\xi}{\nu\mu})^{\frac{1}{2}} = Msin(\Lambda), \quad (22)$$
$$M_y = A_{0y'}b(\frac{\xi}{\nu\mu})^{\frac{1}{2}} = Mcos(\Lambda),$$
$$M = Hartmann\ no. = \left(M_x^2 + M_y^2\right)^{\frac{1}{2}} = A_0 b(\frac{\xi}{\nu\mu})^{\frac{1}{2}}.$$

The Hartmann number is the ratio of magnetic to fluid viscosity. If $M = 0$, the flow field is the classieal laminar pipe flow. If $M \geq 1$, the flow field is determined primarily by the $E \times A$ drift. To uncouple (20) and (21), the functions

$$H_1 = U + A, \quad (23)$$

and

$$H_2 = U - A, \quad (24)$$

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})H_1 + M_x\frac{\partial H_1}{\partial x} + M_y\frac{\partial H_1}{\partial y} = -1, \quad (25)$$

and

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})H_2 - M_x\frac{\partial H_2}{\partial x} - M_y\frac{\partial H_2}{\partial y} = -1, \quad (26)$$

with boundary conditions $H_1 = H_2 = 0, x = \pm\alpha, y = \pm 1$. Thus, if H_1 is solved as $H_1(M_x, M_y)$ from (26), then

$$H_2(M_x, M_y) = H_1(-M_x, -M_y). \quad (27)$$

So that the solution is completely determined when either H_1 or H_2, are known. Having determined H_1 the function H_2 is found from (27) and hence the velocity field U from

$$U = \frac{1}{2}(H_1 + H_2). \quad (28)$$

Now, we will intreduce some numerical resultes for the flow in a square duct with an applied magnetic field parallel to the $x-$axis so that $M_y = 0$. To compare with earlier results [7–9, 14], we give to M, the following values $M_x = 0, 2, 5$ and 8.
By substituting from (4)-(6) in (25) and (26) we get the numerical solutions as follows:

Table 2: U at the centre of the duct

M_x	[7]	[8]	[9]	FDM [16]	Our meth.	Analytic [14]	AE
0	0.298	0.298	0.294	0.294	0.293	0.294	1.1 E-3
2	0.263	0.263	0.258	0.258	0.258	0.258	6.2 E-4
5	0.174	0.174	0.171	0.171	0.171	0.171	1.2 E-4
8	0.120	0.120	0.118	0.118	0.118	0.118	3.0 E-5

In Table 2, the results of the 2-dimensional Extended cubic B-spline method using a mesh of 20×20 were compared with those the numerical [7–9, 16] and also with the analytic solution of Shercliff [14].

In Fig. 3, we show The profile of velocity with Hartmann numbers 0 (top curve), to 8 (bottom curve) at $[-1, 1]$ using a mesh of 20×20 .

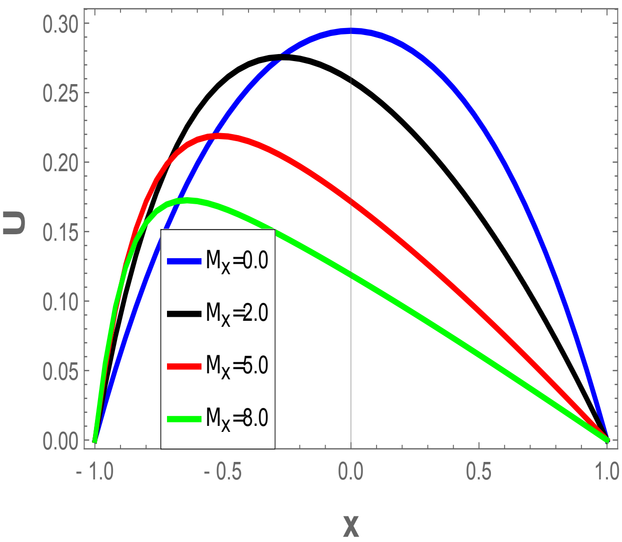

Fig. 3: The profile of velocity with various values of Hartmann numbers

In Table 3, some other results are presented where the period with from [-1, 1] to [-0.5, 0.5] is changed and we also compare these results with Finite difference method [16] and the analytical solution found in the research [14].

Table 3: U at the centre of the duct. Finite difference and analytic simulations compared

M_x	Finite differenc method using a mesh of 50×50 [16]	2-dimensional Extended cubic B-spline method 50×50	Analytic [14]	Absolute error
0	0.073648	0.0736279	0.073671	4.31 E-5
2	0.071109	0.0710908	0.071128	3.72 E-5
5	0.060838	0.0608273	0.060846	1.87 E-5
8	0.049359	0.0493563	0.049363	6.70 E-6

In Fig. 4, we show the profile of the velocity with various values of Hartmann numbers at $[-0.5, 0.5]$ using a mesh of 50×50 . For diverse values of the Hartmann number, the course of action for the speed profile along the $x-$axis has shown up in figs 4 and

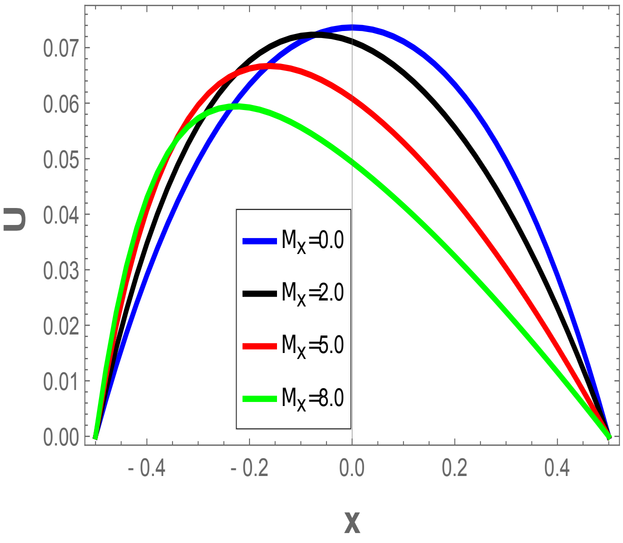

Fig. 4: The profile of velocity with various values of Hartmann numbers.

5. As would be expected, growing the appealing field (growing the Hartmann number) has an affect on the speed of the fluid where it is the speed reduces near to the center of the channel, this clear affect of the alluring field concentrated is as of presently known. In this way, we see that the comes about are totally congruous with the physical meaning of the affect of the alluring field.

The third test problem: [18, 33–36]

We take the third test problem in the 2-dimensional in this form:

$$u_{xx}(x,y)+u_{yy}(x,y)-\sin(\pi x)\sin(\pi y)=0, \quad x,y\in[a,b] \quad (29)$$

the exact solution to that problem given as follows:

$$u(x,y)=-\frac{\sin(\pi x)\sin(\pi y)}{2\pi^2}. \quad (30)$$

We take the boundary conditions to the third problem in this form:

$$u(a,y)=u(x,a)=\alpha, \quad u(b,y)=u(x,b)=\beta. \quad (31)$$

By substitution from (4)-(6) into (29) with (31) we obtain the numerical results as in the next table:

Table 4: The numerical resultes for third problem at $y=0.4, \; x,y\in[0,1]$

x	Numerical results	Exact results	Absolute error
0.2	-0.0282154	-0.0283201	1.04701 E-4
0.4	-0.0456535	-0.0458229	1.69408 E-4
0.6	-0.0456535	-0.0458229	1.69408 E-4
0.8	-0.0282154	-0.0283201	1.04701 E-4

In Table 4, we compared the results of the 2-dimensional Extended cubic B-spline strategy employing at 15×15 and the exact results together. From our results we can say that results are accepted with regard to the exact results. In Figs. 5, 6 we show the numerical results with the exact results at $y=0.5$.

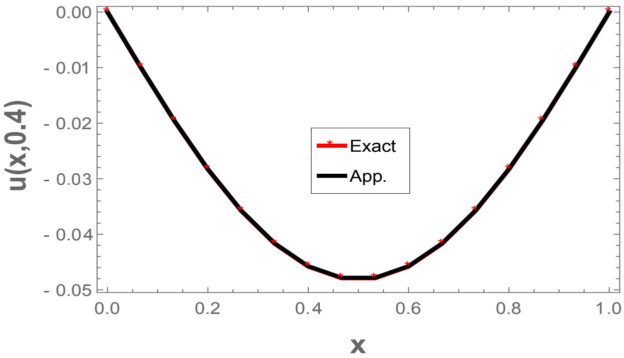

Fig. 5: The numerical results with the exact results at $y=0.4$.

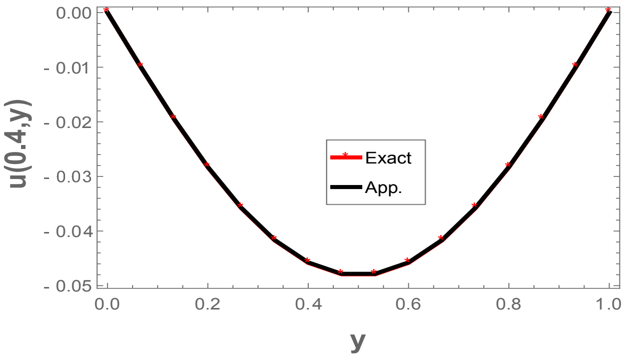

Fig. 6: The numerical results with the exact results at $x=0.4$.

Let 15×15 grid points, we compar between the results of the proposed method and the results of using different methods that shown in Table 6 [18, 33–36].

Table 5: Maximum absolute error according to the method used for third problem.

Our method	QBS method [18]	MCBDQ method [33]	SDQM method [34]	Haar wavelet method [35]	CBS method [36]
1.6 E-4	3.7 E-5	2.1 E-5	1.6 E-4	3.1 E-4	1.6 E-4

The fourth test problem: [18]

We take the fourth test problem in the 3-dimensional in this form:

$$u_{xx}(x,y,z)+u_{yy}(x,y,z)+u_{zz}(x,y,z)-xyz(e^{x+y+z})(3yxz+yx+zx \\ -5x+zy-5y-5z+9)=0, \quad x,y,z\in[a,b] \tag{32}$$

The exact solution to that problem given as follows:

$$u(x,y,z)=(x-x^2)(y-y^2)(z-z^2)e^{x+y+z}. \tag{33}$$

We take the boundary conditions to the fourth problem in this form:

$$\begin{aligned} u(a,y,z)=u(x,a,z)=u(x,y,a)=\alpha,\\ u(b,y,z)=u(x,b,z)=u(x,y,b)=\beta. \end{aligned} \tag{34}$$

By substitution from (12) into (32) with (34) we obtain the numerical results as in the next table:

Table 6: The numerical resultes for test problem at $z=y=0.5, \quad x,y,z\in[0,1]$

x	Numerical solution	Exact solution	Absolute error	Quadratic B-spline method [18]
0.1	0.0168635	0.0168984	3.48852 E-5	3.24947 E-5
0.2	0.0331304	0.0332012	7.07378 E-5	6.49943 E-5
0.3	0.0480531	0.0481595	1.06445 E-4	9.65554 E-5
0.4	0.0606859	0.0608280	1.42149 E-4	1.27075 E-4
0.5	0.0698464	0.0700264	1.79967 E-4	1.57835 E-4
0.6	0.0740704	0.0742955	2.25088 E-4	1.92337 E-4
0.7	0.0715583	0.0718456	2.87275 E-4	2.37433 E-4
0.8	0.0601139	0.0604965	3.82576 E-4	3.04639 E-4
0.9	0.0370736	0.0376082	5.34586 E-4	4.11161 E-4

In Table 6, we compared the results of the 3-dimensional Extended cubic B-spline strategy employing at 20×20 and the exact results together. From our results we can say that results are accepted with regard to the exact results. In Fig. 7, we show the numerical results with the exact results at $y=z=0.5$.

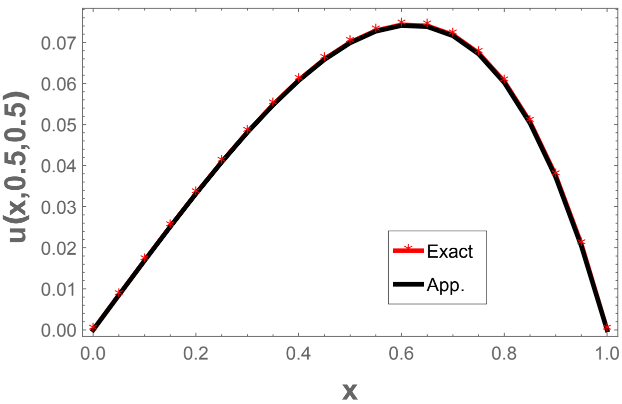

Fig. 7: The numerical results with the exact results at $y=z=0.5$.

5 Conclusion

Perhaps by the end of this work, we will have made a clear contribution to solving some of the problems facing most researchers in various fields through how to deal with mathematical models of different dimensional. The topic studied is very important and we believe that most researchers are waiting for its results. Thinking about this work came after we followed what was presented by some researchers in solutions of partial differential equations in one, two and three dimensional, and we noticed how difficult it is for them to deal with these models as the dimension increases. So we thought to develop the Extended cubic B-spline method that was used previously in solving one-dimensional mathematical problems and we were able to present a shape for this method in two and three dimensional. We tested the accuracy and effectiveness of the derived shapes by providing some numerical examples with different dimensional. The numerical results were compared with the real solution, and the inferred formulas were found effective and accurate. From this perspective, we can say that a clear contribution has been made to overcome the problems of partial differential equations of different dimensional. Amid long-term work, we are going moreover generalize a few other B-Splines shapes to serve as a solution to differential equations in n-dimensional.

Conflict of interest

There is no conflict of interest.

Author contribution

All author worked on this paper equally.

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Kama R. Raslan received the M.Sc. and Ph.D. degrees from the Faculty of Science, Menoufia University and Al-Azhar University, Egypt, in 1996 and 1999, respectively. He is currently a full Professor of Mathematics with the Faculty of Science, Al-Azhar University, Egypt. He has authored/coauthored over 114 scientific papers in top-ranked International Journals and Conference Proceedings. His research interests include Numerical Analysis, Finite Difference Methods, Finite Element Methods, Approximation Theory, and Computational Mathematics.

Khalid K. Ali MSc in pure Mathematics, Mathematics Department 2015 from Faculty of Science, Al-Azhar University, Cairo, Egypt. Ph.D. in pure Mathematics, Mathematics Department entitled 2018 from Faculty of Science, Al-Azhar University, Cairo, Egypt. He has authored/co-authored over 80 scientific papers in top-ranked International Journals and Conference Proceedings. His research interests include Numerical Analysis, Finite Difference Methods, Finite Element Methods, and Computational Mathematics.

Hilal Mohammed Yousif Albayatti received the MSC from University College London, University of London, UK, and PhD from Loughborough University of Technology, UK in Computer Science. Professor Hilal Albayatti's experience extends for more than 40 years in the field of university teaching, scientific research and university administration. Prof. Hilal Albayatti has supervised successfully more than 20 PhD students and 50 MSc Computer Science students. He published more than 60 research papers in reputed international journals and conferences in the field of Computer Science. His area of research interest parallel algorithms, design and analysis of computer algorithms, computer and data security, cyber security, databases, software engineering, IoT.