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On Solutions of Fully Fuzzy Linear Fractional Programming Problems Using Close Interval Approximation for Normalized Heptagonal Fuzzy Numbers

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Abstract: This paper attempts to solve the linear fractional programming problem with fully fuzzy normalized heptagonal fuzzy numbers using the close interval approximation of normalized heptagonal fuzzy number, which is one of the best interval approximations. The maximization (minimization) problem with interval objective function is converted into multi- objective based on order relations introduced by the decision makers' preference between interval profits (costs). Finally, an example is presented to illustrate the proposed method.

Keywords: Normalized heptagonal fuzzy number, Fully fuzzy fractional programming, Close interval approximation, multiple criteria fractional programming, Decision making

1 Introduction

Fractional programming problem (FPP) is defined as a decision making problem that arises to optimize the ratio subject to constraints. Fractional programming can be applied in the fields of traffic planning [1], network flows [2], ...etc. [3] proposed the fractional programming problems and their duality theory. In the meantime, some applications of fractional programming and the algorithms to solve this kind of problems were presented by [4, 5]. [6] showed that if the LFP problem has positive or negative denominator, it is sufficient to solve only one of the equivalent linear programming based on the sign of the denominator. [7] proposed the multiple objective linear fractional programming. [8] developed some fuzzy approaches to solve the multiple objective linear fractional optimization. [9] proposed an interactive fuzzy satisficing method for multi objective linear fractional programming problems. Many authors employed the fuzzy goal programming technique to solve the multi-level multi-objective linear programming problems, such as [10–12] and others. Few decades ago, the multiple objective fractional programming problems were proposed by many authors [13, 15, 41].

In literature, [16] proposed the philosophy of fuzzy sets. [17] introduced fuzzy programming and linear programming with multiple objective functions. Later several researchers addressed fuzzy set theory. [18] studied the theory and applications of fuzzy sets and systems. [19] investigated several fuzzy applications to engineering and management sciences. Linear programming problems with fuzzy random variable coefficient and their applications in the area of distribution problems were presented by [20]. [21] presented the fuzzy linear programming problem with fuzzy numbers. [22] studied the effect of tolerance in fuzzy linear fractional programming. [23] presented a comment on some fuzzy approaches for multiple criteria linear fractional optimization. [24] presented a fuzzy programming technique to solve the fuzzy linear fractional programming with fuzzy coefficients. [25]

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presented an approach to solve the fully fuzzified linear fractional programming problems.

[26] presented the interval numbers and some techniques to handle the applications. [27] explored the multiple objective decision making techniques theoretically and provided their methodology. [28] derived a model for expected value of fuzzy variable and fuzzy expected value models. There are many approaches proposed for solving MOLFP problems [29–38].

In this paper, a linear fractional programming problem with normalized heptagonal fuzzy number in all of the parameters is introduced. A close interval approximation for the normalized heptagonal fuzzy number is defined. A solution procedure for solving the problem is proposed.

The remainder of the paper is organized as follows: Section 2 introduces some preliminaries required in this paper. Section 3 introduces fully fuzzy linear fractional programming formulation. In Section 4, solution procedure is suggested. Section 5, a numerical example is provided for illustration. Conclusion is presented in Section 6.

2 Preliminaries

This section introduces some basic concepts and results related to fuzzy numbers, heptagonal fuzzy numbers, close interval approximation and their arithmetic operations.

Definition 2.1. [37]. A fuzzy set $\tilde{P}$ defined on $\Re$ is said to be fuzzy numbers if its membership function

$$\mu_{\tilde{P}} : \Re \to [0,1],$$

has the following properties:

1. $\mu_{\tilde{P}}(x)$ is an upper semi-continuous membership function;
2. $\tilde{P}$ is convex fuzzy set, i.e. $\mu_{\tilde{P}}(\delta x + (1-\delta)y \geq \min\{\mu_{\tilde{P}}(x), \mu_{\tilde{P}}(y)\}$
3. $\tilde{P}$ is normal, i.e. $exists\, x_0 \in Re$ for which $\mu_{\tilde{P}}(x_0) = 1$;
4. $\mathrm{Supp}(\tilde{P}) = \{x \in \Re : \mu_{\tilde{P}}(x) > 0\}$ is the support of $\tilde{P}$, and the closure $cl(\mathrm{Supp}(\tilde{P}))$ is compact set.

Definition 2.2. [30]. A fuzzy number $\tilde{C}_H = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ is a heptagonal fuzzy number (HFN), whereas $c_1 \leq c_2 \leq c_3 \leq c_4 \leq c_5 \leq c_6 \leq c_7 \in \Re$ and its membership function are defined by:

$$\mu_{\tilde{C}_H} = \begin{cases} \frac{1}{3}\left(\frac{x-c_1}{c_2-c_1}\right) & \text{for } c_1 \leq x \leq c_2, \\ \frac{1}{3} + \frac{1}{3}\left(\frac{x-c_2}{c_3-c_2}\right) & \text{for } c_2 \leq x \leq c_3, \\ \frac{2}{3} + \frac{1}{3}\left(\frac{x-c_3}{c_4-c_3}\right) & \text{for } c_3 \leq x \leq c_4, \\ 1 - \frac{1}{3}\left(\frac{x-c_4}{c_5-c_4}\right) & \text{for } c_4 \leq x \leq c_5, \\ \frac{2}{3} - \frac{1}{3}\left(\frac{x-c_5}{c_6-c_5}\right) & \text{for } c_5 \leq x \leq c_6, \\ \frac{1}{3}\left(\frac{x-c_6}{c_7-c_6}\right) & \text{for } c_6 \leq x \leq c_7, \\ 0 & \text{for } x < c_1 \text{ and } x > c_7. \end{cases}$$

A HFN can be characterized by the so called interval of

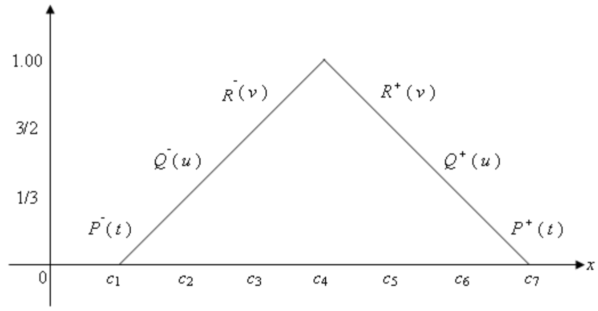

Fig. 1: Graphical representation of normal heptagonal fuzzy numbers

confidence at level α as

$$\tilde{C}_{H\alpha}(x) = \{x \in X : \mu_{\tilde{C}_H} \geq \alpha\}$$
$$= \begin{cases} [P^-(t), P^+(t)] & \text{for } \alpha \in \left[0, \frac{1}{3}\right], \\ [Q^-(u), Q^+(u)] & \text{for } \alpha \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ [R^-(v), R^+(v)] & \text{for } \alpha \in \left[\frac{2}{3}, 1\right], \end{cases}$$

Definition 2.3. An interval approximation $[C] = [c_\alpha^-, c_\alpha^+]$ of a HFN $\tilde{C}$ is called closed interval approximation if

$$c_\alpha^- = \inf\{x \in \Re : \mu_{\tilde{C}} \geq \frac{1}{3}\},$$

and

$$c_\alpha^+ = \sup\{x \in \Re : \mu_{\tilde{C}} \geq \frac{1}{3}\}.$$

Definition 2.4. The center of HFN corresponding to the closed interval approximation $[C] = [c_\alpha^-, c_\alpha^+]$ is defined as

$$C^C = \frac{c_\alpha^- + c_\alpha^+}{2}.$$

Definition 2.5. The associated ordinary (crips) number corresponding to the HFN $\tilde{c}_H = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ is defined by $\hat{a}_H = \frac{c_1, c_2, c_3, c_4, c_5, c_6, c_7}{8}$.

Definition 2.6. Let $[A] = [a_\alpha^-, a_\alpha^+]$, and $[B] = [b_\alpha^-, b_\alpha^+]$ be two interval approximations of HFN. Then the arithmetic operations are:

1. Addition: $[A](+)[B] = [a_\alpha^- + b_\alpha^-, a_\alpha^+ b_\alpha^+]$,
2. Subraction: $[A](-)[B] = [a_\alpha^- b_\alpha^+, a_\alpha^+ b_\alpha^-]$,
3. Scalar multiplication: $k[A] = \begin{cases} [ka_\alpha^-, ka_\alpha^+], & k > 0, \\ [ka_\alpha^-, ka_\alpha^+], & k < 0, \end{cases}$

   $k \in \Re$,
4. Multiplication: $[A](\times)[B] = \left[\frac{a_\alpha^+ b_\alpha^- + a_\alpha^- b_\alpha^+}{2}, \frac{a_\alpha^- b_\alpha^- + a_\alpha^+ b_\alpha^+}{2}\right]$,
5. Division:

$$[A](\div)[B] = \begin{cases} \left[\frac{2a_\alpha^-}{b_\alpha^- + b_\alpha^+}, \frac{2a_\alpha^+}{b_\alpha^- + b_\alpha^+}\right], [B] > 0, & b_\alpha^- + b_\alpha^+ \neq 0, \\ \left[\frac{2a_\alpha^+}{b_\alpha^- + b_\alpha^+}, \frac{2a_\alpha^-}{b_\alpha^- + b_\alpha^+}\right], [B] < 0, & b_\alpha^- + b_\alpha^+ \neq 0, \end{cases}$$

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6. The order relation $(\leq)$

$$[A](\leq)[B] \Leftrightarrow a_\alpha^- \leq b_\alpha^- \wedge a_\alpha^+ \leq b_\alpha^+ \vee a_\alpha^- + a_\alpha^+ \leq b_\alpha^- + b_\alpha^+.$$

Definition 2.7. Let $\tilde{C} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ and $\tilde{D} = (d_1, d_2, d_3, d_4, d_5, d_6, d_7)$. Then

1. Addition:

$$\begin{aligned}\tilde{C}_H(+)\tilde{D}_H &= (c_1,c_2,c_3,c_4,c_5,c_6,c_7) \oplus (d_1,d_2,d_3,d_4,\\ &\quad d_5,d_6,d_7)\\ &= (c_1+d_1, c_2+d_2, c_3+d_3, c_4+d_4, c_5+d_5,\\ &\quad c_6+d_6, c_7+d_7),\end{aligned}$$

2. Subraction:

$$\begin{aligned}\tilde{C}_H(-)\tilde{D}_H &= (c_1,c_2,c_3,c_4,c_5,c_6,c_7) \ominus (d_1,d_2,d_3,d_4,\\ &\quad d_5,d_6,d_7)\\ &= (c_1-d_7, c_2-d_6, c_3-d_5, c_4-d_4, c_5-d_3,\\ &\quad c_6-d_2, c_7-d_1),\end{aligned}$$

3. Scalar multiplication:

$$k\tilde{C}_H = \begin{cases} k(c_1,c_2,c_3,c_4,c_5,c_6,c_7), & k \geq 0,\\ k(c_7,c_6,c_5,c_4,c_3,c_2,c_1), & k < 0.\end{cases}$$

3 Problem Statement

A heptagonal fuzzy linear fractional programming problem

$$\max \text{ (or min)} \tilde{Z} = \frac{\sum_{j=1}^{n} \tilde{c}_j x_j + \tilde{c}_0}{\sum_{j=1}^{n} \tilde{d}_j x_j + \tilde{d}_0}$$

subject to

$$\tilde{M} = \left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i, \quad i = 1,2,\cdots,r_0, \right. \tag{1}$$
$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \geq \tilde{b}_i, \quad i = r_0+1, r_0+2, \cdots, m,$$
$$\left. x_j \geq 0,\ j = 1,2,\cdots,n \right).$$

Problem (1) can be rewritten using the close interval approximation form as follows:

$$\max \text{ (or min)} Z = \frac{\sum_{j=1}^{n} [c_j] x_j + [c_0]}{\sum_{j=1}^{n} [d]_j x_j + [d_0]}$$

subject to

$$M = \left(\sum_{j=1}^{n} [a_{ij}] x_j \leq [b_i], \quad i = 1,2,\cdots,r_0, \right. \tag{2}$$
$$\sum_{j=1}^{n} [a_{ij}] x_j \geq [b_i], \quad i = r_0+1, r_0+2, \cdots, m,$$
$$\left. x_j \geq 0,\ j = 1,2,\cdots,n \right).$$

Where $[a_{ij}] = [(a_{ij})_\alpha^-, (a_{ij})_\alpha^+]$, $[b_i] = [(b_i)_\alpha^-, (b_i)_\alpha^+]$, $[c_j] = [(c_j)_\alpha^-, (c_j)_\alpha^+]$, $[d_j] = [(d_j)_\alpha^-, (d_j)_\alpha^+]$, are closed interval numbers. Assume that all of $[a_{ij}]$, $[b_i]$, $[c_j]$, $[d_j]$, $[c_0]$, and $[d_0] \in F(\mathfrak{R})$, where $F(\mathfrak{R})$ denotes the set of all closed intervals on $\mathfrak{R}$.

Definition 2.8. A point x_j, $j = 1,2,\cdots,n$ which satisfies the constraints in problem (2) is called a feasible solution. Any feasible solution $x_j^* \in M_\alpha$ (the set of all feasible solutions of problem (2)) is said to be an optimal solution if:

$$\frac{[c_j] x_j^*}{[d_j] x_j^*} (\geq \text{ or } \leq) \frac{[c_j] x_j}{[d_j] x_j};\ \forall\ j = 1,2\cdots,n.$$

Proposition 2.1. Problem (1) and problem (2) are equivalent.

Proof. Assume that M_1 and M_2 are two sets of feasible solutions of problem (1) and problem (2), respectively. Then $x \in M_1$ if and only if $\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i$, $i = 1,2,\cdots,r_0$, $\sum_{j=1}^{n} [a_{ij}] x_j \geq [b_i]$, $i = r_0+1, r_0+2, \cdots, m$, if and only if

$$\left\{ \sum_{j=1}^{n} [(a_{ij})_\alpha^-, (a_{ij})_\alpha^+] x_i \right\}, \leq [(b_i)_\alpha^-, (b_i)_\alpha^+], \quad i = 1,2,\cdots,r_0,$$

$$\left\{ \sum_{j=1}^{n} [(a_{ij})_\alpha^-, (a_{ij})_\alpha^+] x_i \right\}, \leq [(b_i)_\alpha^-, (b_i)_\alpha^+],$$
$$i = r_0+1, r_0+2, \cdots, m.$$

if and only if

$$\left\{ \sum_{j=1}^{n} [(a_{ij}]_\alpha^-, (a_{ij})_\alpha^+] x_i \right\}, \quad i = 1,2,\cdots,r_0,$$

$$\left\{ \sum_{j=1}^{n} [(a_{ij}]_\alpha^-, (a_{ij})_\alpha^+] x_i \right\}, \quad i = r_0+1, r_0+2, \cdots, m.$$

If and only if

$$\sum_{j=1}^{n} (a_{ij})_\alpha^- x_j \leq (b_i)_\alpha^-,$$

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$$\sum_{j=1}^{n}(a_ij)^{+}_{\alpha}x_j \le (b_i)^{+}_{\alpha}, \quad i=1,2,\cdots,r_0,$$

$$\sum_{j=1}^{n}(a_{ij})^{-}_{\alpha}x_j \ge (b_i)^{+}_{\alpha},$$

$$\sum_{j=1}^{n}(a_ij)^{+}_{\alpha}x_j \le (b_i)^{-}_{\alpha}, \quad i=r_0+1,r_0+2,\cdots,m.$$

$$\sum_{j=1}^{n}[a_{ij}]x_j \le [b_i], \quad i=1,2,\cdots,r_0,$$

$$\sum_{j=1}^{n}[a_{ij}]x_j \ge [b_i], \quad i=r_0+1,r_0+2,\cdots,m.$$

If and only if $x \in M_2$. Thus $M_1 \cong M_2$.
On the other hand, suppose that x_j^* be an optimal solution for problem (1), then

$$\frac{\sum_{j=1}^{n}\tilde{c}_j x_j^* + \tilde{c}_0}{\sum_{j=1}^{n}\tilde{d}_j x_j^* + \tilde{d}_0}; \quad \forall\, x \in M_1.$$

If and only if

$$\frac{\sum_{j=1}^{n}[(c_{ij})^{-}_{\alpha},(c_j)^{+}_{\alpha}]x_j^* + [c_0]}{\sum_{j=1}^{n}[(d_j)^{-}_{\alpha},(d_j)^{+}_{\alpha}]x_j^* + [d_0]}(\ge \text{ or } \le)\frac{\sum_{j=1}^{n}[(c_j)^{-}_{\alpha},(c_j)^{+}_{\alpha}]x_j + [c_0]}{\sum_{j=1}^{n}[(d_j)^{-}_{\alpha},(d_j)^{+}_{\alpha}x_j + [d_0]}.$$

If and only if

$$\frac{\sum_{j=1}^{n}[c_j]x_j^* + [c_0]}{\sum_{j=1}^{n}[d_j]x_j^* + [d_0]}(\ge \text{ or } \le)\frac{\sum_{j=1}^{n}[c_j]x_j + [c_0]}{\sum_{j=1}^{n}[d_j]x_j + [d_0]}.$$

Thus, x_j^* is an optimal solution of problem (2).

For minimization problem, the solution can be obtained as the pareto optimal solution of the following multiobjective linear fractional programming problem (Ishibuchi and Tanaka, 1990, [16]).

$$\min((Z^C, Z^U) : x \in M \subset \Re^n). \tag{3}$$

Where Z^C is defined as in Definition 2.4.
Similarly, for maximization problem the optimal solution is the pareto optimal solution of

$$\max((Z^C, Z^L) : x \in M \subset \Re^n). \tag{4}$$

Definition 2.9. A point $x \in M$ is a solution of problem (2) if and only if there is no $x^\circ \in M$ which satisfies $Z(x) \le_{LC} Z(x^\circ)$

Problem (4) can be rewritten as:

$$\max\left(Z^c = \frac{\sum_{j=1}^{n}\frac{(c_j)^{-}_{\alpha}+(c_j)^{+}_{\alpha}}{2}x_j + \frac{(c_0)^{-}_{\alpha}+(c_0)^{+}_{\alpha}}{2}}{\sum_{j=1}^{n}\frac{(d_j)^{-}_{\alpha}+(d_j)^{+}_{\alpha}}{2}x_j + \frac{(d_0)^{-}_{\alpha}+(d_0)^{+}_{\alpha}}{2}}, \right.$$

$$\left. Z^L = \frac{\sum_{j=1}^{n}(c_j)^{-}_{\alpha}x_j + (c_0)^{-}_{\alpha}}{\sum_{j=1}^{n}(d_j)^{+}_{\alpha}x_j + (d_0)^{+}_{\alpha}} \right).$$

subject to (5)

$$M = \left(\sum_{j=1}^{n}(a_{ij})^{-}_{\alpha}x_j \le (b_i)^{-}_{\alpha}, \right.$$
$$\sum_{j=1}^{n}(a_{ij})^{+}_{\alpha}x_j \le (b_i)^{+}_{\alpha}, \quad i=1,2,\cdots,r_0,$$
$$\sum_{j=1}^{n}(a_{ij})^{-}_{\alpha}x_j \ge (b_i)^{+}_{\alpha}, \quad \sum_{j=1}^{n}(a_{ij})^{+}_{\alpha}x_j \ge (b_i)^{-}_{\alpha},$$
$$i=1,2,\cdots,r_0+1,r_0+2,\cdots,m,$$
$$\left. x_j \ge 0, \quad j=1,2,\cdots,n \right).$$

For solving problem (5), let us apply the method proposed by Guzel (2013) [41], as:

$$\max\left(\sum_{j=1}^{n}\frac{(c_j)^{-}_{j}+(c_j)^{+}_{j}}{2}x_j + \frac{(c_0)^{-}_{\alpha}+(c_0)^{+}_{\alpha}}{2} - Z^C_{\text{opt}} \right.$$
$$\left(\sum_{j=1}^{n}\frac{(d_j)^{-}_{\alpha}+(d_j)^{+}_{\alpha}}{2}x_j + \frac{(d_0)^{-}_{\alpha}+(d_0)^{+}_{\alpha}}{2} \right)$$
$$+ \sum_{j=1}^{n}(c_j)^{-}_{j}x_j + (c_0)^{-}_{\alpha} - Z^L_{\text{opt}}\left(\sum_{j=1}^{n}(d_j)^{+}_{\alpha}x_j \right.$$
$$\left. \left. + (d_0)^{+}_{\alpha} \right) \right) \tag{6}$$

subject to

$$M = \left(\sum_{j=1}^{n}(a_{ij})^{-}_{\alpha}x_j \le (b_i)^{-}_{\alpha} \right.$$
$$\sum_{j=1}^{n}(a_{ij})^{+}_{\alpha}x_j \le (b_i)^{+}_{\alpha}, \quad i=1,2,\cdots,r_0,$$
$$\sum_{j=1}^{n}(a_{ij})^{-}_{\alpha}x_j \ge (b_i)^{+}_{\alpha}, \quad \sum_{j=1}^{n}(a_{ij})^{+}_{\alpha}x_j \ge (b_i)^{-}_{\alpha},$$
$$i=1,2,\cdots,r_0+1,r_0+2,\cdots,m,$$
$$\left. x_j \ge 0, \quad j=1,2,\cdots,n \right).$$

Where

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$$Z^C_{\text{opt}} = \frac{\sum\limits_{j=1} n\frac{(c_j)^-_\alpha+(c_j)^+_\alpha}{2}x^*_j + \frac{(c_0)^-_\alpha+(c_0)^+_\alpha}{2}}{\sum\limits_{j=1}^{n} \frac{(d_j)^-_\alpha+(d_j)^+_\alpha}{2}x^*_j + \frac{(d_0)^-_\alpha+(d_0)^+_\alpha}{2}}$$

$$= \max\left\{\frac{\sum\limits_{j=1} n\frac{(c_j)^-_\alpha+(c_j)^+_\alpha}{2}x_j + \frac{(c_0)^-_\alpha+(c_0)^+_\alpha}{2}}{\sum\limits_{j=1}^{n} \frac{(d_j)^-_\alpha+(d_j)^+_\alpha}{2}x_j + \frac{(d_0)^-_\alpha+(d_0)^+_\alpha}{2}},\ x \in M\right\}, \tag{7}$$

and

$$Z^L_{\text{opt}} = \frac{\sum\limits_{j=1} n(c_j)^-_\alpha x^*_j + (c_0)^-_\alpha}{\sum\limits_{j=1}^{n} (d_j)^-_\alpha + x^*_j + (d_0)^+_\alpha}$$

$$= \max\left\{\frac{\sum\limits_{j=1}^{n} (c_j)^-_\alpha x_j + (c_0)^-_\alpha}{\sum\limits_{j=1}^{n} (d_j)^+_\alpha x_j + (d_0)^+_\alpha},\ x \in M\right\}. \tag{8}$$

4 Solution Approach

In this section, a solution method for solving fuzzy linear fractional programming problem is illustrated as in the following steps:

Step 1: Consider the linear fractional programming (HFNLPEP) problem with normalized pentagonal fuzzy parameters in the objective function and constraints.

Step 2: Covert the HFNLFP problem into the corresponding close interval approximation linear fractional Programming (IALFP) problem.

Step 3: Apply th multiobjective optimization for IALFP problem.

Step 4: Find Z^C_{opt} and Z^L_{opt} as in (7) and (8), respectively.

Step 5: Formulate the equivalent linear programming of problem (5) as in (6).

Step 6: Solve problem (6) using Software GAMS to obtain the optimal solution which is the efficient solution of problem (5).

5 Numerical Example

Consider a HFNLFP problem as:

Step 1:

$$\begin{aligned}
&\max \tilde{Z} = (1,2,3,5,7,8,9)x_1(+)(0.5,1,2,3,4,6,7)x_2 \\
&\quad /((1,2,3,4,6,8,9)x_1(+)(0.5,1,1.5,2,2.5,3,4)x_2 \\
&\quad (+)(0.25,0.5,0.751,1.5,2,3)) \\
&\text{subject to} \\
&\quad (0.5,1,2,3,4,5,6)x_1(+)(1,2,4,5,6,8,9)x_2 \leq \\
&\quad (12,13,14,15,16,17,18), \\
&\quad (0.5,12,4,7,8,6)x_1(+)(0.5,1,2,3,4,5,6)x_2 \leq \\
&\quad (6,8,9,10,12,13,15), \\
&\quad x_1 \geq 0 \ \text{ and } \ x_2 \geq 0.
\end{aligned} \tag{9}$$

Step 2: Applying the close interval approximation for problem (9), we have

$$\begin{aligned}
&\max Z = \frac{[2,8]x_1(+)[1,6]x_2}{[2,8]x_1(+)[1,3]x_2(+)[0.5,2]} \\
&\text{subject to} \\
&\quad [2,5]x_1(+)[2,8]x_2 \leq [13,17], \\
&\quad [1,8]x_1(+)[1,5]x_2 \leq [8,13], \\
&\quad x_1 \geq 0 \ \text{ and } \ x_2 \geq 0.
\end{aligned} \tag{10}$$

Step 3: Multiobjective optimization for problem (10) is

$$\begin{aligned}
&\max\left(Z^C = \frac{5x_1+3.5x_2}{5x_1+3x_2+1.25}, Z^L = \frac{2x_1+x_2}{8x_1+3x_2+0.5}\right) \\
&\text{subject to} \\
&\quad x_1+2x_2 \ \leq 13, \\
&\quad 5x_1+8x_2 \leq 17, \\
&\quad x_1+x_2 \ \leq 8, \\
&\quad 8x_1+5x_2 \leq 13, \\
&\quad x_1 \geq 0 \ \text{ and } \ x_2 \geq 0.
\end{aligned} \tag{11}$$

Step 4: $Z^C_{\text{opt}} = 0.9754$ at $x^*_1 = 0,\ x^*_2 = 2.126$ and $Z^L_{\text{opt}} = 0.3091$ at $x^*_1 = 0, x^*_2 = 2.124$.

Steps 5, 6: Formulate and solve the following linear programming:

$$\begin{aligned}
&\max Z = -0.349x_1 + 0.6465x_2 - 1.3738 \\
&\text{subject to} \\
&\quad x_1+2x_2 \ \leq 13, \\
&\quad 5x_1+8x_2 \leq 17, \\
&\quad x_1+x_2 \ \leq 8, \\
&\quad 8x_1+5x_2 \leq 13, \\
&\quad x_1 \geq 0 \ \text{ and } \ x_2 \geq 0.
\end{aligned}$$

The optimal solution is $Z^C_{\text{opt}} = -0.0019$ at $x^*_1 = 0,\ x^*_2 = 2.125$.
Hence the optimal solution of problem (10) is $Z^*_{\text{opt}} = [0.2537, 48571]$ at $x^*_1 = 0,\ x^*_2 = 2.125$, and the

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fuzzy optimum value is equal to $\tilde{Z} = (0.0435, 0.1194, 0.2936, 0.5714, 1.0159, 2.2857, 5.333)$ with crisp value equals 1.2792.

It is obvious that the results obtained by the proposed method are more satisfactory than those obtained by Guzel (2013) [41].

6 Conclusions

In this paper, close interval approximation for normalized heptagonal fuzzy linear fractional programming problem has been developed. The close interval approximation LFP problem has converted into multiobjective LFP problem with two objectives: One is the maximization of the lower interval and the other is the maximization of the center. Hence the method proposed by Guzel (2013) [41] is applied for solving the multiobjective LFP problems. The solution set of the close interval approximation objective functions is defined as the efficient solution of the corresponding multiobjective fractional programming problems. The solution method is illustrated through a numerical example.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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