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General Power Inequalities for Generalized Euclidean Operator Radius

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Abstract: In this paper, we establish generalizations and refinements for some results including upper bounds for the general Euclidean operator radius and the numerical radius.

Keywords: Numerical radius, Euclidean operator radius, Usual operator norm.

1 Introduction

Let $(H,\langle .,. \rangle)$ be a complex Hilbert space and $B(H)$ be the C^*algebra of all bounded linear operators on H. The numerical radius $w(.)$ and the usual operator norm $||.||$ for $T \in B(H)$ are defined by $w(T) = \sup_{||x||=1} |\langle Tx, x\rangle|$ and $||T|| = \sup_{||x||=1} \langle Tx, Tx\rangle^{\frac{1}{2}}$.

It is Known that the numerical radius and the usual operator norm are equivalent norms and satisfy the following sharp inequality

$$\frac{1}{2}||T|| \leq w(T) \leq ||T||,$$

for $T \in B(H)$.

Let $T_1,\ldots,T_n \in B(H)$ and $p \geq 1$. The Euclidean operator radius $w_e(T_1,\ldots,T_n)$ and the general Euclidean operator radius $w_p(T_1,\ldots,T_n)$ are defined by

$$w_e(T_1,\ldots,T_n) = \sup_{||x||} \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^2 \right)^{\frac{1}{2}},$$

and

$$w_p(T_1,\ldots,T_n) = \sup_{||x||} \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p \right)^{\frac{1}{p}}.$$

Note that $w_e(.)$ and $w_p(.)$ are norms and if $n = 1$, then $w_p(T_1) = w(T_1)$, see [6,9,10] for more information. Let $a,b \geq 0$ and $0 \leq \alpha \leq 1$. Then we have

• The classical Young inequality:

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \tag{1}$$

• The authors in [1] gave a refinement for (1) as follows:

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b - r_0(\sqrt{a}-\sqrt{b})^2 \quad \text{where } r_0 = \min\{\alpha, 1-\alpha\} \tag{2}$$

• The authors in [2] presented a generalization for (2) as follows:

$$(a^\alpha b^{1-\alpha})^k \leq (\alpha a + (1-\alpha)b)^k - r_0^k (a^{\frac{k}{2}} - b^{\frac{k}{2}})^2 \quad \text{where } r_0 = \min\{\alpha, 1-\alpha\}, \tag{3}$$

for every $k \in \mathbb{N}$.

• Recently, Choi [3] has given another refinement for Young inequality as follows:

$$(a^\alpha b^{1-\alpha})^k \leq (\alpha a + (1-\alpha)b)^k - (2r_0)^k \left(\left(\frac{a+b}{2}\right)^k - (ab)^{\frac{k}{2}} \right) \quad \text{where } r_0 = \min\{\alpha, 1-\alpha\}, \tag{4}$$

for every $k \in \mathbb{N}$.

• The following inequality can be found in [4]:

$$\alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}, \tag{5}$$

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for every $r \geq 1$. Let $A_i, B_i, T_i \in B(H)$ $(i = 1, 2, ..., n)$ and f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ $(t \geq 0)$. Moslehian et al. [5] established the following upper bound for w_p :

$$w_p^{rp}(A_1^* T_1 B_1, ..., A_n^* T_n B_n) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} \left((B_i^* f^2(|T_i|) B_i)^{rp} + (A_i^* g^2(|T_i^*|) A_i)^{rp} \right) \right\|, \quad (6)$$

for every $r, p \geq 1$.
In [6], the authors refined inequality (6) as follows:

$$w_p^{rp}(A_1^* T_1 B_1, ..., A_n^* T_n B_n) \leq \frac{n^{1-\frac{1}{r}}}{2^{\frac{1}{r}}} \left\| \sum_{i=1}^{n} \left((B_i^* f^2(|T_i|) B_i)^{rp} + (A_i^* g^2(|T_i^*|) A_i)^{rp} \right) \right\|^{\frac{1}{r}} - \inf_{||x||} \xi(x), \quad (7)$$

for every $r, p \geq 1$ where $\xi(x) = \frac{1}{2} \sum_{i=1}^{n} \left(\left\langle (B_i^* f^2(|T_i|) B_i)^p x, x \right\rangle^{\frac{1}{2}} - \left\langle (A_i^* g^2(|T_i^*|) A_i)^p x, x \right\rangle^{\frac{1}{2}} \right)^2$.
Recently, the authors [7] has given a generalization for inequality (7) as follows:

$$w_p^p(A_1^* T_1 B_1, ..., A_n^* T_n B_n) \leq \frac{n^{1-\frac{k}{r}}}{2^{\frac{k}{r}}} \left\| \sum_{i=1}^{n} \left((B_i^* f^2(|T_i|) B_i)^{\frac{pr}{k}} + (A_i^* g^2(|T_i^*|) A_i)^{\frac{pr}{k}} \right) \right\|^{\frac{k}{r}} - \inf_{||x||} \xi(x), \quad (8)$$

for every $k \in \mathbb{N}$ and $r, p \geq k$ where

$$\xi(x) = \sum_{i=1}^{n} \left[\frac{1}{2^k} \left(\left\langle \left((B_i^* f^2(|T_i|) B_i)^{\frac{p}{k}} + (A_i^* g^2(|T_i^*|) A_i)^{\frac{p}{k}} \right) x, x \right\rangle \right)^k - \left(\left\langle (B_i^* f^2(|T_i|) B_i)^{\frac{p}{k}} x, x \right\rangle \left\langle (A_i^* g^2(|T_i^*|) A_i)^{\frac{p}{k}} x, x \right\rangle \right)^{\frac{k}{2}} \right]$$

In this paper, we present a generalization and a refinement for some results that include upper bounds for $w_p(T_1, ..., T_n)$ and $w(T)$ where $T, T_i \in B(H)$ $(i = 1, 2, ..., n)$.

2 The Main Results

To prove our results, we need inequalities (1)-(5) and the following two well-known lemmas. The first lemma is a result of the spectral Theorem together with Jensen's inequality (see[8]).

Lemma 1. *Let $T \in B(H)$ be a positive operator and $x \in H$ be any vector. Then*
(a) $\langle Tx, x \rangle^s \leq ||x||^{2s-2} \langle T^s x, x \rangle$ for $s \geq 1$;
(b) $\langle T^s x, x \rangle \leq ||x||^{2-2s} \langle Tx, x \rangle^s$ for $0 < s \leq 1$.

The second lemma is also a consequence of the spectral Theorem [8].

Lemma 2. *Let $T \in B(H)$ and f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ $(t \geq 0)$. Then $\langle Tx, y \rangle \leq \|f(|T|)x\| \, \|g(|T^*|)y\|$ for any $x, y \in H$.*

Our first result generalizes (8).

Theorem 1. *Let $A_i, B_i, T_i \in B(H)$ $(i = 1, 2, ..., n)$ and f and g be nonnegative continuous functions such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for every $k, m \in \mathbb{N}$ and $r, p \geq k$,*

$$w_p^p(A_1^* T_1^m B_1, ..., A_n^* T_n^m B_n) \leq \frac{n^{1-\frac{k}{r}}}{2^{\frac{k}{r}} m} \sum_{j=1}^{m} \left\| \sum_{i=1}^{n} \left((U_{ij})^{\frac{pr}{k}} + (V_{ij})^{\frac{pr}{k}} \right) \right\|^{\frac{k}{r}} - \inf_{||x||=1} \xi(x), \quad (9)$$

where $U_{ij} = A_i^* f^2(|T_i^{*^j}|) A_i$, $V_{ij} = B_i^* T_i^{*^{m-j}} g^2(|T_i^j|) T_i^{m-j} B_i$ and

$$\xi(x) = \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{1}{2^k} (\langle ((U_{ij})^{\frac{p}{k}} + (V_{ij})^{\frac{p}{k}}) x, x \rangle)^k - (\langle (U_{ij})^{\frac{p}{k}} x, x \rangle \langle (V_{ij})^{\frac{p}{k}} x, x \rangle)^{\frac{k}{2}} \right). \quad (10)$$

Proof. Let $x \in H$ be any unit vector. Then

$$\sum_{i=1}^{n} |\langle A_i^* T_i^m B_i x, x \rangle|^p = \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left| \left\langle T_i^{*^j} A_i x, T_i^{m-j} B_i x \right\rangle \right|^p$$

$$\leq \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} ||f(|T_i^{*^j}|) A_i x||^p ||g(|T_i^j|) T_i^{m-j} B_i x||^p$$

(by Lemma 2)

$$= \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \langle U_{ij} x, x \rangle^{\frac{p}{2}} \langle V_{ij} x, x \rangle^{\frac{p}{2}}$$

$$\leq \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\langle (U_{ij})^{\frac{p}{k}} x, x \rangle^{\frac{1}{2}} \langle (V_{ij})^{\frac{p}{k}} x, x \rangle^{\frac{1}{2}} \right)^k$$

(by Lemma 1 part (a))

$$\leq \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{\langle (U_{ij})^{\frac{p}{k}} x, x \rangle^r + \langle (V_{ij})^{\frac{p}{k}} x, x \rangle^r}{2} \right)^{\frac{k}{r}}$$

$$-\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left(\frac{\langle (U_{ij})^{\frac{p}{k}}x,x\rangle+\langle (V_{ij})^{\frac{p}{k}}x,x\rangle}{2}\right)^{k}-\left(\langle (U_{ij})^{\frac{p}{k}}x,x\rangle\langle (V_{ij})^{\frac{p}{k}}x,x\rangle\right)^{\frac{k}{2}}\right).$$

(by inequalities (4) and (5))

$$\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{\langle (U_{ij})^{\frac{pr}{k}}x,x\rangle+\langle (V_{ij})^{\frac{pr}{k}}x,x\rangle}{2}\right)^{\frac{k}{r}}$$

$$-\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left(\frac{\langle (U_{ij})^{\frac{p}{k}}x,x\rangle+\langle (V_{ij})^{\frac{p}{k}}x,x\rangle}{2}\right)^{k}-\left(\langle (U_{ij})^{\frac{p}{k}}x,x\rangle\langle (V_{ij})^{\frac{p}{k}}x,x\rangle\right)^{\frac{k}{2}}\right).$$

(by Lemma 1 part (a))

$$\leq \frac{n^{1-\frac{k}{r}}}{2^{\frac{k}{r}}m}\sum_{j=1}^{m}\left\langle\left(\sum_{i=1}^{n}\left((U_{ij})^{\frac{pr}{k}}+(V_{ij})^{\frac{pr}{k}}\right)\right)x,x\right\rangle^{\frac{k}{r}}$$

$$-\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left(\frac{\langle (U_{ij})^{\frac{p}{k}}x,x\rangle+\langle (V_{ij})^{\frac{p}{k}}x,x\rangle}{2}\right)^{k}-\left(\langle (U_{ij})^{\frac{p}{k}}x,x\rangle\langle (V_{ij})^{\frac{p}{k}}x,x\rangle\right)^{\frac{k}{2}}\right).$$

(by concavity of $t^{\frac{k}{r}}$ $(t\geq 0)$).

We get the desired bound by taking the supremum over all unit vectors $x\in H$.

The inequality (8) can be directly obtained g by letting $m=1$ in Theorem 1. Also Corollary 2.2 in [7] is obtained by setting $m=1$, $f(t)=g(t)=t^{\frac{1}{2}}$ and $T_i=I$ for $i=1,2,\dots,n$ in Theorem 1.

An application of Theorem 1 can be seen in the following result. It includes a general Euclidean operator radius inequality for powers of operators.

Corollary 1.

Let $T_i\in B(H)$ *and* $m\in\mathbb{N}$. *Then,*

$$w_p^p(T_1^m,\dots,T_n^m)\leq \frac{n^{1-\frac{k}{r}}}{2^{\frac{k}{r}}m}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n}|T_i^{*^j}|^{\frac{pr}{k}}+(T_i^{*^{m-j}}|T_i^j|T_i^{m-j})^{\frac{pr}{k}}\right\|^{\frac{k}{r}}-\inf_{||x||=1}\xi(x),$$

for every $k\in\mathbb{N}$ *and* $r,p\geq k$, *where*

$$\xi(x)=\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{2^k}\left\langle(|T_i^{*^j}|^{\frac{p}{k}}+(T_i^{*^{m-j}}|T_i^j|T_i^{m-j})^{\frac{p}{k}})x,x\right\rangle^{k}-\left(\left\langle|T_i^{*^j}|^{\frac{p}{k}}x,x\right\rangle\left\langle(T_i^{*^{m-j}}|T_i^j|T_i^{m-j})^{\frac{p}{k}}x,x\right\rangle\right)^{\frac{k}{2}}\right).$$

The upcoming simpler form follows from Corollary 2.4 by setting $r=k$.

Corollary 2. *Let* $T_i\in B(H)$ $(i=1,2,\dots,n)$. *Then for every* $m,k\in\mathbb{N}$ *and* $p\geq k$,

$$w_p^p(T_1^m,\dots,T_n^m)\leq \frac{1}{2m}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n}|T_i^{*^j}|^p+(T_i^{*^{m-j}}|T_i^j|T_i^{m-j})^p\right\|-\inf_{||x||=1}\xi(x)$$

where

$$\xi(x)=\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle(|T_i^{*^j}|^{\frac{p}{k}}+(T_i^{*^{m-j}}|T_i^j|T_i^{m-j})^{\frac{p}{k}})x,x\right\rangle^{k}-\left(\left\langle|T_i^{*^j}|^{\frac{p}{k}}x,x\right\rangle\left\langle(T_i^{*^{m-j}}|T_i^j|T_i^{m-j})^{\frac{p}{k}}x,x\right\rangle\right)^{\frac{k}{2}}\right)$$

Following the same arguments used in Theorem 1, we achieve the following.

Theorem 2. *Let* $A_i,B_i,T_i\in B(H)$ $(i=1,2,\dots,n)$ *and* f *and* g *be nonnegative continues functions such that* $f(t)g(t)=t$ *for all* $t\in[0,\infty)$. *Then for every* $\alpha\in[0,1]$, $k,m\in\mathbb{N}$ *and* $r,p\geq k$ *we have* $w_p^p(A_1^*T_1^mB_1,\dots,A_n^*T_n^mB_n)$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\|\alpha U_{ij}^{\frac{pr}{\alpha k}}+(1-\alpha)V_{ij}^{\frac{pr}{(1-\alpha)k}}\right\|^{\frac{k}{r}}-\inf_{||x||=1}\xi(x)\right)^{\frac{1}{2}}$$

where U_{ij} *and* V_{ij} *as in Theorem 2.1,*

$$\xi(x)=\lambda_k\left(\left\langle\frac{U_{ij}^{\frac{p}{\alpha k}}+V_{ij}^{\frac{p}{k(1-\alpha)}}}{2}x,x\right\rangle^{k}-(\left\langle U_{ij}^{\frac{p}{\alpha k}}x,x\right\rangle\left\langle V_{ij}^{\frac{p}{k(1-\alpha)}}x,x\right\rangle)^{\frac{k}{2}}\right),$$

and $\lambda_k=(2\min\{\alpha,1-\alpha\})^k$.

Proof. Let $x\in H$ be any unit vector and let $\beta=1-\alpha$. Then

$$\sum_{i=1}^{n}|\langle A_i^*T_i^mB_ix,x\rangle|^p$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\langle U_{ij}x,x\rangle^{\frac{p}{2}}\langle V_{ij}x,x\rangle^{\frac{p}{2}}$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}((\left\langle U_{ij}^{\frac{1}{\alpha}}x,x\right\rangle^{\frac{p\alpha}{k}}\left\langle V_{ij}^{\frac{1}{\beta}}x,x\right\rangle^{\frac{p\beta}{k}})^{k})^{\frac{1}{2}}$$

(by Lemma 1 part(b))

$$\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}((\left\langle U_{ij}^{\frac{p}{k\alpha}}x,x\right\rangle^{\alpha}\left\langle V_{ij}^{\frac{p}{k\beta}}x,x\right\rangle^{\beta})^{k})^{\frac{1}{2}}$$

(by Lemma 1 part(a))

$$\le \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\Bigg((\alpha\left\langle U_{ij}^{\frac{p}{k\alpha}}x,x\right\rangle^{r} + \beta\left\langle V_{ij}^{\frac{p}{k\beta}}x,x\right\rangle^{r})^{\frac{k}{r}} - \lambda_k\Bigg((\frac{\left\langle U_{ij}^{\frac{p}{k\alpha}}x,x\right\rangle+\left\langle V_{ij}^{\frac{p}{k\beta}}x,x\right\rangle}{2})^{k} - (\left\langle U_{ij}^{\frac{p}{k\alpha}}x,x\right\rangle\left\langle V_{ij}^{\frac{p}{k\beta}}x,x\right\rangle)^{\frac{k}{2}}\Bigg)\Bigg)^{\frac{1}{2}}$$

(by inequalities (4) and (5))

$$\le \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\Bigg((\alpha\left\langle U_{ij}x,x\right\rangle^{\frac{pr}{k\alpha}} + \beta\left\langle V_{ij}x,x\right\rangle^{\frac{pr}{k\beta}})^{\frac{k}{r}} - \lambda_k\Bigg((\frac{\left\langle U_{ij}^{\frac{p}{k\alpha}}x,x\right\rangle+\left\langle V_{ij}^{\frac{p}{k\beta}}x,x\right\rangle}{2})^{k} - (\left\langle U_{ij}^{\frac{p}{k\alpha}}x,x\right\rangle\left\langle V_{ij}^{\frac{p}{k\beta}}x,x\right\rangle)^{\frac{k}{2}}\Bigg)\Bigg)^{\frac{1}{2}}$$

(by Lemma 1 part(a)).
The proof of the theorem is finished by taking the supremum over all unit vectors $x \in H$.

The next corollary is a refinement for Theorem 2 in [9].

Corollary 3. *Let $A,B \in B(H)$, $\alpha \in [0,1]$ and $r \ge 1$. Then*

$$w(A^*B) \le \left(\left\|\alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right\|^{\frac{1}{r}} - \inf_{\|x\|=1}\xi(x)\right)^{\frac{1}{2}},$$

where

$$\xi(x) = 2\min\{\alpha,1-\alpha\}\left\langle \frac{|A|^{\frac{2}{\alpha}}+|B|^{\frac{2}{1-\alpha}}}{2}x,x\right\rangle - \left(\left\langle |A|^{\frac{2}{\alpha}}x,x\right\rangle\left\langle |B|^{\frac{2}{1-\alpha}}x,x\right\rangle\right)^{\frac{1}{2}}.$$

Proof. The result follows by letting $n=m=p=k=1$, $A_1 = A$, $B_1 = B$, $T_1 = I$ and $f(t) = g(t) = t^{\frac{1}{2}}$.

The next corollary is obtained by setting $f(t) = g(t) = t^{\frac{1}{2}}$ and $T_i = I$ for $i = 1,2,...,n$ in Theorem 2.

Corollary 4. *Let $A_i,B_i \in B(H)$ $(i = 1,2,...,n)$. Then for any $\alpha \in [0,1]$, $k,m \in \mathbb{N}$ and $r,p \ge k$ we have*

$$w_p^p(A_1^*B_1,...,A_n^*B_n) \le \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\|\alpha|A_i|^{\frac{pr}{\alpha k}} + (1-\alpha)|B_i|^{\frac{pr}{(1-\alpha)k}}\right\|^{\frac{k}{r}} - \inf_{\|x\|=1}\xi(x)\right)^{\frac{1}{2}}$$

where

$$\xi(x) = \lambda_k\left(\left\langle \frac{|A_i|^{\frac{p}{k\alpha}}+|B_i|^{\frac{p}{k(1-\alpha)}}}{2}x,x\right\rangle^{k} - \left(\left\langle |A_i|^{\frac{p}{k\alpha}}x,x\right\rangle\left\langle |B_i|^{\frac{p}{k(1-\alpha)}}x,x\right\rangle\right)^{\frac{k}{2}}\right).$$

The next result is a generalization of Theorem 2.2 in [7].

Theorem 3. *Let $T_i \in B(H)$ $(i = 1,2,...,n)$. Then for $0 \le \alpha \le 1$, $k \in \mathbb{N}$ and $p \ge 2k$,*

$$w_p^p(T_1^m,...,T_n^m) \le \frac{1}{m}\left\|\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}\right\|^{\alpha p}\left(\alpha|T_i^j|^{\frac{p}{k}} + \beta(T_i^{m-j}|T_i^{*^j}|T_i^{*^{m-j}})^{\frac{p}{k}}\right)^{k}\right\| - \inf_{\|x\|=1}\xi(x),\ \textit{where}$$

$$\xi(x) = \frac{(\lambda_k)^k}{m}\times\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}\right\|^{\alpha p}\left[\left\langle\frac{|T_i^j|^{\frac{p}{k}} + (T_i^{m-j}|T_i^{*^j}|T_i^{*^{m-j}})^{\frac{p}{k}}}{2}x,x\right\rangle^{k} - \left(\left\langle|T_i^j|^{\frac{p}{k}}x,x\right\rangle\left\langle(T_i^{m-j}|T_i^{*^j}|T_i^{*^{m-j}})^{\frac{p}{k}}x,x\right\rangle\right)^{\frac{k}{2}}\right].$$

Proof. Let $x \in H$ be a unit vector. Then

$$\sum_{i=1}^{n}|\langle T_i^m x,x\rangle|^p = \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left|\left\langle T_i^j x, T_i^{*^{m-j}}x\right\rangle\right|^p$$

$$\le \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle|T_i^j|^{2\alpha}x,x\right\rangle\left\langle T_i^{m-j}|T_i^{*^j}|^{2(1-\alpha)}T_i^{*^{m-j}}x,x\right\rangle\right)^{\frac{p}{2}}$$

(by Lemma 2)

$$\le \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}x\right\|^{\alpha p}\times\left(\left\langle|T_i^j|^{\frac{p}{k}}x,x\right\rangle^{\alpha}\left\langle(T_i^{m-j}|T_i^{*^j}|T_i^{*^{m-j}})^{\frac{p}{k}}x,x\right\rangle^{1-\alpha}\right)^{k}$$

(by Lemma 1)

$$\le \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}\right\|^{\alpha p}\times\left(\alpha\left\langle|T_i^j|^{\frac{p}{k}}x,x\right\rangle + (1-\alpha)\left\langle(T_i^{m-j}|T_i^{*^j}|T_i^{*^{m-j}})^{\frac{p}{k}}x,x\right\rangle\right)^{k} - \frac{(2\min\{\alpha,1-\alpha\})^k}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}\right\|^{\alpha p}$$

$$\left[\left(\frac{\left\langle |T_i^j|^{\frac{p}{k}}x,x\right\rangle + \left\langle (T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}})^{\frac{p}{k}}x,x\right\rangle}{2}\right)^k - \left(\left\langle |T_i^j|^{\frac{p}{k}}x,x\right\rangle \left\langle (T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}})^{\frac{p}{k}}x,x\right\rangle\right)^{\frac{k}{2}}\right]$$

(by inequality (4))

$$\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}\right\|^{\alpha p} . \left\langle \left(\alpha|T_i^j|^{\frac{p}{k}} + (1-\alpha)(T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}})^{\frac{p}{k}}\right)^k x,x\right\rangle - \frac{(2\min\{\alpha,1-\alpha\})^k}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}\right\|^{\alpha p}$$

$$\left[\left\langle \frac{|T_i^j|^{\frac{p}{k}} + (T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}})^{\frac{p}{k}}}{2}x,x\right\rangle^k - \left(\left\langle |T_i^j|^{\frac{p}{k}}x,x\right\rangle \left\langle (T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}})^{\frac{p}{k}}x,x\right\rangle\right)^{\frac{k}{2}}\right]$$

(by Lemma 1 part (a)).
Taking the supremum over all unit vectors $x \in H$, we finish the proof.

Choosing $\alpha = \frac{1}{2}$ in Theorem 3 we get the following corollary.

Corollary 5. *Let $T_i \in B(H)$ $(i = 1,2,...,n)$. Then for any $m,k \in \mathbb{N}$ and $p \geq 2k$,*

$$w_p^p(T_1^m,...,T_n^m) \leq$$

$$\frac{1}{2^k m}\left\|\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}\right\|^{\frac{p}{2}}\left(|T_i^j|^{\frac{p}{k}} + (T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}})^{\frac{p}{k}}\right)^k\right\| - \inf_{||x||}\xi(x)$$

where

$$\xi(x) =$$

$$\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|T_i^{*^{m-j}}\right\|^{\frac{p}{2}}\left(\left\langle \frac{|T_i^{*j}|^{\frac{p}{k}} + (T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}})^{\frac{p}{k}}}{2}x,x\right\rangle^k - \left(\left\langle |T_i^{*j}|^{\frac{p}{k}}x,x\right\rangle \left\langle (T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}})^{\frac{p}{k}}x,x\right\rangle\right)^{\frac{k}{2}}\right)$$

The next result provides a generalization for Sheikhhosseini et al. estimate ([6], Theorem 2.14).

Theorem 4. *Let $T_i \in B(H)$ $(i = 1,2,...,n)$. Then for every $m \in \mathbb{N}$, $p \geq 1$ and $0 \leq \alpha \leq 1$,*

$$w_p^p(T_1^m,...,T_n^m) \leq$$

$$\frac{1}{2^p m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\||T_i^j|^{2\alpha} + T_i^{m-j}|T_i^{*j}|^{2(1-\alpha)}T_i^{*^{m-j}}\right\| - \inf_{||x||=1}\xi(x)\right)^p$$

, where

$$\xi(x) = \left(\left\langle |T_i^j|^{2\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle T_i^{m-j}|T_i^{*j}|^{2(1-\alpha)}T_i^{*^{m-j}}x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Proof. Let $x \in H$ be a unit vector. Then

$$\sum_{i=1}^{n}|\langle T_i^m x,x\rangle|^p \leq$$

$$\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle |T_i^j|^{2\alpha}x,x\right\rangle^{\frac{1}{2}}\left\langle T_i^{m-j}|T_i^{*j}|^{2(1-\alpha)}T_i^{*^{m-j}}x,x\right\rangle^{\frac{1}{2}}\right)^p$$

(by Lemma 2)

$$\leq \frac{1}{2^p m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left[\left\langle |T_i^j|^{2\alpha}x,x\right\rangle + \left\langle T_i^{m-j}|T_i^{*j}|^{2(1-\alpha)}T_i^{*^{m-j}}x,x\right\rangle - \left(\left\langle |T_i^j|^{2\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle T_i^{m-j}|T_i^{*j}|^{2(1-\alpha)}T_i^{*^{m-j}}x,x\right\rangle^{\frac{1}{2}}\right)^2\right]^p$$

(by inequality 3)).

$$= \frac{1}{2^p m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left[\left\langle\left(|T_i^j|^{2\alpha} + T_i^{m-j}|T_i^{*j}|^{2(1-\alpha)}T_i^{*^{m-j}}\right)x,x\right\rangle - \left(\left\langle |T_i^j|^{2\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle T_i^{m-j}|T_i^{*j}|^{2(1-\alpha)}T_i^{*^{m-j}}x,x\right\rangle^{\frac{1}{2}}\right)^2\right]^p$$

The proof is complete by taking the supremum over all unit vectors $x \in H$.

The final result in this paper follows from Theorem 4 by letting $\alpha = \frac{1}{2}$.

Corollary 6. *Let $T_i \in B(H)$ $(i = 1,2,...,n)$. Then for $m \in \mathbb{N}$ and $p \geq 1$,*

$$w_p^p(T_1^m,...,T_n^m) \leq$$

$$\frac{1}{2^p m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\||T_i^j| + T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}}\right\| - \inf_{||x||=1}\xi(x)\right)^p$$

, where

$$\xi(x) = \left(\left\langle |T_i^j|x,x\right\rangle^{\frac{1}{2}} - \left\langle T_i^{m-j}|T_i^{*j}|T_i^{*^{m-j}}x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

3 Conclusion

In this paper, general inequalities for the generalized Euclidean operator radius were presented. Several existing upper bounds for the numerical radius and the generalized Euclidean operator radius were obtained as special cases from our results. Also, we gave a general refinement for the well-known Dragomir upper bound for the numerical radius of bounded linear operators.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] F. Kittaneh , Y. Manasrah. Improved Young and Heinz inequalities for matrices. J Math Appl, 361,262–269(2009).
[2] Y. Manasrah , F. Kittaneh . A generalization of two refined Young inequalities. Positivity,19, 757-768(2015).
[3] D. Choi . A generalization of Younge-type inequalities. Math Inequal Appl, 21, 99-106 (2018).
[4] GH. Hardy . Littlewood JE, pólya G. Inequalities, 2nd ed. Cambridge: Cambridge University Press (1988).
[5] MS. Moslehian , M. Sattari, K. Shebrawi . Extensions of Euclidean operator radius inequalitie. Math Scaned,120(1),129–144 (2017).
[6] A. Sheikhhosseini , MS. Moslehian , K. Shebrawi . Inequalities for generalized Euclidean operator radius via Younǵs inequality. J. Math Anal Appl,445,1516–1529(2017).
[7] S. Sahoo , NC. Rout , M.Sababheh . Some extended numerical radius inequalities. Linear Multilinear A. (2019). https://doi.org/10.1080/03081087.1698510(2019).
[8] F. Kittaneh. Notes on some inequalities for Hilbert space operators. Publ. Res. I. Math. Sci. 1988;24:283–293.
[9] S. Dragomir . Power inequalities for the numerical radius of a product of two operators in Hilbert spaces. Seraj. J. math,18,269–278(2009).
[10] S.S. Dragomir, A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces, Banach J. Math. Anal, 1(2), 154-175(2007).

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