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Estimation on Kumaraswamy-Inverse Weibull Distribution with Constant Stress Partially Accelerated Life Tests

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Abstract: This paper presents a method of estimation parameters and acceleration factor of Kumaraswamy-Inverse Weibull Distribution based on constant stress partially accelerated life tests. Depending on progrssive type-II censoring, we present the maximum likelihood, Bayes, and two parametric bootstrap methods. In addition, we use the asymptotic variance covariance matrix of the estimators to construct the approximate confidence intervals, bootstrap and credible confidence intervals. Furthermore, we apply Markov Chain Monte Carlo method to compute the Bayes estimators. Also, generating Markov Chain Monte Carlo samples from the posterior density functions using Gibbs within the Metropolis-Hasting algorithm is studied. Finally, a numerical example is discussed to illustrate different methods of point estimation and confidence intervals.

Keywords: Kumaraswamy-Inverse Weibull Distribution, Bootstrap Confidence Intervals, Bayesian Estimation, MCMC.

Abbreviations:

ALT	Accelerated Life Test
PALT	Partially Accelerating Life–Tests
CSALT	Constant-Stress ALT
CSPALT	Constant-Stress Partially Accelerating Life–Tests
SSALT	Step-Stress Accelerated Life Test
SSPALT	Step-Stress Partially Accelerated Life Test
Kum-IW	Kumaraswamy-Inverse Weibull Distribution
PDF	Probability Density Function
CDF	Cumulative Distribution Function
HRF	Hazard Rate Function
SF	Survival Function
MLE	Maximum Likelihood Estimation
ProgT-II-C	Progrssive Type-II Censored
CIs	Confidence Intervals
ACIs	Approximate Confidence Intervals
BP-CI	Percentile Bootstrap Confidence Interval
BT-CI	Bootstrap-T Confidence Interval

1 Introduction

Recently, there has been a growing interest in improving the performance of products. The manufacureres make great efforts to increase the demand and create trust with the consumers. Thus, they challenge difficulty in controlling the failure of the product during the available test time. Accelerated life testing is a very effective method for improvement the performance of the products, and identifying the causes of failures in a short life time.

The continuous improvement in modern industries creates a problem in obtaining information about the lifetime of some products and materials with high reliability at the time of testing under normal conditions. Under such conditions, life testing becomes very expensive and time consuming. To get enough failures data in a short time, we need accelerated conditions, such as stresses, voltage, temperature, pressure, ... etc. This type of testing is called accelerated life testing, where products are run higher than usual stress conditions to induce early failures in a short time. The life data from the high stresses are used to estimate the life distribution at design condition, see [1].

ALT generally deals with three types of stress loadings: constant stress, step stress and progressive stress. Constant stress is the most common type of stress loading, where every item is tested under a constant level of the stress, which is higher than normal level. In this kind of testing, we may have several stress levels, which are applied for different groups of the tested items. This

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means that every item is subjected to only one stress level until the item fails or the test stops for other reasons. If stress level of the test is not high enough, many of the tested items will not fail during the available time and one has to be prepared to handle several censored data. To avoid this problem, step-stress testing can be applied, where all items are first subjected to a specified constant stress for a specified period of time. Items that do not fail will be subjected to a higher level of stress for another specified time. Level of stress increases step by step until all items fail or the test stops for other reasons. Progressive-stress loading is quite similar to the step of stress testing with the difference that the stress level increases continuously. These three types can reduce the testing time and save much material and money, see [2] and [3].

The main assumption in ALT is that the mathematical model related to the lifetime of the unit and stress is known or can be assumed. In some cases, such life stress relationships are unknown and cannot be assumed, i.e, the data obtained from ALT cannot be extrapolated to use condition. Hence, in such cases, another approach can be used; it is partially accelerated life tests. In PALT, test units are run at both usual and higher-than usual stress conditions see, [4].

Many authors have studied ALT, see for example [5, 6, 7, 8]. For step ALT, we change the test condition at a given time or the fixed number of failures. [9] and [10] studied optimal designs of SSALT for many distributions. [11] and [2] investigated statistical inference for SSPALT model based on ProgT-II-C data from Lomax and Rayleigh distribution.

In this paper, we deal only with the PALT. There are two kinds of PALT: CSPALT and SSPALT. The items are run at the accelerated and normal conditions. The SSPALT allows the test to change from normal to accelerated condition at a pre-determined time. However, in CSPALT, we divide sample size into two parts: One of them runs under nomal conditions and the other under accelerated conditions, see for example [12, 13, 14, 15].

In life-tests, some units may fail, so this sample is called censored sample. There are different methods of censoring: The first one is ”Type I censored sample”, and the second is ”Type-II censored sample”. In this paper, we only deal with Type-II censored sample. Schematically, a progressively Type-II censored sample can be described, as follows: Suppose that n independent items are put on a life test with continuous identically distributed failure times $X_1, X_2, \ldots, X_n$. Also, suppose that a censoring scheme $(R_1, R_2, \ldots, R_m)$ is previously fixed such that immediately following the first failure X_1, R_1 surviving items are removed from the experiment at random, and immediately following the second failure X_2, R_2 surviving items are removed from the experiment at random. This process continues until, at the time of the m-th observed failure X_m, the remaining R_m surviving items are removed from the test. The m ordered observed failure times denoted by $X_{1:m:n}^{(R1,\ldots,Rm)}$, $X_{2:m:n}^{(R1,\ldots,Rm)}$, $\ldots, X_{m:m:n}^{(R1,\ldots,Rm)}$ are called progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, \ldots, R_m)$. This censoring was explored by several authors, see [16, 17, 18].

Kum-IW was introduced by [19]. This distribution is an extension of the inverse Weibull distribution. The PDF, CDF, SF, and HRF of the Kum-IW(a,b,c,d) are, as follows:

$$f_1(y) = abcdy^{-(d+1)}\exp\{-acy^{-d}\}(1-\exp\{-acy^{-d}\})^{b-1}, \quad y>0; a,b,c,d>0, \tag{1}$$

$$F_1(y) = 1-(1-\exp\{-acy^{-d}\})^{b}, \tag{2}$$

$$G_1(y) = (1-\exp\{-acy^{-d}\})^{b}, \tag{3}$$

and

$$h_1(y) = abcdy^{-(d+1)}\exp\{-acy^{-d}\}(1-\exp\{-acy^{-d}\})^{-1}. \tag{4}$$

Some basic properties and applications for Kum-IW are studied by [19]. If $c = 1$ and $d = 2$, the resulting distribution is called Kumaraswamy-Inverse Rayleigh distribution, see [20].

The rest of the paper is organized, as follows: Section **2** describes the model and the basic assumptions. The maximum-likelihood estimators and corresponding approximate confidence intervals of the unknown parameters are presented in Section **3**. Two parametric bootstrap confidence intervals for the parameters are presented in Section **4**. Section **5** is devoted to the **MCMC** method. A numerical example is analysed in Section **6**. Section **7** is dedicated to conclusion.

2 Assumptions

We start by investigating the experiment, as follows: We divide the items into two groups: The first group contains n_1 items which are randomly chosen from n test items placed on normal condition and the second group contains $n_2 = n - n_1$; the remaining items under accelerated condition. Based on ProgT-II-C, in group $\mathbf{j}, \mathbf{j} = \mathbf{1}, \mathbf{2}$ the time of the first failure, S_{j_1} items are randomly withdrawn from the remaining $n_1 - 1$ surviving j_1 items. At the second failure,S_{j_2} items from the remaining $n_j - 2 - S_{j_1}$ are randomly withdrawn. The test continues

until the m_j^{th} failure $T_{jm_j:m_J:n_j}^{S_m}$ at which time, all remaining $S_{jm_j} = n_j - m_j - \sum_{k=1}^{m_j-1} S_{j_k}$ for $j = 1, 2$.

The S_{j_i} , $i = 1, \ldots, m$ are fixed before the study. The lifetime of an item tested at normal condition follows Kum-IW(a,b,c,d) distribution with PDF, CDF, SF and HRF given in (1), (2), (3) and (4). The hazard rate of an item tested at accelerated condition is given by $h_2(y) = \lambda h_1(y)$, where λ is an acceleration factor satisfying $(\lambda > 1)$. Thus, the HRF, SF, CDF and PDF under accelerated condition as follows respectively:

$$h_2(y) = \lambda abcdy^{-(d+1)}\exp\{-acy^{-d}\}(1-\exp\{-acy^{-d}\})^{-1}, \tag{5}$$

$$G_2(y) = \exp\left\{-\int_0^y h_2(w)\right\}dw = (1-\exp\{-acy^{-d}\})^{\lambda b}, \tag{6}$$

$$F_2(y) = 1-(1-\exp\{-acy^{-d}\})^{\lambda b}, \tag{7}$$

$$f_2(y) = \lambda abcdy^{-(d+1)}\exp\{-acy^{-d}\}(1-\exp\{-acy^{-d}\})^{\lambda b-1}. \tag{8}$$

If the failure times of the n_j items originally in the test are from a continuous population with CDF $F_j(x)$ and PDF $f_j(x)$, the joint PDF for $Y^{S_j}_{j1:m_j:n_j} < Y^{S_j}_{j2:m_j:n_j} < \ldots < Y^{S_j}_{ji:m_j:n_j}$ and $j = 1,2$

$$L(a,b,c,d,\lambda \mid \underset{\sim}{y}) = \prod_{j=1}^{2} A_j \left\{\prod_{i=1}^{m_j} f_j(y_{ji:m_j:n_j})[1-F_j(y_{ji:m_j:n_j})]^{S_{ji}}\right\}, \tag{9}$$

where $y_j = (y_{j_1}, y_{j_2}, \ldots, y_{jm_j})$, $j = 1,2$, and

$$A_j = n_j(n_j-1-S_{j_1})(n_j-2-S_{j_1}-S_{j_2})\ldots(n_j-m_j-\sum_{k=1}^{m_j-1} S_{j_k}).$$

3 Maximum likelihood Inference

The maximum likelihood function plays an important role in statistical estimation. Thus, we use maximum likelihood parameter estimation to define the parameter that maximizes the likelihood of the sample data. Let, for $j = 1,2$, $Y^{(S_{j_1},\ldots,S_{jm_j})}_{j_1:m_j:n_j} < Y^{(S_{j_1},\ldots,S_{jm_j})}_{j_2:m_j:n_j} < \ldots < Y^{(S_{j_1},\ldots,S_{jm_j})}_{jm_j:m_j:n_j}$ denote two progressively type-II censored samples from two populations whose CDFs and PDFs are as given in (1), (2), (7) and (8) with $(S_{j_1},\ldots,S_{jm_j})$. The log-likelihood function $l(a,b,c,d,\lambda|\underset{\sim}{y}) = \log L(a,b,c,d,\lambda|\underset{\sim}{y})$ based on two progressive Type-II censored samples is given by:

$$\begin{aligned} l(a,b,c,d,\lambda|\underset{\sim}{y}) &= (m_1+m_2)\log(abcd) + m_2\log\lambda \\ &-(d+1)\left(\sum_{i=1}^{m_1}\log(y_{1i:m_1:n_1}) + \sum_{i=1}^{m_2}\log(y_{2i:m_2:n_2})\right) \\ &-\sum_{i=1}^{m_1} acy^{-d}_{1i:m_1:n_1} - \sum_{i=1}^{m_2} acy^{-d}_{2i:m_2:n_2} \\ &+\sum_{i=1}^{m_1}(b(S_{1i}+1)-1)\log(1-\exp\{-acy^{-d}_{1i:m_1:n_1}\}) \\ &+\sum_{i=1}^{m_2}(\lambda b(S_{2i}+1)-1)\log(1-\exp\{-acy^{-d}_{2i:m_2:n_2}\}), \end{aligned} \tag{10}$$

Calculating the first partial derivatives of (10) with respect to the involved parameters and equating each to zero yield

$$\begin{aligned} &\frac{m_1+m_2}{a} - \sum_{i=1}^{m_1} cy^{-d}_{1i:m_1:n_1} - \sum_{i=1}^{m_2} cy^{-d}_{2i:m_2:n_2} \\ &+\sum_{i=1}^{m_1}\frac{(b(S_{1i}+1)-1)(cy^{-d}_{1i:m_1:n_1})\exp\{-acy^{-d}_{1i:m_1:n_1}\}}{(1-\exp\{-acy^{-d}_{1i:m_1:n_1}\})} \\ &+\sum_{i=1}^{m_2}\frac{(b\lambda(S_{2i}+1)-1)(cy^{-d}_{2i:m_2:n_2})\exp\{-acy^{-d}_{2i:m_2:n_2}\}}{(1-\exp\{-acy^{-d}_{2i:m_2:n_2}\})} \\ &= 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &\frac{m_1+m_2}{b} + \sum_{i=1}^{m_1}(S_{1i}+1)\log(1-\exp\{-acy^{-d}_{1i:m_1:n_1}\}) \\ &+\sum_{i=1}^{m_2}(\lambda(S_{2i}+1))\log(1-\exp\{-acy^{-d}_{2i:m_2:n_2}\}) \\ &= 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &\frac{m_1+m_2}{c} - \sum_{i=1}^{m_1} ay^{-d}_{1i:m_1:n_1} - \sum_{i=1}^{m_2} ay^{-d}_{2i:m_2:n_2} \\ &+\sum_{i=1}^{m_1}\frac{(b(S_{1i}+1)-1)(ay^{-d}_{1i:m_1:n_1})\exp\{-acy^{-d}_{1i:m_1:n_1}\}}{(1-\exp\{-acy^{-d}_{1i:m_1:n_1}\})} \\ &+\sum_{i=1}^{m_2}\frac{(b\lambda(S_{2i}+1)-1)(ay^{-d}_{2i:m_2:n_2})\exp\{-acy^{-d}_{2i:m_2:n_2}\}}{(1-\exp\{-acy^{-d}_{2i:m_2:n_2}\})} \\ &= 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{m_1+m_2}{d} - \sum_{i=1}^{m_1}\log(y_{1i:m_1:n_1}) - \sum_{i=1}^{m_2}\log(y_{2i:m_2:n_2}) \\ &+(ac)\left(\sum_{i=1}^{m_1} y^{-d}_{1i:m_1:n_1}\log(y_{1i:m_1:n_1}) + \sum_{i=1}^{m_2} y^{-d}_{2i:m_2:n_2}\log(y_{2i:m_2:n_2})\right) \\ &-\sum_{i=1}^{m_1}\frac{(b(S_{1i}+1)-1)(acy^{-d}_{1i:m_1:n_1}\log(y_{1i:m_1:n_1}))(\exp\{-acy^{-d}_{1i:m_1:n_1}\})}{(1-\exp\{-acy^{-d}_{1i:m_1:n_1}\})} \\ &-\sum_{i=1}^{m_2}\frac{(b\lambda(S_{2i}+1)-1)(acy^{-d}_{2i:m_2:n_2}\log(y_{2i:m_2:n_2}))(\exp\{-acy^{-d}_{2i:m_2:n_2}\})}{(1-\exp\{-acy^{-d}_{2i:m_2:n_2}\})} \\ &= 0, \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\frac{m_2}{\lambda} + \sum_{i=1}^{m_2}(b(S_{2i}+1))\log(1-\exp\{-acy^{-d}_{2i:m_2:n_2}\}) \\ &= 0, \end{aligned} \tag{15}$$

Since Eqs. (11)-(15) cannot be solved analytically, we use Newton-Raphson method to find approximate numerical solution of these equations, see EL-Sagheer [21].

The final estimates of (a,b,c,d,λ) are the MLEs of the parameters, denoted as $\left(\widehat{a},\widehat{b},\widehat{c},\widehat{d},\widehat{\lambda}\right)$.

3.1 Approximate CIs

We use $I_{ij}(\theta) = -\partial^2 l \mid \partial\theta_i\partial\theta_j$ to construct asymptotic confidence intervals of MLEs. The observed Fisher information matrix has second partial derivatives of log-likelihood function (10), with respect to a, b, c, d and λ as the entries, which can be easily obtained, see the Appendix.

$$[I_{ij}(\theta)]^{-1} = \begin{bmatrix} -\frac{\partial^2 l}{\partial a^2} & -\frac{\partial^2 l}{\partial a \partial b} & -\frac{\partial^2 l}{\partial a \partial c} & -\frac{\partial^2 l}{\partial a \partial d} & -\frac{\partial^2 l}{\partial a \partial \lambda} \\ -\frac{\partial^2 l}{\partial b \partial a} & -\frac{\partial^2 l}{\partial b^2} & -\frac{\partial^2 l}{\partial b \partial c} & -\frac{\partial^2 l}{\partial b \partial d} & -\frac{\partial^2 l}{\partial b \partial \lambda} \\ -\frac{\partial^2 l}{\partial c \partial a} & -\frac{\partial^2 l}{\partial c \partial b} & -\frac{\partial^2 l}{\partial c^2} & -\frac{\partial^2 l}{\partial c \partial d} & -\frac{\partial^2 l}{\partial c \partial \lambda} \\ -\frac{\partial^2 l}{\partial d \partial a} & -\frac{\partial^2 l}{\partial d \partial b} & -\frac{\partial^2 l}{\partial d \partial c} & -\frac{\partial^2 l}{\partial d^2} & -\frac{\partial^2 l}{\partial d \partial \lambda} \\ -\frac{\partial^2 l}{\partial \lambda \partial a} & -\frac{\partial^2 l}{\partial \lambda \partial b} & -\frac{\partial^2 l}{\partial \lambda \partial c} & -\frac{\partial^2 l}{\partial \lambda \partial d} & -\frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\hat{a},\hat{b},\hat{c},\hat{d},\hat{\lambda})}$$

$$= \begin{bmatrix} var(\hat{a}) & cov(\hat{a}\hat{b}) & cov(\hat{a}\hat{c}) & cov(\hat{a}\hat{d}) & cov(\hat{a}\hat{\lambda}) \\ cov(\hat{b}\hat{a}) & var(\hat{b}) & cov(\hat{b}\hat{c}) & cov(\hat{b}\hat{d}) & cov(\hat{b}\hat{\lambda}) \\ cov(\hat{c}\hat{a}) & cov(\hat{c}\hat{b}) & var(\hat{c}) & cov(\hat{c}\hat{d}) & cov(\hat{c}\hat{\lambda}) \\ cov(\hat{d}\hat{a}) & cov(\hat{d}\hat{b}) & cov(\hat{d}\hat{c}) & var(\hat{d}) & cov(\hat{d}\hat{\lambda}) \\ cov(\hat{\lambda}\hat{a}) & cov(\hat{\lambda}\hat{b}) & cov(\hat{\lambda}\hat{c}) & cov(\hat{\lambda}\hat{d}) & var(\hat{\lambda}) \end{bmatrix} \quad (16)$$

Thus, the $100(1-\tau)\%$ ACIs for a,b,c,d and λ are obtained as $\hat{a} \pm e_{\frac{\tau}{2}}\sqrt{q_{11}}$, $\hat{b} \pm e_{\frac{\tau}{2}}\sqrt{q_{22}}$, $\hat{c} \pm e_{\frac{\tau}{2}}\sqrt{q_{33}}$ and $\hat{d} \pm e_{\frac{\tau}{2}}\sqrt{q_{44}}$ and $\hat{\lambda} \pm e_{\frac{\tau}{2}}\sqrt{q_{55}}$, respectively, where $q_{11}, q_{22}, q_{33}, q_{44}$ and q_{55} are the elements on the main diagonal of the variance-covariance matrix F^{-1} and $e_{\frac{\tau}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\tau}{2}$.

The $\left(1-\frac{\tau}{2}\right)100\%$ ACIs for for the parameters a,b,c,d and λ take the following forms:

$$\left.\begin{array}{c} (\hat{a}_L,\hat{a}_U) = \hat{a} \pm w_{1-\frac{\tau}{2}}\sqrt{var(\hat{a})} \;\; (\hat{b}_L,\hat{b}_U) = \hat{b} \pm w_{1-\frac{\tau}{2}}\sqrt{var(\hat{b})} \\ (\hat{c}_L,\hat{c}_U) = \hat{c} \pm w_{1-\frac{\tau}{2}}\sqrt{var(\hat{c})} \;\; (\hat{d}_L,\hat{d}_U) = \hat{d} \pm w_{1-\frac{\tau}{2}}\sqrt{var(\hat{d})} \\ (\hat{\lambda}_L,\hat{\lambda}_U) = \hat{\lambda} \pm w_{1-\frac{\tau}{2}}\sqrt{var(\hat{\lambda})} \end{array}\right\}, \quad (17)$$

where $w_{1-\frac{\tau}{2}}$ is the percentile of the standard normal distribution with left-tail probability $1-\frac{\tau}{2}$ and $var(\hat{a}), var(\hat{b}), var(\hat{c}), var(\hat{d}), var(\hat{\lambda})$, represent the asymptotic variances of maximum likelihood estimates which can be calculated using the inverse of the Fisher information matrix.

4 Bootstrap Confidence Intervals

The bootstrap is widely used to estimate CIs and hypothesis tests. [22] and [23] introduced two parametric boatstrap methods: Percentile Bootstrap Confidence Interval and Bootstrap-T Confidence Interval. In this section, the two parametric Bootstrap methods are used to construct CIs for the unknown parameters a,b,c,d and λ. The Bootstrap samples are obtained using the following steps:

1. Let the original progressively Type-II sample, $y_{j_1:m_j:n_j} < y_{j_2:m_j:n_j} < \ldots < y_{j_{m_j}:m_j:n_j}$ for $j = 1,2$, compute $\hat{a},\hat{b},\hat{c},\hat{d}$ and $\hat{\lambda}$.
2. Depending on the values of n_j and m_j $(1 < m_j < n_j)$ with the same censoring scheme in step 1, S_{ji} $(i = 1,2,\ldots,m_j)$, $j = 1,2$, we use the algorithm in [24] to generate two independent progressive samples of sizes m_1 and m_2 from Kum-IW(a,b,c,d,λ), $\underline{y}^* = (y^*_{j1;m_j,n_j} < y^*_{j2;m_j,n_j} < \ldots < y^*_{jmj:m_j:n_j})$.
3. Also, in step 1, based on $\underline{y}^*$, compute the bootstrap sample estimates of $\hat{a},\hat{b},\hat{c},\hat{d}$ and $\hat{\lambda}$. say $\hat{a}^*,\hat{b}^*,\hat{c}^*,\hat{d}^*$ and $\hat{\lambda}^*$.
4. Repeat steps 2 and 3 L times representing L different bootstrap samples. The value of L has been taken to be 1000.
5. Rearrange all $\hat{a}^*,\hat{b}^*,\hat{c}^*,\hat{d}^*$ and $\hat{\lambda}^*$ in ascending order to obtain the bootstrap sample $(\hat{\vartheta}_k^{*[1]},\hat{\vartheta}_k^{*[2]},\ldots,\hat{\vartheta}_k^{*[L]})$, $k = 1,2,3,4,5$, where $\hat{\vartheta}_1^* = \hat{a}^*$, $\hat{\vartheta}_2^* = \hat{b}^*$, $\hat{\vartheta}_3^* = \hat{c}^*$, $\hat{\vartheta}_4^* = \hat{d}^*$ and $\hat{\vartheta}_5^* = \hat{\lambda}^*$.

4.1 BP-CI

Let $\Psi(g) = P(\hat{\vartheta}_k^* \le g)$ be the CDF of $\hat{\vartheta}_k^*$. Define $\hat{\vartheta}^*_{kBoot} = \Psi^{-1}(g)$ for given g. The approximate bootstrap-p $100(1-\tau)\%$ CI of $\hat{\vartheta}_k^*$ is given by

$$[\hat{\vartheta}^*_{k\,Boot}(\frac{\tau}{2}), \hat{\vartheta}^*_{k\,Boot}(1-\frac{\tau}{2})] \quad (18)$$

4.2 BT-CI

Let $\widehat{\zeta}_w^{*[1]} \le \widehat{\zeta}_w^{*[2]} \le \ldots \le \widehat{\zeta}_w^{*[L]}$be the order statistics where

$$\widehat{\zeta}_k^{*[j]} = \frac{\sqrt{L}*(\hat{\vartheta}_k^{*[j]} - \hat{\vartheta}_w)}{\sqrt{Var(\hat{\vartheta}_w^{*[j]})}}, \; j = 1,2,\ldots,L; \; w = 1,2,3,4,5. \quad (19)$$

Where $\hat{\vartheta}_k = \hat{a}$, $\hat{\vartheta}_k = \hat{b}$, $\hat{\vartheta}_k = \hat{c}$, $\hat{\vartheta}_k = \hat{d}$ and $\hat{\vartheta}_k = \hat{\lambda}$, while $Var(\hat{\vartheta}_k^{*[j]})$ is obtained from inverse of the Fisher information matrix. Let $V(g) = P(\zeta_w^* < g)$, $w = 1,2,3,4,5$ be the CDF of ζ_w^*. For a given g, define

$$\hat{\vartheta}^*_{wBoot-t} = \hat{\vartheta}_w + L^{-\frac{1}{2}}\sqrt{Var(\hat{\vartheta}_w^*)}V^{-1}(g). \quad (20)$$

Thus, the approximate bootstrap-t $100(1-\tau)\%$ CI of $\hat{\vartheta}_w^*$ is given by

$$[\hat{\vartheta}^*_{wBoot-t}(\frac{\tau}{2}), \hat{\vartheta}^*_{wBoot-t}(1-\frac{\tau}{2})]. \quad (21)$$

5 Bayesian Estimation Using MCMC

Bayesian estimation has been important recently. It deals with a wide variety of problems in many scientific and engineering areas. We also use non-Bayesian principles as maximum likelihood when we need some conclusions from observed data. For example, we use it in statistics, signal processing, speech analysis, image processing, computer vision, astronomy, telecommunications, neural networks, pattern recognition, machine learning, artificial intelligence, psychology, sociology, medical decision making, econometrics, and biostatistics. The joint prior of the parameters a,b,c,d and λ can be written as $\pi(a,b,c,d,\lambda) \propto (a,b,c,d,\lambda)^{-1}, a>0,\ b>0,\ c>0,\ d>0,\ \lambda>1$.

The joint posterior density function of a,b,c,d and λ, denoted by $\pi^*(a,b,c,d,\lambda|\underline{y})$, can be written as

$$\pi^*(a,b,c,d,\lambda|\underline{y}) \propto (abcd)^{m_1+m_2-1}\lambda^{m_2-1} \times \exp\left\{-(d+1)\left(\sum_{i=1}^{m_1}\log(y_{1i:m_1:n_1})+\sum_{i=1}^{m_2}\log(y_{2i:m_2:n_2})\right)\right\} \times \exp\left\{-(ac)\left(\sum_{i=1}^{m_1}y_{1i:m_1:n_1}^{-d}+\sum_{i=1}^{m_2}y_{2i:m_2:n_2}^{-d}\right)\right\} \times \exp\left\{\sum_{i=1}^{m_1}(b(S_{1i}+1)-1)\log(1-\exp\{-acy_{1i:m_1:n_1}^{-d}\})\right\} \times \exp\left\{\sum_{i=1}^{m_2}(b\lambda(S_{2i}+1)-1)\log(1-\exp\{-acy_{2i:m_2:n_2}^{-d}\})\right\}. \tag{22}$$

The conditional posterior densities function of a,b,c,d and λ are, as follows:

$$\pi_1^*(a|b,c,d,\lambda,\underline{y}) \propto a^{m_1+m_2-1}\prod_{j=1}^{2}\prod_{i=1}^{m_j}\exp\{-acy_{ji:m_j:n_j}^{-d}\} \times(1-\exp\{-acy_{ji:m_j:n_j}^{-d}\})^{\lambda^{j-1}b(S_{ji}+1)-1}, \tag{23}$$

$$\pi_2^*(b|a,c,d,\lambda,\underline{y}) \propto b^{m_1+m_2-1}\prod_{j=1}^{2}\prod_{i=1}^{m_j}(1-\exp\{-acy_{ji:m_j:n_j}^{-d}\})^{\lambda^{j-1}b(S_{j_i}+1)}, \tag{24}$$

$$\pi_3^*(c|a,b,d,\lambda,\underline{y}) \propto c^{m_1+m_2-1}\prod_{j=1}^{2}\prod_{i=1}^{m_j}\exp\{-acy_{ji:m_j:n_j}^{-d}\} \times(1-\exp\{-acy_{ji:m_j:n_j}^{-d}\})^{\lambda^{j-1}b(S_{j_i}+1)-1}, \tag{25}$$

$$\pi_4^*(d|a,b,c,\lambda,\underline{y}) \propto d^{m_1+m_2-1}\prod_{j=1}^{2}\prod_{i=1}^{m_j}y_{ji:m_j:n_j}^{-d-1}\exp\{-acy_{ji:m_j:n_j}^{-d}\} \times(1-\exp\{-acy_{ji:m_j:n_j}^{-d}\})^{\lambda^{j-1}b(S_{j_i}+1)-1}, \tag{26}$$

and

$$\pi_5^*(\lambda|a,b,c,d,\underline{y}) \propto \lambda^{m_2-1}\exp\left\{-\lambda\left(\sum_{i=1}^{m_2}\left[-b\left(S_{ji}+1\right)\right]\log(1-\exp\{-acy_{ji:m_j:n_j}^{-d}\})\right)\right\} \tag{27}$$

1. Start with $a^{(o)},b^{(o)},c^{(o)},d^{(o)}$ and $\lambda^{(o)}$.
2. Put $j=1$.
3. Generate $\lambda^{(j)}$ from
$$Gamma\left(m_2,\sum_{i=1}^{m_2}-b\left(S_{ji}+1\right)\log(1-\exp\{-acy_{ji:m_j:n_j}^{-d}\})\right).$$
4. Use the following M-H algorithm as well as generate $a^{(j)},b^{(j)},c^{(j)} and\ d^{(j)}$ from (23), (24), (25) and (26) with the normal suggested distribution
$N\left(a^{(j-1)},var(a)\right)$, $N\left(b^{(j-1)},var(b)\right)$, $N\left(c^{(j-1)},var(c)\right)$ and $N\left(d^{(j-1)},var(d)\right)$, respectively, where $var(a)$, $var(b)$, $var(c)$ and $var(d)$ can be obtained from the main diagonal in the inverse fisher information matrix.

(i) Generate a proposal a^* from $N\left(a^{(j-1)},var(a)\right)$, b^* from $N\left(b^{(j-1)},var(b)\right)$, c^* from $N\left(c^{(j-1)},var(c)\right)$ and d^* from $N\left(d^{(j-1)},var(d)\right)$.

(ii) Evaluate the acceptance probabilities

$$\left.\begin{array}{l}\rho_a=\min\left[1,\frac{\pi_1^*(a^*|b^{(j)},c^{(j-1)},d^{(j-1)},\lambda^{(j-1)},\underline{y})}{\pi_1^*(a^{(j-1)}|b^{(j)},c^{(j-1)},d^{(j-1)},\lambda^{(j-1)},\underline{y})}\right],\\ \rho_b=\min\left[1,\frac{\pi_2^*(b^*|a^{(j)},c^{(j)},d^{(j)},\lambda^{(j-1)},\underline{y})}{\pi_2^*(b^{(j-1)}|a^{(j)},c^{(j)},d^{(j)},\lambda^{(j-1)},\underline{y})}\right],\\ \rho_c=\min\left[1,\frac{\pi_3^*(c^*|a^{(j)},b^{(j)},d^{(j)},\lambda^{(j-1)},\underline{y})}{\pi_3^*(c^{(j-1)}|a^{(j)},b^{(j)},d^{(j)},\lambda^{(j-1)},\underline{y})}\right],\\ \rho_d=\min\left[1,\frac{\pi_4^*(d^*|a^{(j)},b^{(j)},d^{(j)},\lambda^{(j)},\underline{y})}{\pi_4^*(d^{(j-1)}|a^{(j)},b^{(j)},d^{(j)},\lambda^{(j)},\underline{y})}\right].\end{array}\right\}.$$

5. Compute $a^{(j)},b^{(j)},c^{(j)},d^{(j)}$ and $\lambda^{(j)}$.
6. Put $j=j+1$.
7. Repeat Steps 3-6 L times.
8. In order to guarantee the convergence and to remove the influence of the selection of initial values, the first B simulated varieties are ignored. Then the selected samples are $a^{(j)},b^{(j)},c^{(j)}$ and $d^{(j)}$, $j=B+1,....,L$, for sufficiently large L, which forms a set of approximate posterior samples that can be used to obtain the Bayes MCMC point estimates of a,b,c,d and λ as

$$\left.\begin{array}{ll}\hat{a}_{MCMC}=\frac{1}{L-B}\sum_{j=B+1}^{L}a^{(j)}, & \hat{b}_{MCMC}=\frac{1}{L-B}\sum_{j=B+1}^{L}b^{(j)}\\ \hat{c}_{MCMC}=\frac{1}{L-B}\sum_{j=B+1}^{L}c^{(j)}, & \hat{d}_{MCMC}=\frac{1}{L-B}\sum_{j=B+1}^{L}d^{(j)}\\ \multicolumn{2}{c}{\hat{\lambda}_{MCMC}=\frac{1}{L-B}\sum_{j=B+1}^{L}\lambda^{(j)},}\end{array}\right\}.$$

9.To calculate the credible interval (CRIs) of Ω_k where $\Omega_1 = a, \Omega_2 = b, \Omega_3 = c, \Omega_4 = d$ and $\Omega_5 = \lambda$, we take the quantiles of the sample as the endpoints of the interval. Sort $\left\{\Omega_k^{B+1}, \Omega_k^{B+2}, ..., \Omega_k^{L}\right\}$ as $\left\{\Omega_k^{[1]}, \Omega_k^{[2]}, ..., \Omega_k^{[L-B]}\right\}$. The $100(1-\tau)\%$ symmetric credible interval of Ω_k is

$$\left[\Omega_k(\frac{\tau}{2}(L-B), \Omega_k((1-\frac{\tau}{2})(L-B))\right].$$

6 Numerical Example

Using the algorithm in [24], we generate two samples from Kum-IW(a,b,c,d,λ) with the parameters $(a,b,c,d,\lambda) = (2,2,2.5,3,2)$ using progressive censoring schemes $n_1 = n_2 = n = 50$, $m_1 = 20$, $m_2 = 30$, $S_1 = (5, 0, 0, 5, 0, 0, 3, 0, 0, 5, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0)$ and $S_2 = (3, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 3, 0, 1, 0, 1, 0, 1, 0, 0, 2, 0, 1, 0, 2, 0)$. The two progressively censored data have been presented as follows:

Data 1: 1.1266, 1.1519, 1.2033, 1.2485, 1.2499, 1.3088, 1.3109, 1.3317, 1.3863, 1.4388, 1.4555, 1.5001, 1.5042, 1.5183, 1.5518, 1.6014, 1.6507, 1.9107, 2.1392, 2.2489.

Data 2: 0.9617, 1.0821, 1.1063, 1.1178, 1.1258, 1.1490, 1.1505, 1.1674, 1.1714, 1.2025, 1.2109, 1.2214, 1.2233, 1.3169, 1.3198, 1.3315, 1.3539, 1.3541, 1.3574, 1.3698, 1.3831, 1.3896, 1.4107, 1.4214, 1.5018, 1.5100, 1.5213, 1.6335, 1.6464, 1.6949.

Figure 2 presents the PDFs under normal and accelerated conditions. We compute the MLE of the parameters by solving equations (11-15) using the quasi-Newton-Raphson algorithm. In MCMC apporach, we run the chain for 30000 times and discard the first 5000 values as "burn-in" and we use informative gamma prior distributions. Table 1 presents the MLEs **(-)**$_{ML}$, bootstrap (-)$_{BP}$, bootstrap (-)$_{BT}$ and Bayes MCMC (-)$_{MCMC}$ point estimates of the parametes. Tables (2-3) present 90%, 95%the approximate confidence interval (ACIs), bootstrap confidence intervals (B-CIs) and MCMC confidence intervals.

Table 1: Different point estimates for a, b, c, d and λ.

Table 2: 90% confidence intervals for a, b, c, d and λ.

Table 3: 95% confidence intervals for a, b, c, d and λ.

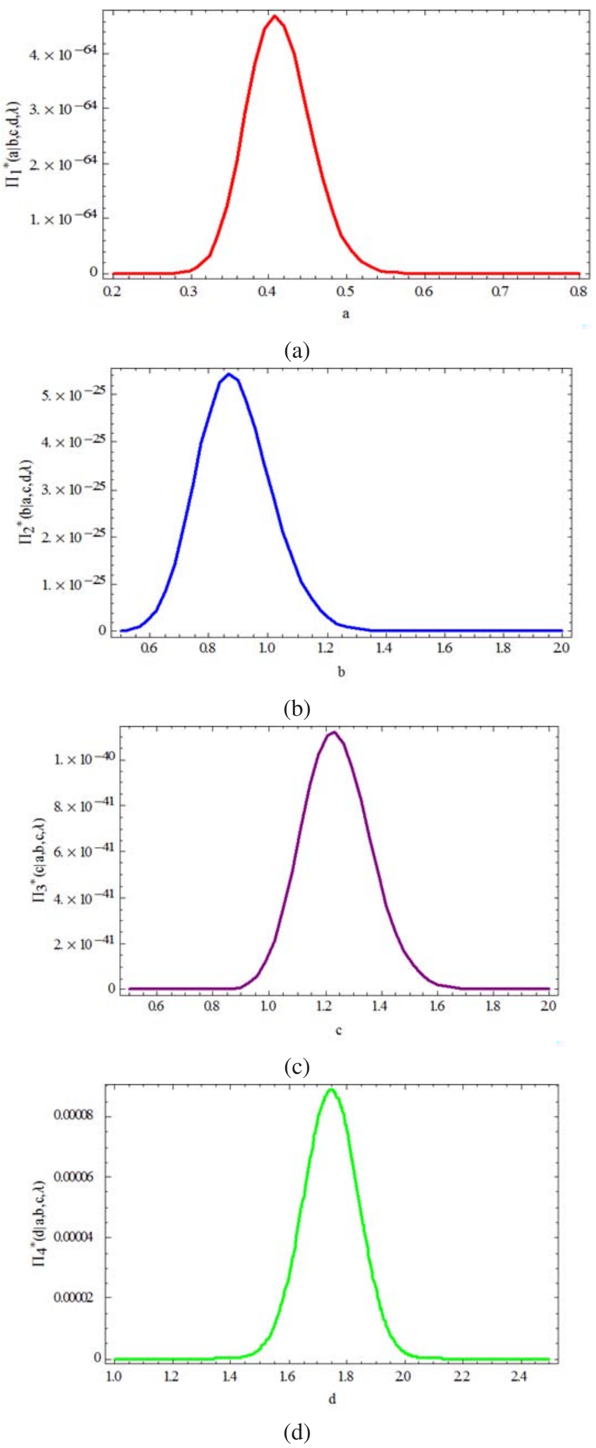

Fig. 1: The conditional posterior densities function of a, b, c and d

Table 1: Different point estimates for a, b, c, d and λ.

Parameters	$(.)_{ML}$	$(.)_{BP}$	$(.)_{BT}$	$(.)_{MCMC}$
a	2.2362	2.3120	2.2242	2.2147
b	1.9866	2.0147	2.0997	1.9989
c	2.7952	2.5759	2.7155	2.4331
d	3.2438	3.3119	3.1994	2.9972
λ	2.2673	2.3564	2.1546	2.1010

Table 2: 90% confidence intervals for a, b, c, d and λ.

Method	a	Length	b	Length	c	Length
ACI	(1.6630,3.0069)	1.3438	(0.2951,3.2819)	2.9868	(2.1807,3.2819)	1.1012
BPCI	(1.3554,3.3832)	2.0278	(1.5516,3.8645)	2.3130	(1.6076,3.7822)	2.1746
BTCI	(1.5247,2.9521)	1.4274	(1.5807,3.8351)	2.2544	(1.7076,3.4839)	1.7763
CRI	(1.4333,3.2394)	1.8061	(0.6753,3.1988)	2.5235	(1.5577,2.9231)	1.3654
Method	d	Length	λ	Length		
ACI	(1.5891,6.6214)	5.0322	(1.7363,4.7351)	2.9988		
BPCI	(2.4832,7.3444)	4.8612	(1.5313,3.9261)	2.3948		
BTCI	(1.8686,6.4285)	4.5599	(1.3378,3.8644)	2.5266		
CRI	(1.4742,5.2388)	3.7646	(1.6135,4.2388)	2.6252		

Table 3: 95% confidence intervals for a, b, c, d and λ.

Method	a	Length	b	Length	c	Length
ACI	(1.6862,3.9779)	2.2916	(0.8048,4.2681)	3.4633	(1.3087,3.3942)	2.0855
BPCI	(1.3462,4.1837)	2.8375	(1.5311,3.9450)	2.4139	(1.0624,2.8370)	1.7746
BTCI	(1.4877,3.8786)	2.3909	(1.5285,3.8874)	2.3589	(1.4878,2.9766)	1.4888
CRI	(1.3292,4.3357)	3.0065	(0.4253,3.9541)	3.5288	(0.9883,3.2999)	2.3116
Method	d	Length	λ	Length		
ACI	(1.3862,7.5908)	6.2046	(1.5772,5.2125)	3.6352		
BPCI	(1.5595,6.5359)	4.9764	(1.4545,5.9777)	4.5232		
BTCI	(1.7614,7.5237)	5.7623	(1.3181,4.9878)	3.6697		
CRI	(1.3059,5.6998)	4.3939	(1.4892,4.7999)	3.3107		

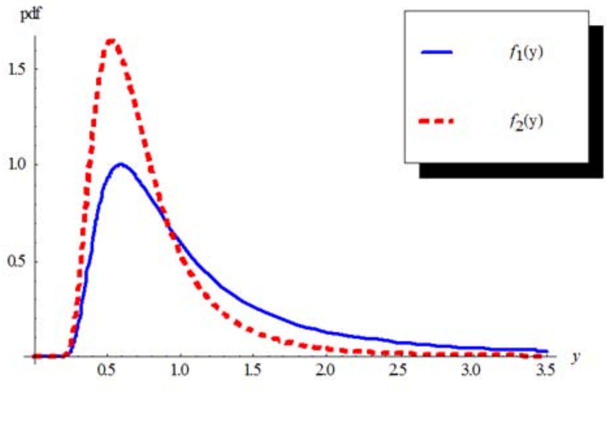

Fig. 2: PDFs under normal and accelerated conditions

7 Conclusion

Based on progressively Type-II censored samples, this paper is related to Bayesian procedures for the analysis of the constant-partially accelerated life testing using the Kum-IW model. Based on the maximum likelihood estimates; Bayes and parametric bootstrap methods, the point estimations and confidence intervals for the distribution parameters and the acceleration factor are obtained. The classical Bayes estimates cannot be obtained in explicit form. One can see the scope of MCMC based Bayesian solutions which make every inferential development routinely available. A numerical example was conducted to examine and compare the performance of the proposed methods, different CSs, different acceleration factors, and different parameter values.

Conflict of Interest

The authors declare that they have no conflict of interest.

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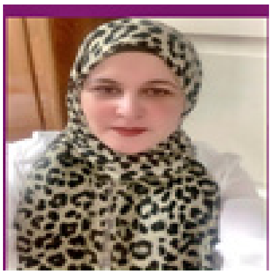

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