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Interpolation of Surfaces with Asymptotic Curves in Euclidean 3-Space

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Abstract: In this paper, we investigate the interpolation of surfaces which are obtained from an isoasymptotic curve in 3D-Euclidean space. We prove that there exists a unique C^0-Hermite surface interpolation related to an isoasymptotic curve under some special conditions on the marching scale functions. Finally, we present some examples and plot their graphs.

Keywords: Interpolation of surfaces , Isoasymptotic curve .
Subject Classification (2010): Primary: 53C50. Secondary: 53A04, 53C40.

1 Introduction

Differential geometry is a branch of mathematics which uses advanced calculus tools in geometry. Recently, it has become an applicable area of mathematics in science and technology. Since the manifold theory has been used in general relativity in the 1900s by Einstein, differential geometry of curves; surfaces and general manifolds have been improving more. From medicine to social science and from artificial intelligence to economy, it is very clear how the differential geometry is applied. In this manner, one can consider that applied mathematics has been changed from numerical and computational methods to differential geometrical tools. For example, to understand the meaning of multiple features data, we use calculus on manifolds in machine learning [1]. Moreover, differential geometry presents us to work on non-euclidean spaces as most real life problems are defined in such spaces. Thus, it is a fundamental tool for understanding events in the universe.

The most important kind of curves is geodesics which play the role of straight lines in Euclidean space on a manifold. Gauss proved that the differential geometry of a surface is different from the geometry of ambient space. The well known example of supporting these ideas is that a geodesic on a unit sphere embedded in a Euclidean space is not a geodesic in a Euclidean space. In this way, the differential geometry of a surface has many significant properties that we can use in applied sciences. The minimal distance between two points on a surface is called a geodesic. This is considered an important idea in many applications [2,3].

A surface could be constructed using a geodesic. In [4], a general surface obtained from a polynomial geodesic. Also, considering a 3-dimensional polynomial curve which is a pregeodesic, the authors constructed ruled cubic patched in [5]. In addition, in [6], authors investigated a developable surface that contains a given Bezier geodesic. Wang at al. defined a parametric surface called surface family using a geodesic curve [7]. They used the Frenet frame of the curve and presented necessary conditions in which the curve is an isogeodesic on a parametric surface considering Frenet aparatus of the curve. Later, Kasap et al. [8] generalized their methods and presented examples. Li et al.[9] investigated the approximation minimal surface with geodesics using the Dirichlet function and they minimized the area of a surface family using Dirichlet approach. This method can be used for obtaining the minimal cost of the material while building surfaces. The family surfaces have been studied, for example as in [10,11,12,13,14,15,16,17,18,19].

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The other special curve, which is as important as geodesics, is the asymptotic curves. An asymptotic curve is a curve always tangent to an asymptotic direction of the surface and it has zero normal curvature. On the other hand, one can be constructed as a surface using an asymptotic curve. Saad et al. [20] approximated the minimal parametric surface with an asymptotic curve by minimizing the Dirichlet function. In [21], the authors examined rational developable surface pencils through an arbitrary parametric curve as its common asymptotic curve. Moreover, Güler at al. constructed a surface interpolating a given curve as the asymptotic curve of it [22]. Similar to geodesics, the asymptotic curves also have many applications in related sciences. In [23] the authors presented a method to design strained grid structures along asymptotic curves to benefit from a high degree of simplification in fabrication and construction. Also, asymptotic curves have several applications in astronomy as in [24,25].

Lee et al. [26] introduced a new method to construct a parametric surface in terms of curves. They defined a surface interpolation associated with a spatial curve passing through some m-points in Euclidean 3-space. Motivated by the above-mentioned studies, we consider a surface interpolation using asymptotic curves in Euclidean space. Firstly, we give some fundamental facts which are used throughout the paper, in Section 2. Then, Section 3 is devoted to the surface interpolations with isoparametric curve and examples with their graphs.

2 Preliminaries

Let $\gamma(\omega)$ be a curve which is arc-length ω in 3D Euclidean space ($\mathbb{E}^3$). Take the Frenet frame of $\gamma(\omega)$ by $\{\mathscr{V}_1(\omega), \mathscr{V}_2(\omega), \mathscr{V}_3(\omega)\}$. Then we have the following well known relations between $\kappa(\omega)$ and $\tau(\omega)$ which are the curvature and the torsion of the curve $\gamma(\omega)$, respectively:

$$\begin{bmatrix} \mathscr{V}_1{}'(\omega) \\ \mathscr{V}_2{}'(\omega) \\ \mathscr{V}_3{}'(\omega) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(\omega) & 0 \\ -\kappa(\omega) & 0 & \tau(\omega) \\ 0 & -\tau(\omega) & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathscr{V}_1(\omega) \\ \mathscr{V}_2(\omega) \\ \mathscr{V}_3(\omega) \end{bmatrix}.$$

Previous equations are called the Frenet apparatus of a curve and are important to understand geometry of the curve. Also, we can classify curves via the Frenet–Serret frames. In [7], Wang et al. defined pencil surface which could be obtained using the Frenet–Serret frames of the curve. This surface is called a surface family or a pencil surface and defined, as follows:

Definition 1. *Let $\gamma(\omega)$ be a curve which is arc-length ω in $\mathbb{E}^3$ and $\{\mathscr{V}_1(\omega), \mathscr{V}_2(\omega), \mathscr{V}_3(\omega)\}$ be the Frenet frame of γ. Then, the map*

$$\Psi(\omega,\eta) = \gamma(\omega) + u(\omega,\eta)\cdot\mathscr{V}_1(\omega) + v(\omega,\eta)\cdot\mathscr{V}_2(\omega) + z(\omega,\eta)\cdot\mathscr{V}_3(\omega), \quad (1)$$

is defined a surface in $\mathbb{E}^3$, where $\Omega \geq \omega \geq 0, \Lambda \geq \eta \geq 0$ for a real-valued constants Ω, Λ, and $u(\omega,\eta), v(\omega,\eta)$ and $z(\omega,\eta)$ are C^1-functions. The surface $\Psi(\omega,\eta)$ is called as surface family or pencil surface [7].

By the following definition we classify some special curves on a parametric surface $\Psi(\omega,\eta)$.

Definition 2. *Take a curve $\gamma(\omega)$ on a parametric surface $\Psi(\omega,\eta)$ that is defined by (1). Then we have following characterizations [27]:*

- *$\gamma(\omega)$ is said to be an isoparametric curve on $\Psi(\omega,\eta)$ if there exists a parameter $\eta_0 \in [0,\Lambda]$ such that $\Psi(\omega,\eta_0) = \gamma(\omega)$.*
- *$\gamma(\omega)$ is an asymptotic curve on a parametric surface $\Psi(\omega,\eta)$ if $\frac{\partial \mathscr{N}(\omega,\eta_0)}{\partial s}.\mathscr{V}_1(\omega) = 0$, where $\mathscr{N}(\omega,\eta)$ is the normal vector of surface $\Psi(\omega,\eta)\cdot$*
- *$\gamma(\omega)$ is called an isoasymptotic of the surface $\Psi(\omega,\eta)$ if it is both a asymptotic curve and an isoparametric curve on the surface $\Psi(\omega,\eta)$.*

With following theorem we have the necessary and sufficient conditions for γ to be an isoasymptotic curve on the surface $\Psi(\omega,\eta)$.

Theorem 1. *[28] Let $\Psi(\omega,\eta)$ be a parametric curve with the marching-scale functions;*

$$u(\omega,\eta) = k(\omega)U(\eta),$$
$$v(\omega,\eta) = m(\omega)V(\eta),$$
$$z(\omega,\eta) = n(\omega)Z(\eta).$$

Then, $\gamma(\omega)$ is an isoasymptotic curve on a parametric surface $\Psi(\omega,\eta)$ if and only if we have

$$\begin{cases} U(\eta_0) = V(\eta_0) = Z(\eta_0) = 0, \\ n(\omega) = 0 \text{ or } \frac{\partial Z(\eta_0)}{\partial \eta} = 0. \end{cases} \quad (2)$$

If we take $k(\omega) = m(\omega) = n(\omega) = 1$ and consider $U(\eta)$, $V(\eta)$ and $Z(\eta)$ as polynomials of the forms in (2), then we have

$$U(\eta) = \sum_{t=1}^{n} a_t(\eta-\eta_0)^t, \quad (3)$$
$$V(\eta) = \sum_{t=1}^{n} b_t(\eta-\eta_0)^t,$$
$$Z(\eta) = \sum_{t=1}^{n} c_t(\eta-\eta_0)^t, \;\; c_1 = 0,$$

respectively, where a_t, b_t, c_t are constants. Then the polynomials $U(\eta)$, $V(\eta)$ and $Z(\eta)$ in (3) satisfy the isoasymptotic condition (2). Thus, we can determine marching-scale functions of a surface family $\Psi(\omega,\eta)$ by the polynomial expressions.

3 Surface Interpolations with Isoasymptotic Curve

In this section, we construct a surface with an isogeodesic curve passing through finite control points lying on $\mathbb{E}^3$.

Now, we give a definition for surface interpolations with isoasymptotic curve passing through some control points on $\mathbb{E}^3$.

Definition 3.*Let $A_1, A_2, ..., A_m$ be different points on $\mathbb{E}^3$ and $\Psi(\omega,\eta) : B \subset \mathbb{R}^2 \to \mathbb{E}^3$ be a parametric surface given by (1). For some different points $(\omega_t, \eta_t) \in B$ $(t = 1,...,m)$, we can construct the surface $\Psi(\omega,\eta)$ such that $\Psi(\omega_t,\eta_t) = A_t$. It is called a surface interpolation associated with the given isoasymptotic curve $\gamma(\omega)$ passing through m-control points A_t $(t = 1,...,m)$, simply, C^0-Hermite surface interpolation with an isoasymptotic curve. In particular, $\{A_1,A_2,...,A_m\}$ is called C^0-Hermite data.*

Polynomials $U(\eta)$, $V(\eta)$ and $Z(\eta)$ with degree n in (3) have n,n and $n-1$ degrees of freedom in terms of coefficients a_t, b_t and c_t respectively. In this case, there are two extra degrees of freedom. To determine a unique parametric surface with isoasymptotic curve, we may assume $a_n = b_n = 0$.

Now, we consider an isogeodesic surface parametrization

$$\Psi(\omega,\eta) = \gamma(\omega) + U(\eta)\cdot\mathscr{V}_1(\omega) + V(\eta)\cdot\mathscr{V}_2(\omega) + Z(\eta)\cdot\mathscr{V}_3(\omega), \quad (4)$$

where $\Omega \geq \omega \geq 0$, $\Lambda \geq \eta \geq 0$, with the marching-scale functions are given in (3) for $a_n = b_n = 0$.

Theorem 2.*Let $A_1, A_2, ..., A_m$ be different points on a parametric surface $\Psi(\omega,\eta)$ given in (4). For $\Psi(\omega_t,\eta_t) = A_t, t = 1,...,m$ there exists a unique C^0-Hermite surface interpolation with an isoasymptotic curve such that the marching-scale functions are given by*

$$U(\eta) = \sum_{t=1}^{n-1} a_t(\eta-\eta_0)^t, \quad V(\eta) = \sum_{t=1}^{n-1} b_t(\eta-\eta_0)^t$$

$$Z(\eta) = \sum_{t=2}^{n} c_t(\eta-\eta_0)^t,$$

and

$$\det\begin{pmatrix} d_{12} & \eta_1-\eta_0 & (\eta_1-\eta_0)^2 & \cdots & (\eta_1-\eta_0)^{n-1} \\ d_{22} & \eta_2-\eta_0 & (\eta_2-\eta_0)^2 & \cdots & (\eta_2-\eta_0)^{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ d_{n2} & \eta_{n-1}-\eta_0 & (\eta_{n-1}-\eta_0)^2 & \cdots & (\eta_{n-1}-\eta_0)^{n-1} \end{pmatrix}$$

where a_t, b_t and c_t are constant and $d_{t2} = V(\eta_{t2})$, $t = 1,2,\ldots,n$.

*Proof.*Let us define $(n-1)$-points of the surface $\Psi(\omega_t,\eta_t)$ by

$$\Psi(\omega_t,\eta_t) = A_t \text{ for } \omega \geq \eta_{n-1}... \geq \eta_1 \geq \eta_0 \geq 0.$$

So, we have

$$\Psi(\omega_t,\eta_t) = A_t = \gamma(\omega_t) + \mathscr{V}_1(\omega_t).U(\eta_t) + \mathscr{V}_2(\omega_t).V(\eta_t) + \mathscr{V}_3(\omega_t).Z(\eta_t).$$

By taking inner product with $\mathscr{V}_1(\omega)$, $\mathscr{V}_2(\omega)$, and $\mathscr{V}_3(\omega)$, respectively we obtain the coefficients as following

$$U(\eta_t) = \langle A_t - \gamma(\omega_t), \mathscr{V}_1(\omega_t)\rangle,$$
$$V(\eta_t) = \langle A_t - \gamma(\omega_t), \mathscr{V}_2(\omega_t)\rangle,$$
$$Z(\eta_t) = \langle A_t - \gamma(\omega_t), \mathscr{V}_3(\omega_t)\rangle.$$

Using

$$U(\eta_t) = c_{t1}, \; V(\eta_t) = c_{t2}, \; Z(\eta_t) = c_{t3}$$

where c_{t1}, c_{t2} and c_{t3} are constant, from (3) and for $a_n = b_n = 0$, we have following matrices;

$$\begin{pmatrix} \eta_1-\eta_0 & \cdots & (\eta_1-\eta_0)^{n-1} \\ \eta_2-\eta_0 & \cdots & (\eta_2-\eta_0)^{n-1} \\ \vdots & \ddots & \vdots \\ \eta_{n-1}-\eta_0 & \cdots & (\eta_{n-1}-\eta_0)^{n-1} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} d_{11} \\ \vdots \\ d_{(n-1)1} \end{pmatrix},$$

$$\begin{pmatrix} \eta_1-\eta_0 & \cdots & (\eta_1-\eta_0)^{n-1} \\ \eta_2-\eta_0 & \cdots & (\eta_2-\eta_0)^{n-1} \\ \vdots & \ddots & \vdots \\ \eta_{n-1}-\eta_0 & \cdots & (\eta_{n-1}-\eta_0)^{n-1} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_{12} \\ \vdots \\ d_{(n-1)2} \end{pmatrix},$$

$$\begin{pmatrix} (\eta_1-\eta_0)^2 & \cdots & (\eta_1-\eta_0)^{n} \\ (\eta_2-\eta_0)^2 & \cdots & (\eta_2-\eta_0)^{n} \\ \vdots & \ddots & \vdots \\ (\eta_{n-1}-\eta_0)^2 & \cdots & (\eta_{n-1}-\eta_0)^{n} \end{pmatrix} \begin{pmatrix} c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_{13} \\ \vdots \\ d_{(n-1)3} \end{pmatrix},$$

for $1 \leq \eta \leq n-1$. Let take

$$M_1 = \begin{pmatrix} \eta_1-\eta_0 & (\eta_1-\eta_0)^2 & \cdots & (\eta_1-\eta_0)^{n-1} \\ \eta_2-\eta_0 & (\eta_2-\eta_0)^2 & \cdots & (\eta_2-\eta_0)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{n-1}-\eta_0 & (\eta_{n-1}-\eta_0)^2 & \cdots & (\eta_{n-1}-\eta_0)^{n-1} \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} (\eta_1-\eta_0)^2 & (\eta_1-\eta_0)^3 & \cdots & (\eta_1-\eta_0)^{n} \\ (\eta_2-\eta_0)^2 & (\eta_2-\eta_0)^3 & \cdots & (\eta_2-\eta_0)^{n} \\ \vdots & \vdots & \ddots & \vdots \\ (\eta_{n-1}-\eta_0)^2 & (\eta_{n-1}-\eta_0)^3 & \cdots & (\eta_{n-1}-\eta_0)^{n} \end{pmatrix}.$$

Then the determinants of M_1 and M_2 is obtained as

$$\det(M_1) = (-1)^{\frac{(n-1)(n-2)}{2}} \prod_{t=1}^{n-1}(\eta_t-\eta_0) \prod_{1\leq t<j\leq n-1}(\eta_t-\eta_j),$$

and

$$\det(M_2) = (-1)^{\frac{(n-1)(n-2)}{2}} \prod_{t=1}^{n-1} (\eta_t - \eta_0)^2 \prod_{1 \le t < j \le n-1} (\eta_t - \eta_j).$$

Since η_t and η_j are non-zero and different from each others, for $1 \le t < j \le n-1$, we get $\det(M_1) \ne 0$ and $\det(M_1) \ne 0$, i.e. $a_1, a_2, ..., a_{n-1}$, $b_1, b_2, ..., b_{n-1}$ and $c_2, c_3, ..., c_n$ have unique solutions. This shows that there exists a uniquely C^0-Hermite surface interpolation with an isoasymptotic curve.

*Example 1.*Consider a curve parametrized by

$$\gamma(\omega) = \left(\frac{\sin\omega}{2}, \frac{\cos\omega}{2}, \frac{\sqrt{3}\omega}{2}\right), \qquad 0 \le \omega \le 2\pi. \tag{5}$$

Curve (5) is shown in Figure 1. By a direct computation, we have

$$\mathscr{V}_1(\omega) = \left(\frac{\cos\omega}{2}, -\frac{\sin\omega}{2}, \frac{\sqrt{3}}{2}\right),$$
$$\mathscr{V}_2(\omega) = (-\sin\omega, -\cos\omega, 0),$$
$$\mathscr{V}_3(\omega) = \left(\frac{\sqrt{3}\cos\omega}{2}, -\frac{\sqrt{3}\sin\omega}{2}, \frac{-1}{2}\right).$$

For $A_1 = (1, 7, \frac{\sqrt{3}\pi}{4})$, the point A_1 lies on the surface pencil with an isoasymptotic curve given by (4). If we take

$$U(\eta) = a_1\eta,\ V(\eta) = b_1\eta,\ W(\eta) = c_2\eta^2$$

then there is only one surface with an isoasymptotic curve passing the point A_1. We take $\omega_1 = \frac{\pi}{2}$, $\eta_1 = 2$, i.e. $\Psi_1(\frac{\pi}{2}, 2) = A_1(1, 7, \frac{\sqrt{3}\pi}{4})$. We obtain the equations:

$$\frac{1}{2} - 2b_1 = 1,$$
$$-a_1 - 2\sqrt{3}c_2 = 7,$$
$$\sqrt{3}a_1 - c_2 = 0,$$

which imply

$$a_1 = -1,\ b_1 = -\frac{1}{2},\ c_2 = -\sqrt{3}.$$

Thus, we can construct the surface with an isoasymptotic curve passing the one point $A_1 = (1, 7, \frac{\sqrt{3}\pi}{4})$ given by

$$\Psi_1(\omega, \eta) = \frac{1}{2}(\sin(\omega) - 3\eta^2\cos(\omega) + \eta\sin(\omega) - \eta\cos(\omega), \tag{6}$$
$$(\eta^2 + \eta)\sin(\omega) + (\eta + 1)\cos(\omega), \sqrt{3}(\omega + \eta^2 - \eta))$$

Surface (6) is shown in Figure 2 and the surface (6) with curve (5) is shown in Figure 3.

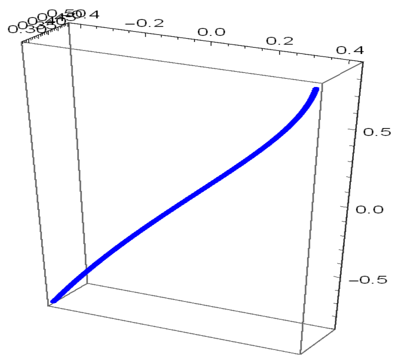

Fig. 1: Graph of γ

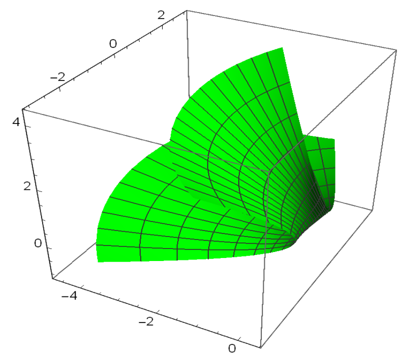

Fig. 2: Graph of surface Ψ_1 is constructed by γ

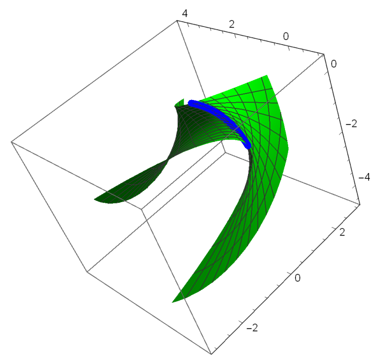

Fig. 3: Graph of surface Ψ_1 with curve γ

*Example 2.*Let the surface with an isoasymptotic curve in Example 1 pass through the additional point $A_2 = (\frac{1}{2}, 0, \frac{\sqrt{3}\pi}{4})$. For the convenience of calculations, taking $\omega_2 = \frac{\pi}{2}$, $\eta_2 = 1$, we have $A_2 = (\frac{1}{2}, 0, \frac{\sqrt{3}\pi}{4})$ and obtain the system of linear equations as follows:

$$\begin{aligned} b_1+2b_2 &= -\frac{1}{4}, \\ a_1+2a_2+2\sqrt{3}(c_2+2c_3) &= -7, \\ \sqrt{3}(a_1+2a_2) &= 2(c_2+2c_3), \\ b_1 &= -b_2, \\ -(a_1+a_2) &= \sqrt{3}(c_2+c_3), \\ \sqrt{3}(a_1+a_2) &= c_2+c_3. \end{aligned}$$

So, we obtain

$$b_1=\frac{1}{4},\ b_2=-\frac{1}{4},\ c_2=\frac{7\sqrt{3}(3-\sqrt{3})}{12},\ c_3=\frac{7\sqrt{3}(\sqrt{3}-3)}{12},$$
$$a_1=\frac{14(3-\sqrt{3})}{12},\ a_2=\frac{14(\sqrt{3}-3)}{12}.$$

Thus the surface with an isoasymptotic curve passing the two points $A_1=(1,7,\frac{\sqrt{3}\pi}{4})$ and $A_2=(\frac{1}{2},0,\frac{\sqrt{3}\pi}{4})$ is uniquely given by

$$\begin{aligned} \Psi_2(\omega,\eta) = (&\frac{\sin(\omega)}{2}+\frac{7\left(3-\sqrt{3}\right)}{12}\left(\eta-\eta^2\right)\cos(\omega) \\ &-\frac{\sin(\omega)}{4}\left(\eta-\eta^2\right) \\ &+\frac{7\left(3-\sqrt{3}\right)}{8}\left(\eta^2-\eta^3\right)\cos(\omega) \\ &+\frac{\cos(\omega)}{2}-\frac{7\left(3-\sqrt{3}\right)}{12}\left(\eta-\eta^2\right)\sin(\omega) \\ &-\left(\eta-\eta^2\right)\frac{\cos(\omega)}{4} \\ &-\frac{7\left(3-\sqrt{3}\right)}{8}\left(\eta^2-\eta^3\right)\sin(\omega) \\ &+\frac{\sqrt{3}s}{2}+\frac{7\left(\sqrt{3}-1\right)}{4}\left(\eta-\eta^2\right) \\ &-\frac{7\left(\sqrt{3}-1\right)}{8}\left(\eta^2-\eta^3\right)). \end{aligned} \quad (7)$$

The surface (7) is shown in Figure 4 and the surface (7) with curve (5) is shown in Figure 5.

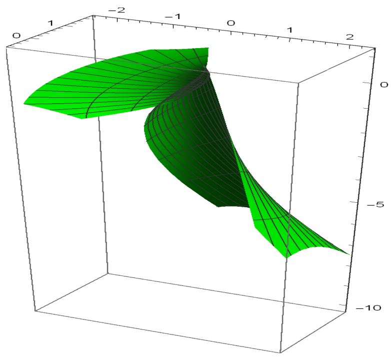

Fig. 4: Graph of surface Ψ_2

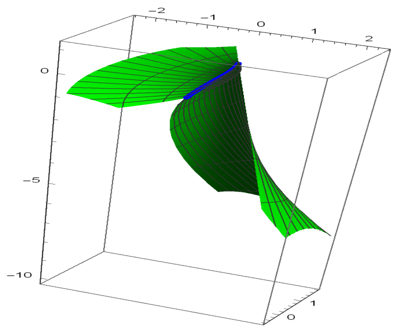

Fig. 5: Graph of surface Ψ_2 with curve γ

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Conflict of Interest

The authors declare that they have no conflict of interest.

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