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Analysis of the Quantum Algorithm based on Entanglement Measure for Classifying Boolean Multivariate Function into Novel Hidden Classes: Revisited

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Abstract: In this work, we revisit the algorithm proposed in [Results in Physics 15 (2019) 102549] for solving an extended variant of the Deutsch-Jozsa problem. This algorithm classifies an arbitrary oracle U_f to one of 2^n classes based on the concurrence measure. Here, we reformulate the mathematical proof of this algorithm in detail based on the first technique of the degree of entanglement quantum computing model.

Keywords: Quantum algorithms, quantum computing, Entanglement, concurrence

1 Introduction

Quantum algorithms have an enormous technological and recent progress to solve the problem that needs high performance computers[1,2]. Nowadays, quantum technologies stand at the crossroads between many areas of study, such as quantum information, computational complexity, machine learning, and quantum statistical mechanics [3,5,4,6][16]-[28]. Boolean functions play a critical role in cryptography, particularly in the design of symmetric key algorithms and information technology [8, 9]. Analyzing these functions can be done via many techniques such as spectral techniques. It was proved that quantum computers can solve some problems that can not be achieved by traditional computers even if those problems are simple [7]. This will make significant progress in information science, quantum chemistry and drug discovery [10].

Deutsch's algorithm is a cornerstone of quantum computing techniques. He suggested the first problem [11,12] that can be solved quantum mechanically quicker than traditional techniques. In 1985, the initial version of Deutsch's algorithm was proposed [11]. This algorithm classifies an oracle U_f, which represents an unknown Boolean function of a single input, into one of two categories: balanced or constant. Therefore, in 1992, the second form is proposed which is called Deutsch–Jozsa algorithm [12]. This algorithm classifies an oracle U_f that represents an unknown Boolean function of n inputs into balanced function or constant function only. Later, in 2019, Zidan et al. proposed a more generalized algorithm [13] for Deutsch–Jozsa algorithm to classify an oracle U_f that represents an unknown Boolean function of n inputs into one of 2^n classes. Although this algorithm was proved in [13], however it is not quite rigorous. In this paper,we will reformulate a rigorous mathematical proof of this algorithm based on the first technique of the degree of entanglement quantum computing model [13] to cover some gaps of the proof that was performed in [14].

The following is a breakdown of the paper's structure: The key steps of the Extended Deutsch–Jozsa algorithm

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are shown in Section 2. In Section 3, the complete analysis of this algorithm is investigated. Section 4 concludes the main findings of the paper.

1.1 The Extended Deutsch–Jozsa algorithm

Here, we present the steps of the Extended Deutsch–Jozsa algorithm [13] based on entanglement measure computing model [14] as follows:

1. Register Preparation: initialize the two quantum registers as tensor product of $|\chi\rangle$ and two ancilla qubits $|rs\rangle$ that are initialized in state $|0\rangle$ as follows

$$|\xi_0\rangle = |\chi\rangle \otimes |rs\rangle = |0\rangle^{\otimes n} \otimes |0\rangle^{\otimes 2}.$$

2. $|\xi_1\rangle = H^{\otimes n}|\chi\rangle \otimes I^{\otimes 2}|rs\rangle$.
3. $|\xi_2\rangle = U_f|\chi, r\rangle \otimes I|s\rangle$.
4. To acquire another copy of $|rs\rangle$, repeat steps 1, 2, and 3 since M_z operator requires two copies to measure the degree of entanglement between the qubits $|rs\rangle$ [14].
Remark: This step does not contradict the non-cloning theorem [15], since when steps 1, 2, and 3 are repeated, a new distinct system is initialized in the first step, and when the second and third steps are applied, a new copy of $|rs\rangle$ is produced separately, without cloning the original state.
5. Apply the operator M_z, illustrated in Figure 1, on the two copies of the qubits $|rs\rangle$ and estimate P_{0011} and P_{1100} to quantify the concurrence value C and estimate the P_{0000} and P_{1111}. Where P_{0000}, P_{0011}, P_{1100} and P_{1111} represent the probabilities of the states $|0000\rangle$, $|0011\rangle$, $|1100\rangle$ and $|1111\rangle$, respectively.
(i) If $P_{0000} > P_{1111}$ then $U_f \in$ class r,

$$r = \frac{N}{2}(1 - \sqrt{1 - C^2}).$$

(a) If $r = 0$ then U_f is the constant function $f(x_1, x_2, \ldots, x_n) = 0$.

(b) If $r = \frac{N}{2}$ then U_f is a balanced function.

(b) If $0 < r$ and $r \neq \frac{N}{2}$ then $U_f \in r$.
(ii) If $P_{0000} < P_{1111}$ then $U_f \in$ class r,

$$r = \frac{N}{2}(1 + \sqrt{1 - C^2}).$$

(a) If $r = 0$ then U_f is the constant function $f(x_1, x_2, \ldots, x_n) = 1$.

(b) If $r = \frac{N}{2}$ then U_f is a balanced function.

(c) If $0 < r$ and $r \neq \frac{N}{2}$ then $U_f \in r$.

The quantum circuit of this algorithm is shown in Figure 2.

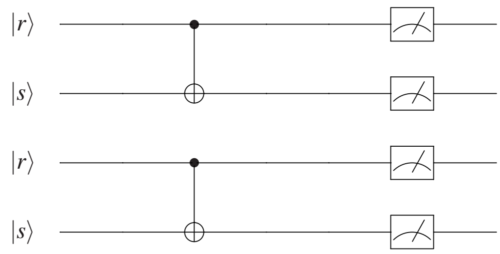

Fig. 1: The circuit model of the operator M_z.

1.2 The Performance analysis of the Extended Deutsch–Jozsa algorithm

Here, we will analyze and prove the Extended Deutsch–Jozsa algorithm via the degree of entanglement based computing model. Particularly, we will develop our prove using the first technique of this model (see Ref. [14]).
In step 1, We initialize the system by the quantum registers $|\chi\rangle$ of size n qubit and two extra qubits $|r\rangle \otimes |s\rangle$, where all the qubits are initialized in the state $|0\rangle$. In step 2, n of Hadamard gates are applied on each qubit of register $|\chi\rangle$ to generate a uniform superposition, that contains all possible values of the independent variables $x_1, x_2, \ldots, x_n$, therefore the state of the quantum system is as follows:

$$|\xi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} |k\rangle|00\rangle.$$

In step 3, the oracle U_f is applied on the register $|\chi\rangle$ and the qubit $|r\rangle$ as $U_f : |\chi, r\rangle = |\chi, r \oplus f(\chi)\rangle$, so the state of the quantum system is as follows:

$$|\xi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} |k, 0 \oplus f(k)\rangle|0\rangle.$$

Consequently, this implies to

$$|\xi_2\rangle = \sqrt{\frac{r}{N}}\, |\beta_1\rangle\, |00\rangle + \sqrt{\frac{r_1}{N}}\, |\beta_2\rangle\, |10\rangle, \tag{1}$$

such that

$$|\beta_1\rangle = \frac{1}{\sqrt{r}} \sum_{k=\{k|f(k)=0\}} |k\rangle, \; |\beta_2\rangle = \frac{1}{\sqrt{r_1}} \sum_{k=\{k|f(k)=1\}} |k\rangle.$$

Where r represents the number of the states that correspond to $f(k) = 0$, and r_1 represents the number of the states that correspond to $f(k) = 1$, $k = 0, 1, \ldots, 2^n - 1$, therefore

$$N = r + r_1. \tag{2}$$

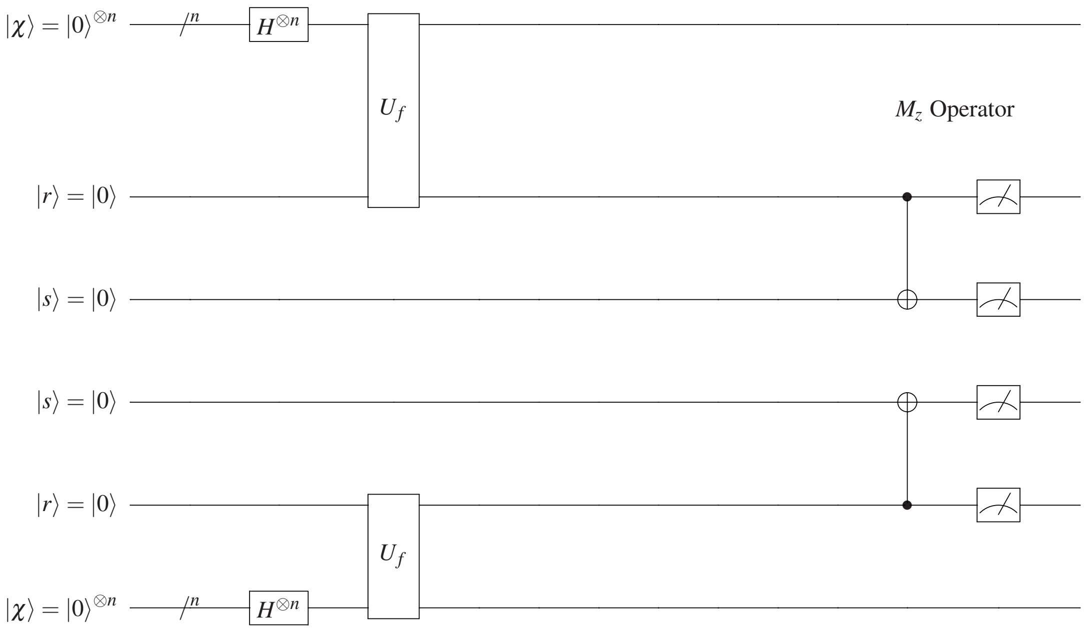

Fig. 2: the Extended Deutsch–Jozsa algorithm that classifies an oracle U_f into one of 2^n classes.

Because the Mz operator requires two copies to quantify the degree of entanglement between the qubits $|r\rangle$ and $|s\rangle$, steps 1, 2 and 3 are repeated in step 4 to produce a copy of the two-qubit system $|rs\rangle$. This step does not contradict the non-copying theorem [15], since when steps 1, 2, and 3 are repeated, a new different system is initialized in the first step, and when the second and third steps are applied, a new copy of $|rs\rangle$ is produced separately, without cloning the original state. As a result of this process, we now have two identical replicas in the form

$$|\xi_3\rangle = |\xi_2^{c1}\rangle \otimes |\xi_2^{c2}\rangle = \frac{r}{N}\,|\beta_1\rangle^{\otimes 2}\,|0000\rangle + \sqrt{\frac{r}{N}}\sqrt{\frac{r_1}{N}}\,|\beta_1\rangle|\beta_2\rangle\,|0010\rangle + \sqrt{\frac{r}{N}}\sqrt{\frac{r_1}{N}}\,|\beta_2\rangle|\beta_1\rangle\,|1000\rangle + \frac{r_1}{N}\,|\beta_2\rangle^{\otimes 2}\,|1010\rangle. \quad (3)$$

Finally, in step 5, the M_z operator is applied on the two copies of $|rs\rangle$, which allows application of two consequence operations. In the first operation, the M_z operator applies the CNOT-gate, for each replica, between the two qubits $|r\rangle$ and $|s\rangle$. After applying this operation, the state of the system is described as follows:

$$|\xi_4\rangle = |\xi_2^{c1}\rangle \otimes |\xi_2^{c2}\rangle = \frac{r}{N}\,|\beta_1\rangle^{\otimes 2}\,|0000\rangle + \sqrt{\frac{r}{N}}\sqrt{\frac{r_1}{N}}\,|\beta_1\rangle|\beta_2\rangle\,|0011\rangle + \sqrt{\frac{r}{N}}\sqrt{\frac{r_1}{N}}\,|\beta_2\rangle|\beta_1\rangle\,|1100\rangle + \frac{r_1}{N}\,|\beta_2\rangle^{\otimes 2}\,|1111\rangle. \quad (4)$$

The Mz operator quantifies the concurrence value C between the two qubits $|r\rangle$ and $|s\rangle$ in the second operation by measuring them and calculating the probability of the states $|0011\rangle$ and/or $|1100\rangle$, then using Eq. (5) to quantify the concurrence value C. Considering that the concurrence is measured via the Mz operator as [14]

$$C = 2\sqrt{P_{0011}},\ C = 2\sqrt{P_{1100}},\ or\ C = \sqrt{2(P_{1100}+P_{1100})}. \quad (5)$$

Therefore the concurrence value C is calculated as follows:

$$C = 2\sqrt{2(P_{1100}+P_{1100})} = 2\frac{\sqrt{r_1 r}}{N} = 2\frac{\sqrt{r(N-r)}}{N}. \quad (6)$$

So, it is clear that this function is a quadratic equation in terms of r as follows:

$$r^2 - Nr + \frac{C^2N^2}{4} = 0.$$

which has the following two roots

$$r = \frac{N}{2}(1 \pm \sqrt{1-C^2}), \quad (7)$$

one of these roots represents the number of states that satisfy $f(x_1,x_2,...,x_n)$=1. The other root represents the number of states that satisfy $f(x_1,x_2,...,x_n)$=0. To determine which root among Eq. (7) represents the class label r of U_f, we need to determine the most likelihood probability among the states $|0000\rangle$ and $|1111\rangle$ in state defined by Eq. (4). Because if the number of states that satisfy $f(x_1,x_2,...,x_n) = 1$ are greater than the number of states that satisfy $f(x_1,x_2,...,x_n) = 0$. This makes the probability of the state $|1111\rangle$ most significant compared with the probability of the state $|0000\rangle$ in Eq. (4). Consequently, the entanglement between the qubit $|r\rangle$ and the qubit $|s\rangle$ exists with degree C. Therefore, $\frac{N}{2}(1+\sqrt{1-C^2})$ represents the class label r for the oracle U_f. On the other hand, if the number of states that satisfy $f(x_1,x_2,...,x_n) = 0$ is greater than the number of states that satisfy $f(x_1,x_2,...,x_n) = 1$ this implies that the probability of the state $|0000\rangle$ is most significant compared with the probability of the state $|1111\rangle$ in Eq. (4). Consequently, the entanglement between the qubit $|r\rangle$ and the qubit $|s\rangle$ exists with degree C. Therefore, $\frac{N}{2}(1-\sqrt{1-C^2})$ represents the class label r of the oracle U_f. It is worth mentioning that, if all the states satisfy that $f(x_1,x_2,...,x_n) = 0$ this implies that the probability of the state $|0000\rangle$ is 1 in Eq. (4). Consequently, the probability of the state $|0000\rangle$ is 1 in Eq. (4) and the concurrence value C vanishes. Therefore, $r = 0$ indicates that the oracle U_f represents the constant function $f(k) = 0$, $\forall k = 0,1,...,2^n-1$. However, if all the states satisfy that $f(x_1,x_2,...,x_n) = 1$, this implies that the probability of the state $|1111\rangle$ is 1 in Eq. (4). Consequently, the probability of the state $|1111\rangle$ is 1 in Eq. (4) and the concurrence value C vanishes as well. Therefore, $r = N$ indicates that the oracle U_f represents the constant function $f(k) = 1$, $\forall k = 0,1,...,2^n-1$. Finally, if the number of states that satisfy $f(x_1,x_2,...,x_n) = 1$ equals to the number of states that satisfy $f(x_1,x_2,...,x_n) = 0$. The probability of the state $|1111\rangle$ is equal to the probability of the state $|0000\rangle$ in Eq. (4). Consequently, the probability of the state $|1111\rangle$ equals the probability of the state $|0000\rangle$ in Eq. (4), Hence the concurrence is maximal. Therefore, $r = \frac{N}{2}$ indicates that the oracle U_f belongs to the balanced function class.

2 Perspective

In this work, the Extended Deutsch–Jozsa algorithm is proofed based on the degree of entanglement based computing model.

Conflict of interest

There is no conflict of interest.

References

[1] L. Huang, H. Zhou, K. Feng, et al., *Quantum random number cloud platform*, npj Quantum Inf 7, 107, 2021

[2] W Liu, Q Wu, J Shen, J Zhao, M Zidan, L Tong, *An optimized quantum minimum searching algorithm with sure-success probability and its experiment simulation with Cirq*, *Journal of Ambient Intelligence and Humanized Computing*, 1-10, 2021

[3] L. Mashhor, R. Salah, A. Heindric, A.-A. A.-H. Yaya, *Non-Classical Properties of Two Mode Dissipative Cavity*, Information Sciences Letters, 10 (2), pp. 197-204, 2021

[4] Z. Ahmad, A, Rashid, *Maximum Entropy Formalism for Zero Truncated Poission and Binomial Distribution*, Journal of Statistics Applications & Probability, 6(2), PP.441- 444, 2017

[5] R. Salah, M. M. Anwer, M. Abdel-Aty, *Pancharatnam Phase of Non-Hermitian Hamiltonian*, Information Sciences Letters, 10(1), PP.25-32, 2021

[6] A. Barr, W. Gispen, A. Lamacraft, *Quantum ground states from reinforcement learning*, Proc. Mach. Learn Res 107, 635–653, 2020

[7] Mohammed Zidan, H. Eleuch, M. Abdel-Aty, *Non-classical computing problems: Toward novel type of quantum computing problems*, Results in Physics, 21,103536, 2021

[8] M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press: Cambridge, UK, 2000.

[9] D. M. Nguyen, S. Kim, *Quantum Key Distribution Protocol Based on Modified Generalization of Deutsch-Jozsa Algorithm in d-level Quantum System*, International Journal of Theoretical Physics, 58 (1), PP.71-82, 2019

[10] A. Robert, P.K. Barkoutsos, S. Woerner et al., *Resource-efficient quantum algorithm for protein folding*, npj Quantum Inf, 7, 38, 2021

[11] D. Deutsch, Quantum theory, *the Church-Turing principle and the universal quantum computer*, Proc. R. Soc. of Lond. A. 400, 97 (1985).

[12] D. Deutsch and R. Jozsa, *Rapid solution of problems by quantum computation*, Proc. R. Soc. of Lond. A. ,439, 553 (1992).

[13] M. Zidan, A.-H. Abdel-Aty, D. M. Nguyen, A. S. A. Mohamed, Y. Al-Sbou, H. Eleuc, M. Abdel-Aty,*A quantum algorithm based on entanglement measure for classifying Boolean multivariate function into novel hidden classes*, Results in Physics, 15, 102549, 2019

[14] M. Zidan, *A novel quantum computing model based on entanglement degree*, Modern Physics Letters, B 34 (35), 2050401, 2021

[15] W. K. Wootters, W. H. Zurek,*A Single Quantum Cannot be Cloned*, Nature, 299, 802–803, 1982.

[16] M. Abdel-Aty, Influence of a Kerr-like medium on the evolution of field entropy and entanglement in a three-level atom, J. Phys. B: Atomic, Molecular and Optical Physics 33, pp. 2665–2676 (2000).

[17] M. Abdel-Aty, General formalism of interaction of a two-level atom with cavity field in arbitrary forms of nonlinearities, Physica A 313(3-4), pp. 471-487 (2002).

[18] M. Abdel-Aty, G. M. Abd Al-Kader, A.-S. F. Obada, Entropy and entanglement of an effective two-level atom interacting with two quantized field modes in squeezed displaced fock states, Chaos, Solitons and Fractals 12(13), pp. 2455-2470 (2001).

[19] M. Abdel-Aty, Engineering entanglement of a general three-level system interacting with a correlated two-mode nonlinear coherent state, The European Physical Journal D 23(1), pp. 155-165 (2003).

[20] M. Abdel-Aty, An investigation of entanglement and quasiprobability distribution in a generalized Jaynes-Cummings model, Journal of Mathematical Physics 44(4), pp. 1457-1471 (2003).

[21] M. Abdel-Aty, Quantum information entropy and multi-qubit entanglement, Progress in Quantum Electronics 31(1), pp. 1-49 (2007).

[22] T. M. El-Shahat, S. Abdel-Khalek, M. Abdel-Aty, A.-S. F. Obada, Aspects on entropy squeezing of a two-level atom in a squeezed vacuum Chaos, Solitons and Fractals 18(2), pp. 289-298 (2003).

[23] M. Abdel-Aty, M. S. Abdalla, A.-S. F. Obada, Entropy and phase properties of isotropic coupled oscillators interacting with a single atom: one- and two-photon processes, Journal of Optics B: Quantum and Semiclassical Optics 4(3), pp. S133-S141 (2002).

[24] M. Sebawe Abdalla, S. S. Hassan, M. Abdel-Aty, Entropic uncertainty in the Jaynes-Cummings model in presence of a second harmonic generation, Optics Communications 244(1-6), pp. 431-443 (2005).

[25] M. Abdel-Aty, M. S. Abdalla, A.-S. F. Obada, Uncertainty relation and information entropy of a time-dependent bimodal two-level system, Journal of Physics B: Atomic, Molecular and Optical Physics 35(23), pp. 4773-4786 (2002).

[26] A.-S. F. Obada, D. A. M. Abo-Kahla, Metwally, M. Abdel-Aty, The quantum computational speed of a single Cooper-pair box, Physica E: Low-Dimensional Systems and Nanostructures 43(10), pp. 1792-1797 (2011).

[27] S. Korashy, A. S. Abdel-Rady, Abdel-Nasser A. Osman, Influence of Stark shift and Kerr-medium on the interaction of a two-level atom with two quantized field modes: a time-dependent system, Quantum Information Review 5(1), pp. 9-14 (2017).

[28] M. Abdel-Aty, Quantum phase entropy and entanglement of a multiphoton three-level atom near the edge of a photonic band gap, Laser Physics 16, pp. 1381 (2006).

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