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On a Generalized Laplace Transform

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Abstract: In this work we define a generalized Laplace transform and establish some of its fundamental properties, in addition, we show that it contains as particular cases, several known from the literature, including the classical Laplace transform. In addition, its application to the resolution of generalized differential equations is shown.

Keywords: Generalized derivatives and integral, Generalized Laplace transform, Fractional calculus.

1 Introduction

One of the mathematical areas that is in constant development is that of Differential Equations (with different operators and in different functional spaces), and their solution methods, in particular, due to the multiplicity of applications and its own theoretical development, over time, researchers and productions related to this area have been increasing, you can consult in [1,2,3,4] different aspects of this increase and its overlaps with the development of Mathematics itself.

In particular, one of the main difficulties is finding methods to find analytical solutions to some classes of differential equations, within these methods are those that use different integral transformations (Laplace, Mellin and Fourier, for example) some attempts in this direction, to fractional and generalized differential equations can be found in [5,6,7,8,9,10,11,12,13,14].

In [15] (see also [16] and [17]) a generalized fractional derivative was defined in the following way.

Definition 1.*Given a function $f : [0,+\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by*

$$N_F^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon F(t,\alpha)) - f(t)}{\varepsilon} \tag{1}$$

for all $t > 0$, $\alpha \in (0,1)$ being $F(\alpha,t)$ is some function. Here we will use some cases of F defined in function of $E_{a,b}(.)$ the classic definition of Mittag-Leffler function with $Re(a), Re(b) > 0$. Also we consider $E_{a,b}(t^{-\alpha})_k$ is the k-nth term of $E_{a,b}(.)$.

If f is $\alpha-$differentiable in some $(0,\alpha)$, and $\lim_{t\to 0^+} N_F^{\alpha} f(t)$ exists, then define $N_F^{\alpha} f(0) = \lim_{t\to 0^+} N_F^{\alpha} f(t)$, note that if f is differentiable, then $N_F^{\alpha} f(t) = F(t,\alpha) f'(t)$ where $f'(t)$ is the ordinary derivative.

The function $E_a(z)$ was defined and studied by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential function. This generalization was studied by Wiman in 1905, Agarwal in 1953 and Humbert and Agarwal in 1953, and others, additional details and more information can be found in [18,19,20,21,22,23,24,25,26,27].

This generalized differential operator contains many of the known local operators (for example, the conformable derivative of [28] and the non-conformable of [29])) and has shown its usefulness in various applications, as it can be consulted, for example, in [30, 31,32,33,34,35,36,37]. One of the most required properties of a derivative operator is the Chain Rule, to calculate the derivative of compound functions, which does not exist in the case of classical fractional derivatives $N_{\Phi}^{\alpha}(f \circ g)(t) = N_{\Phi}^{\alpha} f(g(t)) = f'(g(t)) N_{\Phi}^{\alpha} g(t)$.

Now, we give the definition of a general fractional integral (cf. [38]). Throughout the work we will consider that the integral operator kernel T defined below is an absolutely continuous function.

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Definition 2. *Let I be an interval $I \subseteq \mathbb{R}$, $a,t \in I$ and $\alpha \in \mathbb{R}$. The integral operator $J^{\alpha}_{T,a+}$, right and left, is defined for every locally integrable function f on I as*

$$J^{\alpha}_{T,a+}(f)(t) = \int_a^t \frac{f(s)}{T(t-s,\alpha)} ds, t > a. \tag{2}$$

$$J^{\alpha}_{T,b-}(f)(t) = \int_t^b \frac{f(s)}{T(s-t,\alpha)} ds, b > t. \tag{3}$$

Remark. As pointed out in [38], many fractional integral operators can be obtained as particular cases of the previous one, under certain choices of the F kernel. For example, if $F(t-s,\alpha) = \Gamma(\alpha)(t-s)^{1-\alpha}$ the right Riemann-Liouville integral is obtained (similarly to the left), further details on Fractional Calculus and fractional integral operators linked to the generalized integral of the previous definition, can be found in [39,40,41,42,43,44,45,46].

Remark. In certain applications, it is necessary to work with the "central" operator defined by $J^{\alpha}_{T,a}(f)(t) = \int_a^t \frac{f(s)}{T(s,\alpha)} ds, t > a$.

Remark. We can define the function space $L^p_{\alpha}[a,b]$ as the set of functions over $[a,b]$ such that $(J^{\alpha}_{T,a+}[f(t)]^p(b)) < +\infty$.

The following property is one of the fundamental ones and links the integral operator with the generalized derivative, defined above.

Proposition 1. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, $0 < \alpha \leq 1$ and f a α-differentiable function on I such that f' is a locally integrable function on I. Then, we have for all $t \in I$*

$$J^{\alpha}_{F,a}\left(N^{\alpha}_{F}(f)\right)(t) = f(t) - f(a).$$

Proposition 2. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$ and $\alpha \in (0,1]$.*

$$N^{\alpha}_{F}\left(J^{\alpha}_{F,a}(f)\right)(t) = f(t),$$

for every continuous function f on I and $a,t \in I$.

Remark. In [28] it is defined the integral operator $J^{\alpha}_{F,a}$ for the choice of the function F given by $F(t,\alpha) = t^{1-\alpha}$, and [28, Theorem 3.1] shows

$$N^{\alpha} J^{\alpha}_{t^{1-\alpha},a}(f)(t) = f(t),$$

for every continuous function f on I, $a,t \in I$ and $\alpha \in (0,1]$. Hence, Proposition 2 extends to any F this important equality.

The following result summarizes some elementary properties of the integral operator $J^{\alpha}_{T,a+}$.

Theorem 1. *Let I be an interval $I \subseteq \mathbb{R}$, $a,b \in I$ and $\alpha \in \mathbb{R}$. Suppose that f,g are locally integrable functions on I, and $k_1,k_2 \in \mathbb{R}$. Then we have*

(1) $J^{\alpha}_{T,a+}\left(k_1 f + k_2 g\right)(t) = k_1 J^{\alpha}_{T,a+} f(t) + k_2 J^{\alpha}_{T,a+} g(t)$,

(2) if $f \geq g$, then $J^{\alpha}_{T,a+} f(t) \geq J^{\alpha}_{T,a+} g(t)$ for every $t \in I$ with $t \geq a$,

(3) $\left|J^{\alpha}_{T,a+} f(t)\right| \leq J^{\alpha}_{T,a+} |f|(t)$ for every $t \in I$ with $t \geq a$,

(4) $\int_a^b \frac{f(s)}{T(s,\alpha)} ds = J^{\alpha}_{T,a+} f(t) - J^{\alpha}_{T,b-} f(t) = J^{\alpha}_{T,a+} f(t)(b)$ for every $t \in I$.

Let $C^1[a,b]$ be the set of functions f with first ordinary derivative continuous on $[a,b]$, we consider the following norms on $C^1[a,b]$:

$$\|F\|_C = \max_{[a,b]} |f(t)|, \quad \|F\|_{C^1} = \left\{ \max_{[a,b]} |f(t)| + \max_{[a,b]} |f'(t)| \right\}$$

The Propositions 1 and 2 were obtained under the case that the kernel of both operators coincide (as is the case with local operators), we will give some results in the event that this does not happen.

Theorem 2. *For a function $f \in C^1[a,b]$ and $x \in [a,b]$, we have*

$$\left|N^{\alpha}_{F,a+} f(t)\right| \leq K(\alpha)\|F\|_C \max_{t\in[a,x]} |f(t)|. \tag{4}$$

$$\left|N^{\alpha}_{F,b-} f(t)\right| \leq K(\alpha)\|F\|_C \max_{t\in[x,b]} |f(t)|. \tag{5}$$

Remark. The constant $K(\alpha)$ of the theorem can depend on other parameters, as in the case of the Katugampola operator, where some parameter ρ will appear.

Theorem 3. *The fractional derivatives $N^{\alpha}_{F,a+} f(t)$ and $N^{\alpha}_{F,b-} f(t)$ are bounded operators from $C^1[a,b]$ to $C[a,b]$ with*

$$\left|N^{\alpha}_{F,a+} f(t)\right| \leq K\|F\|_C \|f\|_{C^1}, \tag{6}$$

$$\left|N^{\alpha}_{F,b-} f(t)\right| \leq K\|F\|_C \|f\|_{C^1}, \tag{7}$$

where the constant K, may be depend of derivative frame considered.

Remark. From previous results we obtain that the derivatives $N^{\alpha}_{F,a+} f(t)$ and $N^{\alpha}_{F,b-} f(t)$ are well defined.

Theorem 4. *(Integration by parts) Let $f,g : [a,b] \to \mathbb{R}$ differentiable functions and $\alpha \in (0,1]$. Then, the following property hold*

$$J^{\alpha}_{F,a+}((f)(N^{\alpha}_{F,a+} g(t))) = [f(t)g(t)]_a^b - J^{\alpha}_{F,a+}((g)(N^{\alpha}_{F,a+} f(t))). \tag{8}$$

Theorem 5. *If $f:[a,b]\to\mathbb{R}$ is a continuous function and $\alpha\in(0,1]$ then, the following inequality is fulfilled*

$$\left|J^{\alpha}_{F,a+}(f)(t)\right|\le J^{\alpha}_{F,a+}|f|(t). \tag{9}$$

Taking into account the ideas of [47] we can define the generalized partial derivatives as follows.

Definition 3. *Given a real valued function $f:\mathbb{R}^n\to\mathbb{R}$ and $\overrightarrow{a}=(a_1,\cdots,a_n)\in\mathbb{R}^n$ a point whose ith component is positive. Then the generalized partial N-derivative of f of order α in the point $\overrightarrow{a}=(a_1,\cdots,a_n)$ is defined by*

$$N^{\alpha}_{F_i,t_i}f(\overrightarrow{a}) = \lim_{\varepsilon\to 0}\frac{f(a_1,\cdots,a_i+\varepsilon F_i(a_i,\alpha),\cdots,a_n)-f(a_1,\cdots,a_i,\cdots,a_n)}{\varepsilon} \tag{10}$$

if it exists, is denoted $N^{\alpha}_{F_i,t_i}f(\overrightarrow{a})$, and called the ith generalized partial derivative of f of the order $\alpha\in(0,1]$ at $\overrightarrow{a}$.

Remark. If a real valued function f with n variables has all generalized partial derivatives of the order $\alpha\in(0,1]$ at $\overrightarrow{a}$, each $a_i>0$, then the generalized α-gradient of f of the order $\alpha\in(0,1]$ at $\overrightarrow{a}$ is

$$\nabla^{\alpha}_{N}f(\overrightarrow{a})=(N^{\alpha}_{t_1}f(\overrightarrow{a}),\cdots,N^{\alpha}_{t_n}f(\overrightarrow{a})) \tag{11}$$

Taking into account the above definitions, it is not difficult to prove the following result, on the equality of mixed partial derivatives.

Theorem 6. *Under assumptions of Definiton 3, assume that $f(t_1,t_2)$ it is a function for which, mixed generalized partial derivatives exist and are continuous, $N^{\alpha+\beta}_{F_{1,2},t_1,t_2}(f(t_1,t_2))$ and $N^{\beta+\alpha}_{F_{2,1},t_2,t_1}(f(t_1,t_2))$ over some domain of $\mathbb{R}^2$ then*

$$N^{\alpha+\beta}_{F_{1,2},t_1,t_2}(f(t_1,t_2))=N^{\beta+\alpha}_{F_{2,1},t_2,t_1}(f(t_1,t_2)) \tag{12}$$

In this paper, based on the operators of the definitions 1 and 2 define us and study a Generalized Laplace Transform, which contains as particular cases several of those reported in the literature and apply it to the resolution of a generalized differential equation, subject to certain initial conditions.

2 Main Results

The following generalized exponential order will play an important role in our work.

Definition 4. *Let $\alpha\in(0,1]$ and c a real number. We define the generalized exponential order in the following way*

$$E^{N}_{\alpha}(c,t)=exp\left(c\mathscr{F}(t,\alpha)\right).$$

with $\mathscr{F}(t,\alpha)=\int_0^t\frac{ds}{F(s,\alpha)}=J^{\alpha}_{F,0}(1)(t)$.

From Definitions 1, 4 and the Chain Rule, we have $N^{\alpha}_{F}\left\{E^{N}_{\alpha}(c,t)\right\}=cE^{N}_{\alpha}(c,t)$.

Definition 5. *Let $\alpha\in(0,1]$, let g a function and s a real number. We define the Generalized Laplace Transform in the following way*

$$F(s)=\left(\mathscr{L}^{\alpha}_{N}\{g(t)\}\right)(s)=J^{\alpha}_{F,0}\big(E^{N}_{\alpha}(-s,t)g(t)\big)(\infty).$$

and its inverse transform

$$g(t)=\left(\mathscr{L}^{\alpha}_{N}\{G(s)\}^{-1}\right)(t)=J^{\alpha}_{F,0}\left(F(t,\alpha)E^{N}_{\alpha}(s,t)G(s)\right)(\infty)$$

Remark. If $F(t,\alpha)=1$ then we have the usual Laplace Transform, and if $F(t,\alpha)=t^{1-\alpha}$ then we have the Conformable Laplace Transform defined in [39] (also see [48,49,50]). If we put $F(t,\alpha)=\frac{1}{g'(t)}$ then we obtain the generalized Laplace transform of [8] (more details in [51, 52]).

Theorem 7. *The Generalized Laplace Transform has the following properties:*

$$\mathscr{L}^{\alpha}_{N}\{\alpha g(t)+\beta h(t)\}=\alpha\mathscr{L}^{\alpha}_{N}\{g(t)\}+\beta\mathscr{L}^{\alpha}_{N}\{h(t)\} \tag{13}$$

$$\mathscr{L}^{\alpha}_{N}\{N^{\alpha}_{F}g(t)\}=-g(0)-s\mathscr{L}^{\alpha}_{N}\{g(t)\} \tag{14}$$

$$\mathscr{L}^{\alpha}_{N}\{J^{\alpha}_{F,0}(g(s))(t)\}=\frac{1}{s}\mathscr{L}^{\alpha}_{N}\{g(t)\} \tag{15}$$

$$\mathscr{L}^{\alpha}_{N}\{(N^{\alpha}_{F})^{n}g(t)\}=-\sum_{k=1}^{n}(-1)^{k}s^{n-k}\left((N^{\alpha}_{F})^{k-1}\right)g(0) - s^{n}\mathscr{L}^{\alpha}_{N}\{g(t)\} \tag{16}$$

where $(N^{\alpha}_{F})^{n}=\underbrace{N^{\alpha}_{F}\circ N^{\alpha}_{F}\circ\quad .\quad .\quad .\quad\circ N^{\alpha}_{F}}_{n\quad times}$

Proof. Is easy to see that the first equality holds because the same property of the integral.

$$\begin{aligned}\mathscr{L}^{\alpha}_{N}\{\alpha g(t)+\beta h(t)\}&=\int_0^\infty\frac{E^{N}_{\alpha}(-s,t)\left(\alpha g(t)+\beta h(t)\right)}{F(t,\alpha)}dt\\&=\alpha\int_0^\infty\frac{E^{N}_{\alpha}(-s,t)g(t)}{F(t,\alpha)}dt\\&\quad+\beta\int_0^\infty\frac{E^{N}_{\alpha}(-s,t)h(t)}{F(t,\alpha)}dt\end{aligned}$$

then

$$\mathscr{L}^{\alpha}_{N}\{\alpha g(t)+\beta h(t)\}=\alpha\mathscr{L}^{\alpha}_{N}\{g(t)\}+\beta\mathscr{L}^{\alpha}_{N}\{h(t)\}$$

For the second property:

$$\mathscr{L}_N^{\alpha}\{N_F^{\alpha} g(t)\} = \int_0^{\infty} \frac{E_{\alpha}^N(-s,t) N_F^{\alpha} g(t)}{F(t,\alpha)} dt$$
$$= \int_0^{\infty} E_{\alpha}^N(-s,t) g'(t) dt$$

Integrating by parts we obtain the desired equality

For the third property we apply the second property to the function $h(t) = J_{F,0}^{\alpha}(g(s))(t)$ then $h(0) = 0$ and its N-derivative is $N_F^{\alpha} h(t) = g(t)$

Finally to obtain the fourth property we iterate the second property n times.

Theorem 8. *Since the function $\mathscr{F}(t,\alpha)$ has the property $\mathscr{F}'(t,\alpha) > 0$ then the following relation between the Generalized Laplace Transform and the classical one holds:*

$$\left(\mathscr{L}_N^{\alpha}\{g(t)\}\right)(s) = \left(\mathscr{L}\left\{g\left(\mathscr{F}(t,\alpha)^{-1}\right)\right\}\right)(s) \quad (17)$$

Proof. The proof is straightforward with the change of variables $u = \mathscr{F}(t,\alpha)$

$$\left(\mathscr{L}_N^{\alpha}\{g(t)\}\right)(s) = \int_0^{\infty} \frac{E_{\alpha}^N(-s,t) g(t)}{F(t,\alpha)} dt$$
$$= \int_0^{\infty} e^{-su} g\left(\mathscr{F}(t,\alpha)^{-1}\right) du$$
$$= \left(\mathscr{L}\left\{g\left(\mathscr{F}(t,\alpha)^{-1}\right)\right\}\right)(s) \quad (18)$$

Definition 6. *A function $f : [0,\infty) \to \mathscr{R}$ is said to be of $g(t)-$exponential order if and only if there exists non-negative constants M, c, T such that $|f(t)| \le Me^{cg(t)}$ for $t \ge T$.*

Theorem 9. *If $f : [0,\infty) \to \mathscr{R}$ is a piecewise function of $\mathscr{F}(t,\alpha)-$exponential order, then the Generalized Laplace Transform exists for $s > c$.*

Proof.

$$\left|\left(J_{F,0}^{\alpha} E_{\alpha}^N(-s,t) g(t)\right)(\tau)\right| \le \left(J_{F,0}^{\alpha} E_{\alpha}^N(-s,t)|g(t)|\right)(\tau)$$
$$\le M\left(J_{F,0}^{\alpha} e^{-s\mathscr{F}(t,\alpha)} e^{c\mathscr{F}(t,\alpha)}\right)(\tau) \quad (19)$$

Again, we apply the change of variables $u = \mathscr{F}(t,\alpha)$ and obtain

$$= M\int_0^{\tau} e^{(c-s)u} du = \frac{M}{s-c}\left(1 - e^{(c-s)\tau}\right) \xrightarrow{\tau\to+\infty} \frac{M}{s-c} \quad (20)$$

Theorem 10. *If $\alpha \in (0,1]$ then we have*

a) $\mathscr{L}_N^{\alpha}\{1\} = \frac{1}{s}$

b) $\mathscr{L}_N^{\alpha}\left\{E_{\alpha}^N(c,t)\right\} = \frac{1}{s-c}$

c) $\mathscr{L}_N^{\alpha}\left\{g(t) E_{\alpha}^N(c,t)\right\} = g(s-c)$

d) $\mathscr{L}_N^{\alpha}\{sin(c\mathscr{F}(t,\alpha))\} = \frac{c}{s^2+c^2}$

e) $\mathscr{L}_N^{\alpha}\{cos(c\mathscr{F}(t,\alpha))\} = \frac{s}{s^2+c^2}$

f) $\mathscr{L}_N^{\alpha}\{sinh(c\mathscr{F}(t,\alpha))\} = \frac{c}{s^2-c^2}$

g) $\mathscr{L}_N^{\alpha}\{cosh(c\mathscr{F}(t,\alpha))\} = \frac{s}{s^2-c^2}$

Proof. a) $\mathscr{L}_N^{\alpha}\{1\} = \int_0^{\infty} \frac{E_{\alpha}^N(-s,t)}{F(t,\alpha)} dt = \int_0^{\infty} e^{-su} du = \lim_{\tau\to\infty} -\frac{-e^{-s\tau}}{s} = \frac{1}{s}$

b) $\mathscr{L}_N^{\alpha}\left\{E_{\alpha}^N(c,t)\right\} = \int_0^{\infty} \frac{E_{\alpha}^N(-s,t) E_{\alpha}^N(c,t)}{F(t,\alpha)} dt = \int_0^{\infty} e^{-(s-c)u} du = \lim_{\tau\to\infty} -\frac{-e^{-(s-c)\tau}}{s} = \frac{1}{s-c}$

c) $\mathscr{L}_N^{\alpha}\left\{g(t) E_{\alpha}^N(c,t)\right\} = \int_0^{\infty} \frac{E_{\alpha}^N(-s,t) g(t) E_{\alpha}^N(c,t)}{F(t,\alpha)} dt = \int_0^{\infty} \frac{E_{\alpha}^N(-s+c,t) g(t)}{F(t,\alpha)} du = g(s-c)$

d) $\mathscr{L}_N^{\alpha}\{cos(c\mathscr{F}(t,\alpha))\} = \int_0^{\infty} \frac{E_{\alpha}^N(-s,t) cos(c\mathscr{F}(t,\alpha))}{F(t,\alpha)} dt = \int_0^{\infty} \frac{e^{-sF(t,\alpha)} cos(c\mathscr{F}(t,\alpha))}{F(t,\alpha)} dt = \int_0^{\infty} e^{-su} cos(cu) du$

to obtain this result we solve the last integral by parts twice.

e) same as the previous item

f) recall that $sinh(\alpha) = \frac{e^{\alpha}-e^{-\alpha}}{2}$ then

$\mathscr{L}_N^{\alpha}\{sinh(c\mathscr{F}(t,\alpha))\} = \frac{1}{2}\int_0^{\infty} \frac{E_{\alpha}^N(-s,t)\left(e^{c\mathscr{F}(t,\alpha)} - e^{-c\mathscr{F}(t,\alpha)}\right)}{F(t,\alpha)} dt =$
$= \frac{1}{2}\left(\int_0^{\infty} \frac{e^{-(s-c)\mathscr{F}(t,\alpha)}}{F(t,\alpha)} du - \int_0^{\infty} \frac{e^{-(s+c)\mathscr{F}(t,\alpha)}}{F(t,\alpha)} du\right) = \frac{1}{2}\left(\frac{1}{s-c} - \frac{1}{s+c}\right) = \frac{c}{s^2-c^2}$

g) For $cosh(\alpha) = \frac{e^{\alpha}+e^{-\alpha}}{2}$ we proceed in the same way

$\mathscr{L}_N^{\alpha}\{cosh(c\mathscr{F}(t,\alpha))\} = \frac{1}{2}\int_0^{\infty} \frac{E_{\alpha}^N(-s,t)\left(e^{c\mathscr{F}(t,\alpha)} + e^{-c\mathscr{F}(t,\alpha)}\right)}{F(t,\alpha)} dt =$
$= \frac{1}{2}\left(\int_0^{\infty} \frac{e^{-(s-c)\mathscr{F}(t,\alpha)}}{F(t,\alpha)} du + \int_0^{\infty} \frac{e^{-(s+c)\mathscr{F}(t,\alpha)}}{F(t,\alpha)} du\right) = \frac{1}{2}\left(\frac{1}{s-c} + \frac{1}{s+c}\right) = \frac{s}{s^2-c^2}$

To complete the theoretical body the following result is necessary.

Definition 7. *Let f and g be two functions which are piecewise continuous at each interval $[0,T]$ and of generalized exponential order. We define the N-convolution of f and g by*

$$(f*g)_N(t) = \int_0^t f(\tau)g\left[\mathscr{F}^{-1}(\mathscr{F}(t,\alpha)-\mathscr{F}(\tau,\alpha))\right]\frac{d\tau}{F(\tau,\alpha)}, t \leq T. \tag{21}$$

The commutativity of the N-convolution is given in the following result.

Lemma 1. *Let f and g be two functions which are piecewise continuous at each interval $[0,T]$ and of generalized exponential order. Then*

$$(f*g)_N(t) = (g*f)_N(t). \tag{22}$$

Proof. From Definition 7, we can set the proof with the change of variables $\mathscr{F}(u,\alpha) = \mathscr{F}(t,\alpha)-\mathscr{F}(\tau,\alpha)$.

Below we present the N-Laplace transform of the N-convolution.

Theorem 11. *Let f and g be two functions which are piecewise continuous at each interval $[0,T]$ and of generalized exponential order. Then*

$$\mathscr{L}_N^\alpha\{(\mathrm{f}*\mathrm{g})_N\} = \mathscr{L}_N^\alpha\{f\}\mathscr{L}_N^\alpha\{g\}. \tag{23}$$

Proof. It is enough to start from the right member of the previous equality, $\mathscr{L}_N^\alpha\{f\}\mathscr{L}_N^\alpha\{g\}$, choosing u such that $\mathscr{F}(u,\alpha) = \mathscr{F}(t,\alpha)-\mathscr{F}(\tau,\alpha)$ and change the order of integration, to reach the desired result.

3 Applications to generalized differential equations

An interesting application of the Generalized Laplace Transform is the Generalized Cauchy Problem, which we will consider in the form:

$$N_F^\alpha(x(t)) = A(t)x(t) + f(t,x(t))$$

with $x(0) = x_0 + g(x)$, and $0 \leq t \leq T_0$, A is a sectorial operator which generates a strongly analytic semigroup $(T(t))_{t\geq 0}$ on the Banach Space $(X, \| \cdot \|)$

Now we apply the Generalized Laplace Transform in , and we obtain

$$\Lambda_N^\alpha\{x(t)\}(s) = (s-A)^{-1}[x_0+g(x)] + (s-A)^{-1}\Lambda_N^\alpha\{f(t,x(t))\}(s)$$

and taking the inverse Laplace Transform

$$x(t) = T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)[x_0+g(x)] + \int_0^t T\left(\int_0^t \frac{ds}{F(s,\alpha)} - \int_0^s \frac{du}{F(u,\alpha)}\right) f(s,x(s))\,ds$$

Where $T(t) = \int_0^\infty e^{st}(s-A(s))^{-1}ds$

Theorem 12. *With the following assumptions*

a) The functions $f(t,\cdot): X \to X$, $f(\cdot,x): [0,T_0] \to X$ are continuous and there exists a function $\Delta_r \in L([0,T_0],R^+)$ such that $\underset{\|x\|\leq r}{Sup}\|f(t,x)\| \leq \Delta_r(t)$

b) There exists a constant $C > 0$ such that $\|g(y)-g(x)\| < C|y-x|$ for all x, y.

c) $(T(t))_{t>0}$ is compact

the Cauchy Problem has at least one solution.

Proof. Let $r \geq \frac{\underset{t\in[0,T_0]}{Sup}\left|T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)\right|\left[\|x_0\|+\|g(0)\|+\int_0^{T_0}\frac{ds}{F(s,\alpha)}\|\Delta_r\|_{L^\infty}\right]}{1-C\underset{t\in[0,T_0]}{Sup}\left|T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)\right|}$, let B_r the closed ball centered at the origin and radius r and let $x \in B_r$.

we will prove that the operator

$$T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)[x_0+g(x)] + \int_0^t T\left(\int_0^t \frac{ds}{F(s,\alpha)} - \int_0^s \frac{du}{F(u,\alpha)}\right) f(s,x(s))\,ds$$

has at least one fixed point, which is the solution for the Cauchy Problem.

i) The expression $T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)[x_0+g(x)]$ is an contraction operator according with the assumptions a) and b).

ii) Let $\Psi(x)(t) = \int_0^t T\left(\int_0^t \frac{ds}{F(s,\alpha)} - \int_0^s \frac{du}{F(u,\alpha)}\right) f(s,x(s))\,ds$ we will show that $\Psi(x)$ is continuous and compact.

$\Psi(x)$ is continuous: let $(x_n)_{n\in N}$ such that $x_n \to x$ then

$$|\Psi(x_n)(t) - \Psi(x)| \leq \underset{t\in[0,T_0]}{Sup}\left|T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)\right| \times \left[\int_0^{T_0}\frac{ds}{F(s,\alpha)}\|f(s,x_n(s))-f(s,x(s))\|_{L^\infty}\right]$$

for the continuity of the function f we have $\|f(s,x_n(s)) - f(s,x(s))\|_{L^\infty} \to 0$ as $n \to \infty$ hence $|\Psi(x_n)(t) - \Psi(x)| \to 0$ as $n \to \infty$.

$\Psi(x)$ is compact:

Theorem 13. *Under the assumptions*

a)The function $f(\cdot,x):[0,T_0] \to X$ is continuous.

b) There exists a constant $C > 0$ such that $\|g(y) - g(x)\| < C|y-x|$ for all x, y.

c)There exists a constant $L > 0$ such that $\|g(y) - g(x)\| < L\|y-x\|$ for all x, y in X.

the Cauchy Problem has a unique solution.

Proof. Let

$$\Psi(t) = T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)[x_0 + g(x)] + \int_0^t T\left(\int_0^t \frac{ds}{F(s,\alpha)} - \int_0^s \frac{du}{F(u,\alpha)}\right) f(s,x(s))\,ds$$

and let x, y in X, then

$$\Psi(x)(t) - \Psi(y)(t) = T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)[g(x) - g(y)] +$$

$$+\int_0^t T\left(\int_0^t \frac{ds}{F(s,\alpha)} - \int_0^s \frac{du}{F(u,\alpha)}\right) \times [f(s,x(s)) - f(s,y(s))]\,ds$$

taking the norm of the difference and then the supreme for all $t \in [0,T_0]$ we obtain

$$|\Psi(x) - \Psi(y)| \leq (F(T_0,\alpha)L + C) \underset{t\in[0,T_0]}{Sup} \left|T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)\right| |x-y|$$

Because the Banach contraction principle, we see that the operator $\Psi(t)$ has a unique fixed point which is the solution for the Fractional Cauchy Problem.

Theorem 14. *Let x, y be solutions for the Cauchy Problem associated with x_0, y_0 respectively. Suppose that the conditions of the previous theorem are satisfied, then we have the estimate*

$$|y-x| \leq \frac{\alpha \underset{t\in[0,T_0]}{Sup} \left|T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)\right|}{\alpha - \alpha \underset{t\in[0,T_0]}{Sup} \left|T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)\right| \left(C + L\int_0^t \frac{ds}{F(s,\alpha)}\right)}$$

Remark. In this theorem we refer to the stability of the solution for the Cauchy Problem, that is the dependence of the solution to the initial conditions.

Proof. For $t \in [0,T_0]$

$$y(t) - x(t) = T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)[y_0 - x_0 + g(y) - g(x)] +$$

$$+\int_0^t T\left(\int_0^t \frac{ds}{F(s,\alpha)} - \int_0^s \frac{du}{F(u,\alpha)}\right)[f(s,y(s)) - f(s,x(s))]ds$$

taking the supreme on both sides we obtain

$$|y-x| \leq \underset{t\in[0,T_0]}{Sup} \left|T\left(\int_0^t \frac{ds}{F(s,\alpha)}\right)\right| \times \left[\|y_0 - x_0\| + \left(C + L\int_0^t \frac{ds}{F(s,\alpha)}\right)|y-x|\right]$$

from this inequality we find the estimate.

Remark. Others results on the stability of the solutions of certain generalized differential equations can be consulted in [53].

4 Conclusions

Throughout this work we first presented a function that generalizes the exponential function, which we use for the definitions of a Generalized Laplace Transform and the N-convolution (a generalization of the well known convolution of two functions of exponential order). For the Generalized Laplace Transform we prove equivalent to the known properties of the classical Laplace Transform and for the N-convolution we prove an interesting property that relates it to the Generalized Laplace Transform. Finally we present an application to solving a generalized differential equation with its corresponding theorems of existence and uniqueness.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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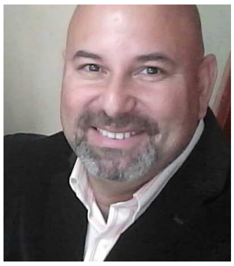

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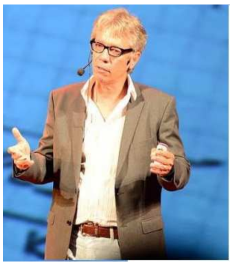

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