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Existence and Uniqueness for Multi-Term Sequential Fractional Integro-Differential Equations with Non-Local Boundary Conditions

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Abstract: In this paper we present a new type of non-local multi-point boundary value problems of Caputo type sequential fractional integro-differential equations. The paper shows that existence and uniqueness results can be obtained using standard tools of fixed point theorem. An application to illustrate the power of the obtained results on quantum information theory is discussed.

Keywords: Sequential fractional derivative, Multi-point, Caputo fractional derivative, Fixed point theorem, Fractional differential equation.

1 Introduction

The subject of fractional calculus received a great attention in the last two decades. Differential equations of fractional order arise in several research areas of science and engineering, such as physics, chemistry, aerodynamics, polymer rheology, economic, control theory, signal and image processing, and biophysic [1]-[47]. Also, more different applications in quantum information may give new features using fractional calculus [18]-[32]. Recently, many researchers have given attention to the existence of solutions of the initial and boundary value problems for fractional differential equations. Some papers addressed the existence of solutions to boundary value problems with two-point, three-point, multi-point or integral boundary conditions (See for examples [9]-[34]).

An important application of the fractional time quantum information is non-unitarity of the quantum evolution and more insights and observations destroy the equivalence between Schrödinger equation and Heisenberg pictures [12]. A review of fundamentals and physical applications of fractional quantum mechanics has been presented [24]. Recently, Iomin [3] discussed the problem of fractional evolution in quantum mechanics and the results can be applied in different models [25]-[30].

Xinwei and Landong [40] reviewed the existence of solutions for the nonlinear fractional differential equation:

$$^{c}D^{\alpha}u(t) = f(t,u(t),^{c}D^{\beta}u(t)), \qquad (0<t<1),$$

with boundary values $u(0) = u'(0) = 0$ or $u'(0) = u(1) = 0$

or $u(0) = u(1) = 0$, where $1 < \alpha \le 2, 0 < \beta \le 1$, and f is continuous on $[0,1]\times\mathbb{R}\times\mathbb{R}$.

Su and Zhang [38] investigated the existence and uniqueness of solutions for the following nonlinear two-point fractional boundary value problem

$$^{c}D^{\alpha}u(t) = f(t,u(t),^{c}D^{\beta}u(t)), \qquad (0<t<1),$$

with boundary values $a_1u(0) - a_2u'(0) = A$ and $b_1u(1) + b_2u'(1) = B$, where $\alpha, \beta, a_i, b_i (i = 1,2)$ satisfy certain conditions.

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Ahmad and Sivasundaram [6] explored the existence of solutions for the nonlinear fractional integro-differential equation

${}^cD^q u(t) = f(t, u(t), (\phi u)(t), (\psi u)(t)), (0 < t < 1, 1 < q \le 2),$

with boundary values $u'(0) + au(\eta_1) = 0$, $bu'(1) + u(\eta_2) = 0$ and $0 < \eta_1 \le \eta_2 < 1$, where ${}^cD^q$ is the Caputo fractional derivative , $a, b \in (0,1)$, $f : [0,1] \times X \times X \times X \to X$ is continuous and the mappings $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ with the property $\sup_{t\in[0,1]} |\int_0^t \lambda(t,s)ds| < \infty$ and $\sup_{t\in[0,1]} |\int_0^t \gamma(t,s)ds| < \infty$, the maps ϕ and ψ are defined by $(\phi u)(t) = \int_0^t \gamma(t,s)u(s)ds$ and $(\psi u)(t) = \int_0^t \lambda(t,s)u(s)ds$. Here,$X$ is a Banach space (see [3]).

In this paper, we study the existence and uniqueness of solutions for the nonlinear fractional integro-differential equation with m-point multi-term fractional integral boundary conditions.

$$\begin{cases} ({}^cD^q + k\,{}^cD^{q-1})u(t) = f(t, u(t), (\phi u)(t), (\psi u)(t), \\ {}^cD^{\beta_1}u(t), \ldots, {}^cD^{\beta_n}u(t)), \\ t \in [0,1] \\ \\ u(0) = 0, \quad \sum_{i=1}^{m-1} a_i u(\xi_i) = \beta \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} u(s)ds, \end{cases} \tag{1}$$

where ${}^cD^q$ is the standard Caputo fractional derivative of order q, with $1 < q \le 2, 0 < \beta_i < 1, k > 0, 0 < \eta < \xi_1 < \xi_2 < \ldots < \xi_{m-1} < 1, \beta, a_i, i = 1, \ldots, m$ are real constants, $f : [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ is continuous and for the mappings $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ with the property $\sup_{t\in[0,1]} |\int_0^t \lambda(t,s)ds| < \infty$ and $\sup_{t\in[0,1]} |\int_0^t \gamma(t,s)ds| < \infty$, the maps ϕ and ψ are defined by $(\phi u)(t) = \int_0^t \gamma(t,s)u(s)ds$ and $(\psi u)(t) = \int_0^t \lambda(t,s)u(s)ds$.

Present paper is arranged as follows : In section 2, we present a basic result that lays the foundation for defining a fixed point problem equivalent to the given problem (1). The main results, based on Banach's contraction mapping principal, fixed point and Krasnoselskii's fixed point theorem, are presented in section 3. Illustrating examples are discussed in section 4.

2 Basic Result

For convenience of the reader, we present some necessary definitions on fractional calculus theory, which can be found in [1].

Definition 2.1. The Riemann-Liouville fractional integral of order q for a continuous function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \quad q > 0,$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

Definition 2.2. For a at least n-times continuously differentiable function $f : (0,\infty) \longrightarrow \mathbb{R}$, the Caputo derivative of order $q > 0$ is defined as

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds,$$

$$n - 1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.1. Let $\alpha > 0$, then the differential equation

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$ and

$$I^{\alpha c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

here $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$ and $n = [\alpha] + 1$.

Lemma 2.2. Let $y(t) \in C([0,1])$ a function $u \in C^2([0,1], \mathbb{R})$ be a solution of linear sequential fractional differential equation

$$({}^cD^q + k\,{}^cD^{q-1})u(t) = y(t),$$

with boundary values $u(0) = 0$ and $\sum_{i=1}^{m-1} a_i u(\xi_i) = \beta \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} u(s)ds$ has the unique solution given by

$$\begin{aligned} u(t) = \frac{(e^{-kt} - 1)}{\Delta} \Big(\beta \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} \Big(\int_0^s e^{-k(s-x)} \\ \times \Big(\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)} y(\tau)d\tau \Big) dx \Big) ds \\ - \sum_{i=1}^{m-1} a_i \int_0^{\xi_i} e^{-k(\xi_i - s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} y(x)dx \Big) ds \Big) \\ + \int_0^t e^{-k(t-s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} y(x)dx \Big) ds, \end{aligned} \tag{2}$$

where

$$\Delta = \sum_{i=1}^{m-1} a_i \left(e^{-k\xi_i} - 1 \right) - \beta \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} e^{-ks} ds \tag{3}$$

$$+\frac{\beta\eta^q}{\Gamma(q+1)} \neq 0.$$

Proof. For $q \in (1,2]$, we consider the following linear fractional differential equation:

$$({}^cD^q + k{}^cD^{q-1})u(t) = y(t), \tag{4}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, we can write its solution as

$$u(t) + k{}^cD^{-1}u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds + c_0 + c_1t, \tag{5}$$

where c_0 and c_1 are arbitrary constants. Now (5) can be expressed as

$$u(t) = -k\int_0^t u(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds + c_0 + c_1t. \tag{6}$$

Differentiating (6), we obtain

$$u'(t) = -ku(t) + \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}y(s)ds + c_1, \tag{7}$$

which can alternatively be written as

$$(u(t)e^{kt})' = e^{kt}\left(\frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}y(s)ds + c_1\right). \tag{8}$$

Integrating from 0 to t, we have

$$u(t) = Ae^{-kt} + \int_0^t e^{-k(t-s)}\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}y(x)dxds + B. \tag{9}$$

Using the data $u(0) = 0$ in (9), we find that $A = -B$. Thus, (9) takes the form

$$u(t) = A(e^{-kt}-1) + \int_0^t e^{-k(t-s)}\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}y(x)dxds. \tag{10}$$

Using the condition $\sum_{i=1}^{m-1} a_iu(\xi_i) = \beta\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}u(s)ds$ in (10), we obtain

$$A = \frac{1}{\Delta}\Big(\beta\int_0^\eta \frac{(\eta-s)^{q-2}}{\Gamma(q-1)} \Big(\int_0^s e^{-k(s-x)}\Big(\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)}y(\tau)d\tau\Big)dx\Big)ds\Big)$$
$$-\sum_{i=1}^{m-1} a_i\int_0^{\xi_i} e^{-k(\xi_i-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}y(x)dx\Big)ds,$$

where Δ is given by (3). Substituting the value of A in (10), we get the solution (2). The converse follows by direct computation. This completes the proof.

In the next lemma, we present some estimates that we need in the sequel.

Lemma 2.3. For $y \in C([0,1],\mathbb{R})$ with $\|y\| = \sup_{t\in[0,1]}|y(t)|$ we have

$$(i)\quad \Big|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\Big(\int_0^s e^{-k(s-x)} \Big(\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)}y(w)d\tau\Big)dx\Big)ds\Big| \leq \frac{\eta^{2q-2}}{(k\Gamma(q))^2}(k\eta + e^{-k\eta}-1)\|y\|.$$

$$(ii)\quad \Big|\sum_{i=1}^{m-1} a_i\int_0^{\xi_i} e^{-k(\xi_i-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}y(x)dx\Big)ds\Big| \leq \sum_{i=1}^{m-1}|a_i|\xi_i^{q-1}(1-e^{-k\xi_i})\frac{\|y\|}{k\Gamma(q)}.$$

$$(iii)\quad \Big|\int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}y(x)dx\Big)ds\Big| \leq \frac{1}{k\Gamma(q)}(1-e^{-k})\|y\|.$$

Proof. (i) Obviously,

$$\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)}d\tau = -\frac{(x-\tau)^{q-1}}{\Gamma(q)}\Big|_0^x = \frac{x^{q-1}}{\Gamma(q)}$$

and

$$\int_0^s e^{-k(s-x)}\frac{x^{q-1}}{\Gamma(q)}dx \leq \frac{s^{q-1}}{\Gamma(q)}\int_0^s e^{-k(s-x)}dx \leq \frac{s^{q-1}}{k\Gamma(q)}(1-e^{-ks}).$$

Hence

$$\Big|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\Big(\int_0^s e^{-k(s-x)} \int_0^\tau \frac{(x-\tau)^{q-2}}{\Gamma(q-1)}y(\tau)d\tau\Big)dx\Big)ds\Big|$$
$$\leq \|y\|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}.\Big(\frac{s^{q-1}}{k\Gamma(q)}\Big)(1-e^{-ks})ds$$

$$\leq \|y\| \frac{\eta^{2q-2}}{(k\Gamma(q))^2} \int_0^\eta (1-e^{-ks})ds$$
$$= \|y\| \frac{\eta^{2q-2}}{(k\Gamma(q))^2} (k\eta + e^{-k\eta} - 1).$$

The proofs of (ii) and (iii) are similar. The proof is complete.

3 Existence and uniqueness results

This section is devoted to the main results concerning the existence and uniqueness of solution for the problem (1). First of all, we fix our terminology.

Let C be the space of all continuous real-valued functions on $I = [0,1]$ and

$$X = \{u : u \in C(I,\mathbb{R}) \quad \text{and} \quad {}^cD^{\beta_i}u \in C(I,\mathbb{R}), \ (0<\beta_i<1), \text{for} \quad i=1,\dots,n\}$$

denote the space equipped with the norm

$\|u\|_X = \|u\| + \sum_{i=1}^n \|{}^cD^{\beta_i}u\| = \max_{t\in I}|u(t)| + \sum_{i=1}^n \max_{t\in I}|{}^cD^{\beta_i}u(t)|$.

It is known that $(X, \|.\|)$ is a Banach space.

To define a fixed point problem equivalent to (1), we make use of the lemma (2.2) to define an operator $F : X \to X$ as

$$Fu(t) = \frac{(e^{-kt}-1)}{\Delta}\Big(\beta\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \Big(\int_0^s e^{-k(s-x)}\Big(\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)} \times f(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau), {}^cD^{\beta_1}u(\tau),\dots,{}^cD^{\beta_n}u(\tau))d\tau\Big)dx\Big)ds$$
$$-\sum_{i=1}^{m-1} a_i \int_0^{\xi_i} e^{-k(\xi_i-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \times f(x,u(x),(\phi u)(x),(\psi u)(x), {}^cD^{\beta_1}u(x),\dots,{}^cD^{\beta_n}u(x))dx\Big)ds\Big)$$
$$+\int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \times f(x,u(x),(\phi u)(w),(\psi u)(w), {}^cD^{\beta_1}u(x),\dots,{}^cD^{\beta_n}u(x))dx\Big)ds. \tag{11}$$

Observe that problem (1) has solutions if the operator defined by (11) has fixed point.

For computational convenience, we set

$$P = \sup_{t\in[0,1]} \frac{|e^{-kt}-1|}{|\Delta|} = \frac{|e^{-k}-1|}{|\Delta|}, \tag{12}$$
$$\tilde{P} = \sup_{t\in[0,1]} \frac{|ke^{-kt}|}{|\Delta|} = \frac{ke^{-k}}{|\Delta|}.$$

$$\Lambda_1 = P\Delta_1 + \frac{(1-e^{-k})}{k\Gamma(q)}, \quad \Lambda_2 = \tilde{P}\Delta_1 + \frac{(2-e^{-k})}{\Gamma(q)}. \tag{13}$$

$$\omega = \zeta(1+\lambda_0+\gamma_0). \tag{14}$$

$$\gamma_0 = \sup_{t\in I}\Big|\int_0^t \gamma(t,s)ds\Big|, \quad \lambda_0 = \sup_{t\in I}\Big|\int_0^t \lambda(t,s)ds\Big|. \tag{15}$$

$$\Delta_1 = |\beta|\frac{\eta^{2q-2}}{(k\Gamma(q))^2}(k\eta + e^{-k\eta}-1) + \sum_{i=1}^m |a_i|\xi_i^{q-1}(1-e^{-k\xi_i})\frac{1}{k\Gamma(q)}. \tag{16}$$

In this section, our first result is based on the Banach fixed point theorem (see [17]).

Theorem 3.1. Assume that $f : [0,1]\times\mathbb{R}^{n+3} \to \mathbb{R}$ is a continuous function satisfying the assumption

(H_1) $|f(t,x,y,w,u_1,u_2,\dots,u_n) - f(t,x',y',w',v_1,v_2,\dots,v_n)|$

$\leq L_1|x-x'| + L_2|y-y'| + L_3|w-w'| + d_1|u_1-v_1| + d_2|u_2-v_2| + \dots + d_n|u_n-v_n|,$

for all $t \in [0,1]$ and $x,y,w,x',y',w',u_1,u_2,\dots,u_n,v_1,v_2,\dots,v_n \in \mathbb{R}$.

where $L_i, d_j > 0, \forall i=1,2,3, \forall j=1,2,\dots,n$ are Lipschitz constants.

Then problem (1) has a unique solution if $\Big(\Lambda_1 + \Lambda_2\sum_{i=1}^n \frac{1}{\Gamma(2-\beta_i)}\Big)\omega < 1$, where $\Lambda_1,\Lambda_2,\omega$ are given by (13), (14), $\zeta_1 = \sup\{L_1,d_1,d_2,\dots,d_n\}, \zeta_2 = \sup\{L_2,L_3\}, \zeta = \sup\{\zeta_1,\zeta_2\}$.

Proof. Let us fix

$$r \geq \frac{\Lambda_1 M + \Lambda_2 M\sum_{i=1}^n \frac{1}{\Gamma(2-\beta_i)}}{1-\Lambda_1\omega - \Lambda_2\omega\sum_{i=1}^n\frac{1}{\Gamma(2-\beta_i)}},$$

where $\Lambda_1,\Lambda_2,\omega$ are given by (13), (14), respectively and $M = \sup_{t\in[0,1]}|f(t,0,\dots,0)|$.

As the first step, we show that $FB_r \subset B_r$ where $B_r = \{u \in X : \quad \|u\|_X \le r\}$.

For $u \in B_r$, using (H_1), we have

$$|Fu(t)| \le \sup_{t\in[0,1]} \frac{|e^{-kt}-1|}{|\Delta|}\Big(|\beta|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \Big(\int_0^s e^{-k(s-x)}\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)} \times(|f(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau), {}^cD^{\beta_1}u(\tau),\ldots,{}^cD^{\beta_n}u(\tau)) -f(\tau,0,\ldots,0)|+M)d\tau\Big)dx\Big)ds$$
$$+\sum_{i=1}^{m-1}|a_i|\int_0^{\xi_i} e^{-k(\xi_i-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \times(|\, f(x,u(x),(\phi u)(x),(\psi u)(x), {}^cD^{\beta_1}u(x),\ldots,{}^cD^{\beta_n}u(x)) -f(x,0,\ldots,0)\,|+M)dxds\Big)$$
$$+\int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \times(|f(x,u(x),(\phi u)(w),(\psi u)(w), {}^cD^{\beta_1}u(x),\ldots,{}^cD^{\beta_n}u(x) -f(x,0,\ldots,0)\,|+M)dx\Big)ds$$
$$\le P\Big(|\beta|\frac{\eta^{2q-2}}{(k\Gamma(q))^2}(k\eta+e^{-k\eta}-1) +\sum_{i=1}^{m-1}|a_i|\xi_i^{q-1}(1-e^{-k\xi_i})\frac{1}{k\Gamma(q)}\Big)$$
$$\times(\sup\{L_1,d_1,\ldots,d_n\}(|u|+|{}^cD^{\beta_1}u|+\ldots+|{}^cD^{\beta_n}u|) +\sup\{L_2,L_3\}(\gamma_0+\lambda_0)|u|+M)$$
$$+\frac{(1-e^{-k})}{k\Gamma(q)}(L_1|u(x)| +L_2\gamma_0|u(x)|+L_3\lambda_0|u(x)| +d_1|{}^cD^{\beta_1}u(x)| +\ldots+d_n|{}^cD^{\beta_n}u(x)|+M)$$
$$\le P\Delta_1(\zeta_1 r+\zeta_2(\gamma_0+\lambda_0)r+M) +\frac{(1-e^{-k})}{k\Gamma(q)}(\zeta_1 r+\zeta_2(\gamma_0+\lambda_0)r+M)$$
$$\le (P\Delta_1+\frac{(1-e^{-k})}{k\Gamma(q)})\zeta(1+\gamma_0+\lambda_0)r +(P\Delta_1+\frac{(1-e^{-k})}{k\Gamma(q)})M$$
$$\le \Lambda_1\omega r+\Lambda_1 M.$$

Also, we have

$$|(Fu)'(t)| \le \sup_{t\in[0,1]} \frac{|-ke^{-kt}|}{|\Delta|}\Big(|\beta|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \Big(\int_0^s e^{-k(s-x)}\Big(\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)} \times|f(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),\ldots,{}^cD^{\beta_n}u(\tau)) -f(\tau,0,\ldots,0)\,|+M)|d\tau\Big)dx\Big)ds$$
$$+\sum_{i=1}^{m-1}|a_i|\int_0^{\xi_i} e^{-k(\xi_i-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \times(|f(x,u(x),(\phi u)(x),(\psi u)(x),{}^cD^{\beta_1}u(x),\ldots,{}^cD^{\beta_n}u(x)) -f(x,0,\ldots,0)\,|+M)dx\Big)ds\Big)+\int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}$$
$$\times(|f(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),\ldots,{}^cD^{\beta_n}u(s)) -f(s,0,\ldots,0)\,|+M)\,|\,ds+k\int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}$$
$$\times(|f(x,u(x),(\phi u)(x),(\psi u)(x),{}^cD^{\beta_1}u(x),\ldots,{}^cD^{\beta_n}u(x)) -f(x,0,\ldots,0)\,|+M)|dx\Big)ds$$
$$\le \tilde{P}\Delta_1(\zeta_1 r+\zeta_2(\gamma_0+\lambda_0)r+M) +\frac{(2-e^{-k})}{\Gamma(q)}(\zeta_1 r+\zeta_2(\gamma_0+\lambda_0)r+M)$$
$$\le \Lambda_2\omega r+\Lambda_2 M.$$

By the definition of the Caputo fractional derivative with $0<\beta_i<1$, we get

$$\Big(\,|\,{}^cD^{\beta_i}(Fu)(t)\,|\,\Big) = \Big|\int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fu)'(s)ds\Big|$$
$$\le \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(\Lambda_2\omega r+\Lambda_2 M)ds$$
$$\le \frac{1}{\Gamma(2-\beta_i)}(\Lambda_2\omega r+\Lambda_2 M).$$

Hence,

$$\begin{aligned}\|F(u)\| &= \|F(u)\| + \sum_{i=1}^{n} \|{}^c D^{\beta_i} F(u)\| \\ &\le \Lambda_1 \omega r + \Lambda_1 M \\ &+ \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)} (\Lambda_2 \omega r + \Lambda_2 M) \\ &\le r.\end{aligned}$$

Thus, $FB_r \subset B_r$. Now, for any $u, v \in X$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}&|(Fu)(t) - (Fv)(t)| \\ &\le \sup_{t\in[0,1]} \frac{|e^{-kt}-1|}{|\Delta|} \\ &\Big(|\beta| \int_0^{\eta} \frac{(\eta-s)^{q-1}}{\Gamma(q)} \\ &\Big(\int_0^{s} e^{-k(s-\tau)} \Big(\int_0^{\tau} \frac{(\tau-w)^{q-2}}{\Gamma(q-1)} \\ &\times |f(w, u(w), (\phi u)(w), (\psi u)(w), \\ &{}^c D^{\beta_1} u(w), \ldots, {}^c D^{\beta_n} u(w)) \\ &- f(w, v(w), (\phi v)(w), (\psi v)(w), \\ &{}^c D^{\beta_1} v(w), \ldots, {}^c D^{\beta_n} v(w))| dw\Big) d\tau\Big) ds \\ &+ \sum_{i=1}^{m-1} |a_i| \int_0^{\xi_i} e^{-k(\xi_i - s)} \Big(\int_0^{s} \frac{(s-x)^{q-2}}{\Gamma(q-1)} \\ &\times |f(x, u(x), (\phi u)(x), (\psi u)(x), \\ &{}^c D^{\beta_1} u(x), \ldots, {}^c D^{\beta_n} u(x)) \\ &- f(x, v(x), (\phi v)(x), (\psi v)(x), \\ &{}^c D^{\beta_1} v(x), \ldots, {}^c D^{\beta_n} v(x))| dx\Big)\Big) \\ &+ k \int_0^{t} e^{-k(t-s)} \Big(\int_0^{s} \frac{(s-x)^{q-2}}{\Gamma(q-1)} \\ &\times |f(x, u(x), (\phi u)(x), (\psi u)(x), \\ &{}^c D^{\beta_1} u(x), \ldots, {}^c D^{\beta_n} u(x)) \\ &- f(x, v(x), (\phi v)(x), (\psi v)(x), \\ &{}^c D^{\beta_1} v(x), \ldots, {}^c D^{\beta_n} v(x))| dx\Big) ds \\ &\le (P\Delta_1 + \frac{1}{k\Gamma(q)}(1-e^{-k}))(L_1|u-v| \\ &+ L_2|\phi u - \phi v| + L_3|\psi u - \psi v| \\ &+ d_1|{}^c D^{\beta_1} u - {}^c D^{\beta_1} v| + \ldots \\ &+ d_n|{}^c D^{\beta_n} u - {}^c D^{\beta_n} v|) \\ &\le (P\Delta_1 + \frac{1}{k\Gamma(q)}(1-e^{-k})) \\ &(\sup\{L_1, d_1, \ldots, d_n\}\|u-v\| \\ &+ \sup\{L_2, L_3\}(\gamma_0 + \lambda_0)\|u-v\|) \\ &\le (P\Delta_1 + \frac{1}{k\Gamma(q)}(1-e^{-k})) \\ &\sup\{\zeta_1, \zeta_2\}(1+\gamma_0+\lambda_0)\|u-v\| \\ &\le (P\Delta_1 + \frac{1}{k\Gamma(q)}(1-e^{-k}))\zeta(1+\gamma_0+\lambda_0)\|u-v\| \\ &\le \Lambda_1 \omega \|u-v\|.\end{aligned}$$

Also, we have

$$|(Fu)'(t) - (Fv)'(t)| \le \Lambda_2 \omega \|u-v\|.$$

Which implies that

$$\begin{aligned}&|{}^c D^{\beta_i}(Fu)(t) - {}^c D^{\beta_i}(Fv)(t)| \le \\ &\int_0^{t} \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \\ &|(Fu)'(s) - (Fv)'(s)| ds \\ &\le \frac{1}{\Gamma(2-\beta_i)} \Lambda_2 \omega \|u-v\|.\end{aligned}$$

From the above inequalities, we have

$$\begin{aligned}\|F(u) - F(v)\| &= \|F(u) - F(v)\| \\ &+ \sum_{i=1}^{n} \|{}^c D^{\beta_i} F(u) - {}^c D^{\beta_i} F(v)\| \\ &\le \Lambda_1 \omega \|u-v\| \\ &+ \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)} \Lambda_2 \omega \|u-v\|. \\ &\le \left(\Lambda_1 + \Lambda_2 \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)}\right) \omega \|u-v\|.\end{aligned}$$

As : $\left(\Lambda_1 + \Lambda_2 \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)}\right) \omega < 1$, F is a contraction. Thus the conclusion of the theorem follows by the contraction mapping principle. This completes the proof.

Now, we prove the existence of solutions of (1) by applying Krasnoselskii's fixed point theorem [1].

Theorem 3.2. Let M be a closed, convex, bounded and nonempty subset of a Banach space X. Let A, B be the operators such that : (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compcat and continuous; and (iii) B is a contraction mapping. Then, there exists $z \in M$ such that : $z = Az + Bz$.

Theorem 3.3. Let $f : [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ be a continuous function satisfying the assumptions (H_1) and

$(H_2)|f(t,x,y,w,u_1,\dots,u_n)| \leq \mu(t), \quad \forall (t,x,y,w,u_1,\dots,u_n) \in [0,1]\times\mathbb{R}^{n+3}$, where $\mu \in C([0,1],\mathbb{R}^+)$.

Then, the boundary value problem (1) has at least one solution on $[0,1]$.

If $\left(P+\tilde{P}\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\right)\Delta_1\omega<1$.

Proof. Let $\|\mu\|=\sup_{t\in[0,1]}|\mu(t)|$, and consider $B_R=\{u\in X:\|u\|\leq R\}$, we fix $R\geq\left(\Lambda_1+\Lambda_2\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\right)\|\mu\|$. We define the operators F_1 and F_2 on B_R as

$$(F_1u)(t)=\int_0^t e^{-k(t-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}f(x,u(x),(\phi u)(x),(\psi u)(x),{}^cD^{\beta_1}u(x),\dots,{}^cD^{\beta_n}u(x))dx\Big)ds,$$

$$\begin{aligned}(F_2u)(t)=\frac{(e^{-kt}-1)}{\Delta}\Big(&\beta\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}\Big(\int_0^s e^{-k(s-x)}\Big(\int_0^x\frac{(x-\tau)^{q-2}}{\Gamma(q-1)}\\&\times f(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),\dots,{}^cD^{\beta_n}u(\tau))d\tau\Big)dx\Big)ds\\&-\sum_{i=1}^{m-1}a_i\int_0^{\xi_i}e^{-k(\xi_i-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}\\&\times f(x,u(x),(\phi u)(x),(\psi u)(x),{}^cD^{\beta_1}u(x),\dots,{}^cD^{\beta_n}u(x))dx\Big)ds\Big).\end{aligned}$$

For $u,v\in B_R$, using the notation (13), we have

$$\begin{aligned}|(F_1u)(t)+(F_2v)(t)|&\leq(P\Delta_1+\frac{1}{k\Gamma(q)}(1-e^{-k}))\|\mu\|\\&\leq\Lambda_1\|\mu\|.\end{aligned}$$

Also

$$\begin{aligned}|(F_1u)'(t)+(F_2v)'(t)|&\leq(\tilde{P}\Delta_1+\frac{1}{\Gamma(q)}(2-e^{-k}))\|\mu\|\\&\leq\Lambda_2\|\mu\|,\end{aligned}$$

which implies that

$$\begin{aligned}|{}^cD^{\beta_i}(F_1u+F_2v)|&\leq\int_0^t\frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}|F_1'u+F_2'v|ds\\&\leq\frac{\Lambda_2}{\Gamma(2-\beta_i)}\|\mu\|.\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}\|F_1u+F_2v\|&=\|F_1u+F_2v\|+\sum_{i=1}^{n}\|{}^cD^{\beta_i}F_1u+{}^cD^{\beta_i}F_2v\|\\&\leq\Lambda_1\|\mu\|+\Lambda_2\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\|\mu\|\\&\leq\left(\Lambda_1+\Lambda_2\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\right)\|\mu\|\leq R.\end{aligned}$$

Thus, $F_1u+F_2v\in B_R$, we prove that F_2 is a contraction mapping.

Let $u,v\in B_R$, we have

$$\begin{aligned}|F_2u(t)-F_2v(t)|\leq&\sup_{t\in[0,1]}\frac{|e^{-kt}-1|}{|\Delta|}\Big(|\beta|\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}\Big(\int_0^s e^{-k(s-x)}\Big(\int_0^x\frac{(x-\tau)^{q-2}}{\Gamma(q-1)}\\&\times|f(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),({}^cD^{\beta_1}u)(\tau),\dots,({}^cD^{\beta_n}u)(\tau)\\&-f(\tau,v(\tau),(\phi v)(\tau),(\psi v)(\tau),({}^cD^{\beta_1}v)(\tau),\dots,({}^cD^{\beta_n}v)(\tau)d\tau\Big)dx\Big)ds\\&-\sum_{i=1}^{m-1}|a_i|\int_0^{\xi_i}e^{-k(\xi_i-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}\\&\times(|f(x,u(x),(\phi u)(x),(\psi u)(x),{}^cD^{\beta_1}u(x),\dots,{}^cD^{\beta_n}u(x))\\&-f(x,v(x),(\phi v)(x),(\psi v)(x),{}^cD^{\beta_1}v(x),\dots,{}^cD^{\beta_n}v(x))|dx\Big)ds\Big)\\&\leq P\Delta_1\zeta(1+\gamma_0+\lambda_0)\|u-v\|\\&\leq P\Delta_1\omega\|u-v\|.\end{aligned}$$

Also,

$$\begin{aligned}|F_2'u(t)-F_2'v(t)|&\leq\tilde{P}\Delta_1\zeta(1+\gamma_0+\lambda_0)\|u-v\|\\&\leq\tilde{P}\Delta_1\omega\|u-v\|,\end{aligned}$$

which implies that

$$\begin{aligned}&|{}^cD^{\beta_i}F_2u(t)-{}^cD^{\beta_i}F_2v(t)|\leq\\&\int_0^t\frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\tilde{P}\Delta_1\omega\|u-v\|ds\\&\leq\frac{1}{\Gamma(2-\beta_i)}\tilde{P}\Delta_1\omega\|u-v\|.\end{aligned}$$

From the above inequalities, we have

$$\begin{aligned}\|F_2u-F_2v\| &= \|F_2u-F_2v\| \\ &+\sum_{i=1}^{n}\|^cD^{\beta_i}F_2u-{}^cD^{\beta_i}F_2v\| \\ &\leq P\Delta_1\omega\|u-v\| \\ &+\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\tilde{P}\Delta_1\omega\|u-v\| \\ &\leq (P+\tilde{P}\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)})\Delta_1\omega\|u-v\|.\end{aligned}$$

As : $\left(P+\tilde{P}\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\right)\Delta_1\omega<1$,
F_2 is contraction mapping.

We prove the continuity of F_1. Let $\{u_n\}_{n=0}^{\infty}\subset M$ and $u\in M$ such that $u_n\longrightarrow u$ as $n\longrightarrow\infty$, so continuity of f implies that

$$\begin{aligned}&\lim_{n\longrightarrow+\infty}f(t,u_n(t),(\phi u_n)(t),(\psi u_n)(t),\\ &{}^cD^{\beta_1}u_n(t),\dots,{}^cD^{\beta_n}u_n(t))\\ &=f(t,u(t),(\phi u)(t),(\psi u)(t),\\ &{}^cD^{\beta_1}u(t),\dots,{}^cD^{\beta_n}u(t)).\end{aligned}$$

Then,

$$\begin{aligned}&\left|F_1u_n(t)-F_1u(t)\right| =\\ &\Big|\int_0^t e^{-k(t-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}\\ &\times(f(t,u_n(t),(\phi u_n)(t),(\psi u_n)(t),\\ &{}^cD^{\beta_1}u_n(t),\dots,{}^cD^{\beta_n}u_n(t)))\\ &-f(t,u(t),(\phi u)(t),(\psi u)(t),\\ &{}^cD^{\beta_1}u(t),\dots,{}^cD^{\beta_n}u(t))dx\Big)ds\Big|\\ &\leq\int_0^t e^{-k(t-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}\\ &\times|f(t,u_n(t),(\phi u_n)(t),(\psi u_n)(t),\\ &{}^cD^{\beta_1}u_n(t),\dots,{}^cD^{\beta_n}u_n(t))\\ &-f(t,u(t),(\phi u)(t),(\psi u)(t),\\ &{}^cD^{\beta_1}u(t),\dots,{}^cD^{\beta_n}u(t))|dx\Big)ds\\ &\leq\frac{1}{k\Gamma(q)}(1-e^{-k})\\ &\times|f(t,u_n(t),(\phi u_n)(t),(\psi u_n)(t),\\ &{}^cD^{\beta_1}u_n(t),\dots,{}^cD^{\beta_n}u_n(t))\\ &-f(t,u(t),(\phi u)(t),(\psi u)(t),\\ &{}^cD^{\beta_1}u(t),\dots,{}^cD^{\beta_n}u(t))|.\end{aligned}$$

Also, we have

$$\begin{aligned}&|F_1'u_n(t)-F_1'u(t)|=|\int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}\\ &\times(f(s,u_n(s),(\phi u_n)(s),(\psi u_n)(s),\\ &{}^cD^{\beta_1}u_n(s),\dots,{}^cD^{\beta_n}u_n(s))\\ &-f(s,u(s),(\phi u)(s),(\psi u)(s),\\ &{}^cD^{\beta_1}u(s),\dots,{}^cD^{\beta_n}u(s))ds|\\ &-k\int_0^t e^{-k(t-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}\\ &\times(f(s,u_n(s),(\phi u_n)(s),(\psi u_n)(s),\\ &{}^cD^{\beta_1}u_n(s),\dots,{}^cD^{\beta_n}u_n(s))\\ &-f(s,u(s),(\phi u)(s),(\psi u)(s),\\ &cD^{\beta_1}u(s),\dots,{}^cD^{\beta_n}u(s)))dx\Big)ds\Big|\\ &\leq\frac{(2-e^{-k})}{\Gamma(q)}\Big|f(s,u_n(s),(\phi u_n)(s),(\psi u_n)(s),\\ &{}^cD^{\beta_1}u_n(s),\dots,{}^cD^{\beta_n}u_n(s))\\ &-f(s,u(s),(\phi u)(s),(\psi u)(s),\\ &{}^cD^{\beta_1}u(s),\dots,{}^cD^{\beta_n}u(s))\Big|.\end{aligned}$$

By the definition of the Caputo fractional derivative with $0<\beta_i<1$, we get

$$\begin{aligned}&\left|{}^cD^{\beta_i}F_1u_n(t)-{}^cD^{\beta_i}F_1u(t)\right|\leq\\ &\int_0^t\frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\frac{(2-e^{-k})}{\Gamma(q)}\\ &\times|f(s,u_n(s),(\phi u_n)(s),(\psi u_n)(s),\\ &{}^cD^{\beta_1}u_n(s),\dots,{}^cD^{\beta_n}u_n(s))\\ &-f(s,u(s),(\phi u)(s),(\psi u)(s),\\ &{}^cD^{\beta_1}u(s),\dots,{}^cD^{\beta_n}u(s))|ds\\ &\leq\frac{(2-e^{-k})}{\Gamma(q)\Gamma(2-\beta_i)}\\ &\times|f(s,u_n(s),(\phi u_n)(s),(\psi u_n)(s),\\ &{}^cD^{\beta_1}u_n(s),\dots,{}^cD^{\beta_n}u_n(s))\\ &-f(s,u(s),(\phi u)(s),(\psi u)(s),\\ &{}^cD^{\beta_1}u(s),\dots,{}^cD^{\beta_n}u(s))|.\end{aligned}$$

Hence,

$$\|F_1u_n(t)-F_1u(t)\| = \|F_1u_n(t)-F_1u(t)\| + \sum_{i=1}^{n}\|{}^cD^{\beta_i}F_1u_n(t) - {}^cD^{\beta_i}F_1u(t)\|$$
$$\leq \frac{1}{k\Gamma(q)}(1-e^{-k})\,|\,f(s,u_n(s),(\phi u_n)(s),(\psi u_n)(s), {}^cD^{\beta_1}u_n(s),\ldots,{}^cD^{\beta_n}u_n(s)) - f(s,u(s),(\phi u)(s),(\psi u)(s), {}^cD^{\beta_1}u(s),\ldots,{}^cD^{\beta_n}u(s))\,|$$
$$+\sum_{i=1}^{n}\frac{(2-e^{-k})}{\Gamma(q)\Gamma(2-\beta_i)} \times|f(s,u_n(s),(\phi u_n)(s),(\psi u_n)(s), {}^cD^{\beta_1}u_n(s),\ldots,{}^cD^{\beta_n}u_n(s)) - f(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s), \ldots,{}^cD^{\beta_n}u(s))| \longrightarrow 0,$$

it follows that $\|F_1u_n(t)-F_1u(t)\|\longrightarrow 0$ as $n\longrightarrow\infty$, which implies the continuity of F_1.
Also, F_1 is uniformly bounded on B_R as

$$\|F_1u\|\leq\frac{(1-e^{-k})}{k\Gamma(q)}\|\mu\|,$$

$$\|F_1'u\|\leq\frac{(2-e^{-k})}{\Gamma(q)}\|\mu\|,$$

$$\|{}^cD^{\beta_i}F_1u\|\leq\frac{1}{\Gamma(2-\beta_i)}\frac{(2-e^{-k})}{\Gamma(q)}\|\mu\|,$$

and

$$\|F_1u\|\leq\frac{(1-e^{-k})}{k\Gamma(q)}\|\mu\|+\frac{(2-e^{-k})}{\Gamma(q)}\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\|\mu\|.$$

Now, we prove the compactness of the operator F_1, we define

$$C_0 = \sup_{(t,u,\psi u,\phi u,{}^cD^{\beta_1},\ldots,{}^cD^{\beta_n})\in[0,1]\times B_r}|f(t,u,(\phi u)(t),(\psi u)(t), {}^cD^{\beta_1}u(t),\ldots,{}^cD^{\beta_n}u(t)|.$$

For $0<t_1<t_2<1$, we have

$$\Big|(F_1u)(t_2)-(F_1u)(t_1)\Big| = \Big|\int_0^{t_2}e^{-k(t_2-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)} \times f(x,u(x),(\phi u)(x),(\psi u)(x), {}^cD^{\beta_1}u(x),\ldots,{}^cD^{\beta_n}u(x)dx\Big)ds$$
$$-\int_0^{t_1}e^{-k(t_1-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)} \times f(x,u(x),(\phi u)(x),(\psi u)(x),{}^cD^{\beta_1}u(x), \ldots,{}^cD^{\beta_n}u(x)dx\Big)ds\Big|$$
$$\leq\int_0^{t_1}|e^{-k(t_2-s)}-e^{-k(t_1-s)}|\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)} \times|f(x,u(x),(\phi u)(x),(\psi u)(x),{}^cD^{\beta_1}u(x), \ldots,{}^cD^{\beta_n}u(x)|dxds$$
$$+\int_{t_1}^{t_2}e^{-k(t_2-s)}\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)} \times|f(x,u(x),(\phi u)(x),(\psi u)(x),{}^cD^{\beta_1}u(x), \ldots,{}^cD^{\beta_n}u(x)|dxds$$
$$\leq\frac{C_0}{k\Gamma(q)}\Big(|t_2^q-t_1^q|+|t_2^qe^{-kt_2}-t_1^qe^{-kt_1}|\Big)$$

and :

$$|{}^cD^{\beta_i}F_1u(t_2)-{}^cD^{\beta_i}F_1u(t_1)|\leq \int_0^{t_1}\frac{(t_2-s)^{-\beta_i}-(t_1-s)^{-\beta_i}}{\Gamma(1-\beta_i)}|(F_1u)'(s)|ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}|(F_1u)'(s)|ds$$
$$\leq\frac{C_0}{\Gamma(1-\beta_i)}\frac{(2-e^{-k})}{\Gamma(q)}\Big\{\int_0^{t_1}\frac{|(t_1-s)^{\beta_i}-(t_2-s)^{\beta_i}|}{(t_1-s)^{\beta_i}(t_2-s)^{\beta_i}}ds + \int_{t_1}^{t_2}(t_2-s)^{-\beta_i}ds\Big\}$$

Clearly, $|F_1u(t_2)-F_1u(t_1)|\longrightarrow 0$ and $|{}^cD^{\beta_i}F_1u(t_2)-{}^cD^{\beta_i}F_1u(t_1)|\longrightarrow 0$ independent of u as $t_2\longrightarrow t_1$. Thus, F_1 is relatively compact on B_R. Hence, by the Arzela-Ascoli theorem, F_1 is compact on B_R. Thus all the assumptions of theorem (3.2) are satisfied and the conclusion of theorem (3.2) implies that the boundary value problem (1) has at least one solution on $[0,1]$. This completes the proof.

Our next existence result is based on the following fixed point theorem [1].

Theorem 3.4. Let E be a Banach space. Assume that $F:E\longrightarrow E$ is completely continuous operator and the set $\Omega=\{u\in E: u=\mu Fu,\quad 0<\mu<1\}$ is bounded. Then F has a fixed point in E.

Theorem 3.5. Assume that there exists $C>0$ such that

$$|f(t,u(t),(\phi u)(t),(\psi u)(t),{}^{c}D^{\beta_1}u(t),\ldots,{}^{c}D^{\beta_n}u(t))| \leq C,$$
$$\forall t\in[0,1],(u,\phi u,\psi u,{}^{c}D^{\beta_1}u,\ldots,{}^{c}D^{\beta_n}u)\in\mathbb{R}^{n+3}.$$

Then the problem (1) has at least one solution on $[0,1]$.

Proof. We show that the operator F is completely continuous. Let B be a bounded set in X. Then, there exists $C>0$ such that $|f(t,u(t),(\phi u)(t),(\psi u)(t),{}^{c}D^{\beta_1}u(t),\ldots,{}^{c}D^{\beta_n}u(t))| \leq C, \forall t\in[0,1], u\in B$, we get:

$$\begin{aligned}|(Fu)(t)| &\leq (P\Delta_1+\frac{1}{k\Gamma(q)}(1-e^{-k}))C\\ &\leq \Lambda_1 C.\end{aligned}$$

Also

$$\begin{aligned}|(Fu)'(t)| &\leq (\tilde{P}\Delta_1+\frac{1}{\Gamma(q)}(2-e^{-k}))C\\ &\leq \Lambda_2 C,\end{aligned}$$

which implies that

$$\begin{aligned}\left|{}^{c}D^{\beta_i}(Fu)(t)\right| &= \Big|\int_0^t\frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fu)'(s)ds\Big|\\ &\leq \frac{1}{\Gamma(2-\beta_i)}\Lambda_2 C.\end{aligned}$$

From the above inequalities, we get

$$\|Fu\|_X \leq \Lambda_1 C+\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\Lambda_2 C<\infty,$$

which implies that $\|Fu\|\leq\infty$. Hence $F(B)$ is uniformly bounded.

It is easy to verify that the operator F is continuous since f is continuous. Next, we show that F is equicontinuous on bounded subsets of X.

Now, let $t_1,t_2\in[0,1], t_1<t_2; u\in B$, we have the following facts:

$$\begin{aligned}|Fu(t_2)-Fu(t_1)| \leq{}& \frac{|e^{-kt_2}-e^{-kt_1}|}{|\Delta|}\Big(|\beta|\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}\\ &\Big(\int_0^s e^{-k(s-x)}\Big(\int_0^x\frac{(x-\tau)^{q-2}}{\Gamma(q-1)}\\ &\times|f(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),\\ &{}^{c}D^{\beta_1}u(\tau),\ldots,{}^{c}D^{\beta_n}u(\tau)|d\tau\Big)dx\Big)ds\\ &+\sum_{i=1}^{m-1}|a_i|\int_0^{\xi_i}e^{-k(\xi_i-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}\\ &\times|f(x,u(x),(\phi u)(x),(\psi u)(x),\\ &{}^{c}D^{\beta_1}u(x),\ldots,{}^{c}D^{\beta_n}u(x)|dx\Big)ds\\ &+\int_0^{t_1}|e^{-k(t_2-s)}-e^{-k(t_1-s)}|\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}\\ &\times|f(x,u(x),(\phi u)(x),(\psi u)(x),\\ &{}^{c}D^{\beta_1}u(x),\ldots,{}^{c}D^{\beta_n}u(x)|dx\Big)ds\Big)\\ &+\int_{t_1}^{t_2}e^{-k(t_2-s)}\Big(\int_0^s\frac{(s-x)^{q-2}}{\Gamma(q-1)}\\ &\times|f(x,u(x),(\phi u)(x),(\psi u)(x),\\ &{}^{c}D^{\beta_1}u(x),\ldots,{}^{c}D^{\beta_n}u(x)|dx\Big)ds\\ \leq{}& \frac{|e^{-kt_2}-e^{-kt_1}|}{|\Delta|}\{|\beta|\frac{\eta^{2q-2}}{(k\Gamma(q))^2}(k\eta+e^{-k\eta}-1)\\ &+\sum_{i=1}^{m-1}|a_i|\xi_i^{q-1}(1-e^{-k\xi_i})\frac{1}{k\Gamma(q)}\}C\\ &+\frac{C}{k\Gamma(q)}(|t_2^q-t_1^q|+|t_2^q e^{-kt_2}-t_1^q e^{-kt_1}|)\\ \leq{}& \Big(\frac{|e^{-kt_2}-e^{-kt_1}|}{|\Delta|}\Delta_1+\frac{1}{k\Gamma(q)}(|t_2^q-t_1^q|+\\ &|t_2^q e^{-kt_2}-t_1^q e^{-kt_1}|)\Big)C.\end{aligned}$$

Also, we have

$$\begin{aligned}&|{}^{c}D^{\beta_i}(Fu)(t_2)-{}^{c}D^{\beta_i}(Fu)(t_1)| =\\ &\Big|\int_0^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fu)'(s)ds\\ &-\int_0^{t_1}\frac{(t_1-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fu)'(s)ds\Big|\\ &\leq\int_0^{t_1}\frac{|(t_2-s)^{-\beta_i}-(t_1-s)^{-\beta_i}|}{\Gamma(1-\beta_i)}|(Fu)'(s)|ds\\ &+\int_{t_1}^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}|(Fu)'(s)|ds\\ &\leq\frac{C\Lambda_2}{\Gamma(1-\beta_i)}\Big\{\int_0^{t_1}\frac{|(t_1-s)^{\beta_i}-(t_2-s)^{\beta_i}|}{(t_1-s)^{\beta_i}(t_2-s)^{\beta_i}}\\ &+\int_{t_1}^{t_2}|(t_2-s)^{-\beta_i}|ds\Big\}.\end{aligned}$$

From the above inequalities, we get

$$\|Fu\| \leq \Big(\frac{|e^{-kt_2}-e^{-kt_1}|}{|\Delta|}\Delta_1 + \frac{1}{k\Gamma(q)}(|t_2^q - t_1^q|+ |t_2^q e^{-kt_2} - t_1^q e^{-kt_1}|)\Big)C + \sum_{i=1}^{n}\Big(\frac{C\Lambda_2}{\Gamma(1-\beta_i)}\Big\{\int_0^{t_1}\frac{|(t_1-s)^{\beta_i}-(t_2-s)^{\beta_i}|}{(t_1-s)^{\beta_i}(t_2-s)^{\beta_i}} + \int_{t_1}^{t_2}|(t_2-s)^{-\beta_i}|ds\Big\}\Big) \longrightarrow 0 \quad as \quad t_2 \longrightarrow t_1.$$

Thus, the operator F is equicontinuous. Hence, by Arzela-Ascoli theorem,
$F : X \longrightarrow X$ is completely continuous.

Now, consider the set $\Omega = \{u \in X : u = \mu F u, \quad 0 < \mu < 1\}$. In order to show that Ω is bounded, let $u \in \Omega$, then $u = \mu F u, 0 < \mu < 1$, for $t \in [0,1]$, we have

$$\begin{aligned}|u(t)| &= |\mu(Fu)(t)| \\ &\leq \mu|(Fu)(t)| \\ &\leq \Big(\Lambda_1 + \Lambda_2\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\Big)C < \infty.\end{aligned}$$

It follows that the set Ω is bounded independently of μ. From theorem (3.4), the operator F has at least a fixed point, which implies that the problem (1) has at least one solution on $[0,1]$.

4 Examples

*Example 1.*Consider the four-point problem

$$\begin{cases}({}^cD^{\frac{7}{4}} + \frac{1}{4}{}^cD^{\frac{3}{4}})u(t) = \omega_1, \quad t \in [0,1] \\ u(0) = 0, \\ \frac{1}{6}u(\frac{1}{10}) + \frac{1}{3}u(\frac{2}{10}) + \frac{1}{5}u(\frac{3}{10}) = \int_0^{\frac{1}{20}}\frac{(\frac{1}{20}-s)^{\frac{3}{4}}}{\Gamma(\frac{7}{4})}u(s)ds.\end{cases}$$

Where $\omega_1 = f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{3}}u(t),{}^cD^{\frac{2}{3}}u(t))$, $q = \frac{7}{4}, k = \frac{1}{4}, \beta_1 = \frac{1}{3}, \beta_2 = \frac{2}{3}, a_1 = \frac{1}{6}, a_2 = \frac{1}{3}, a_3 = \frac{1}{5}$, $\xi_i = \frac{i}{10}, i = 1,2,3, \beta = 1, \eta = \frac{1}{20}$, $(\phi u)(t) = \int_0^t \frac{e^{-2(s-t)}}{2}u(s)ds$ and $(\psi u)(t) = \int_0^t \frac{e^{-3(s-t)}}{2}u(s)ds$ with $\gamma_0 = \frac{e^2-1}{4}$ and $\lambda_0 = \frac{e^3-1}{6}$, with the given value of f has at least one solution on $[0,1]$, we choose

$$f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{3}}u(t),{}^cD^{\frac{2}{3}}u(t)) = \frac{|u(t)|}{36(1+|u(t)|)} + \frac{e^{-\frac{\pi}{2}t}\sin(\frac{\pi}{2}t)}{64(1+t^3)}\Big((\phi u)(t) + \frac{|{}^cD^{\frac{1}{3}}u(t)|}{1+|{}^cD^{\frac{1}{3}}u(t)|}\Big) + \frac{\cos t + e^t}{81(1+t^2)}\Big((\psi u)(t) + \frac{|{}^cD^{\frac{2}{3}}u(t)|}{1+|{}^cD^{\frac{2}{3}}u(t)|}\Big),$$

we have that

$$|f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{3}}u(t),{}^cD^{\frac{2}{3}}u(t)) - f(t,v(t),(\phi v)(t),(\psi v)(t),{}^cD^{\frac{1}{3}}v(t),{}^cD^{\frac{2}{3}}v(t))| \leq \frac{1}{36}|u(t)-v(t)| + \frac{1}{64}|\phi u(t) - \phi v(t)| + \frac{1}{81}|\psi u(t) - \psi v(t)| + \frac{1}{64}|{}^cD^{\frac{1}{3}}u(t) - {}^cD^{\frac{1}{3}}v(t)| + \frac{1}{81}|{}^cD^{\frac{2}{3}}u(t) - {}^cD^{\frac{2}{3}}v(t)|,$$

with the given values, it is found that

$\zeta_1 = \frac{1}{36}, \zeta_2 = \frac{1}{64}, \zeta = \frac{1}{36}, \Delta \approx -3.4808 \times 10^{-2} \neq 0$,
$\Delta_1 \approx 2.5495 \times 10^{-2}$,

$\Lambda_1 \approx 1.1247, \Lambda_2 \approx 1.4714, \omega \approx 0.16051$,

finally we have that $\Big(\Lambda_1 + \Lambda_2\sum_{i=1}^{2}\frac{1}{\Gamma(2-\beta_i)}\Big)\omega = 0.70663 < 1$. According to all the conditions of theorem (3.1), we conclude that there exists a unique solution for the problem (1) on $[0,1]$.

*Example 2.*Consider the problem

$$\begin{cases}({}^cD^{\frac{9}{5}} + \frac{1}{11}{}^cD^{\frac{4}{5}})u(t) = \omega_2, \quad t \in [0,1] \\ u(0) = 0, \\ \frac{1}{6}u(\frac{1}{4}) + \frac{1}{3}u(\frac{1}{3}) + \frac{1}{5}u(\frac{1}{5}) = \int_0^{\frac{1}{30}}\frac{(\frac{1}{30}-s)^{\frac{4}{5}}}{\Gamma(\frac{9}{5})}u(s)ds,\end{cases}$$

where $\omega_2 = f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t)), q = \frac{9}{5}, k = \frac{1}{11}, a_1 = \frac{1}{6}, a_2 = \frac{1}{3}, a_3 = \frac{1}{5}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{3}, \xi_3 = \frac{1}{2}, \eta = \frac{1}{30}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{3}{4}$, $\beta = 1, (\phi u)(t) = \int_0^t \frac{e^{-2(s-t)}}{2}u(s)ds$ and

$(\psi u)(t) = \int_0^t \frac{e^{-3(s-t)}}{2}u(s)ds$ with $\gamma_0 = \frac{e^2-1}{4}$ and $\lambda_0 = \frac{e^3-1}{6}$. Now we illustrate the obtained results by choosing different values of $f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t))$. Let us first consider

$$f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t)) = \frac{e^{-\pi t}(1+\sin^2(\pi t))|u(t)|}{6(t+6)^3(1+|u(t)|)} + \frac{e^{-\pi t}\sin(\pi t)}{7(1+t^2)}\Big((\phi u)(t)+\frac{|{}^cD^{\frac{1}{2}}u(t)|}{1+|{}^cD^{\frac{1}{2}}u(t)|}\Big) + \frac{1+\sin^2(\pi t)}{8(1+t^2)}\Big((\psi u)(t)+\frac{|{}^cD^{\frac{3}{4}}u(t)|}{1+|{}^cD^{\frac{3}{4}}u(t)|}\Big).$$

Then,

$$\begin{aligned}&|f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t)) - f(t,v(t),(\phi v)(t),(\psi v)(t),{}^cD^{\frac{1}{2}}v(t),{}^cD^{\frac{3}{4}}v(t))|\\ &\le \frac{1}{6}|u(t)-v(t)| + \frac{1}{7}|\phi u(t)-\phi v(t)| + \frac{1}{8}|\psi u(t)-\psi v(t)| + \frac{1}{7}|{}^cD^{\frac{1}{2}}u(t)-{}^cD^{\frac{1}{2}}v(t)| + \frac{1}{8}|{}^cD^{\frac{3}{4}}u(t)-{}^cD^{\frac{3}{4}}v(t)|.\end{aligned}$$

Finally, we have that $\zeta = \frac{1}{6}, \Delta \approx -2.2581\times 10^{-2}, \Delta_1 \approx 6.0286\times 10^{-2}, P \approx 3.8483, \tilde{P} \approx 3.6761, \omega \approx 0.96303$ Thus, $\Big(P+\tilde{P}\sum_{i=1}^{2}\frac{1}{\Gamma(2-\beta_i)}\Big)\Delta_1\omega = 0.69970 < 1$. Hence by theorem (3.3), the problem (1) has at least one solution on $[0,1]$.

Next, we show the applicability of theorem (3.3) with the nonlinear function f given by

$$f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t)) = \frac{e^{-\pi t}(2+|u(t)|)}{t+2+2|u(t)|}e^{3t} + \frac{e^{-t}\sin(\pi t)}{1+t^2}\Big(\frac{|(\phi u)(t)|}{1+|(\phi u)(t)|}+\frac{|{}^cD^{\frac{1}{2}}u(t)|}{1+|{}^cD^{\frac{1}{2}}u(t)|}\Big)e^{3t} + \frac{1+\sin^2 t}{1+t^2}\Big(\frac{|(\psi u)(t)|}{1+|(\psi u)(t)|}+\frac{|{}^cD^{\frac{3}{4}}u(t)|}{1+|{}^cD^{\frac{3}{4}}u(t)|}\Big)e^{3t}.$$

It is easy to see that $|f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t))| \le 7e^{3t} = \mu(t)$. Then, by the condition (H_2) with $\mu(t) = 7e^{3t}$. In consequence, the conclusion of theorem (3.3) implies that the problem (1) has at least one solution on $[0,1]$.

*Example 3.*Consider the following fractional four-point boundary problem

$$\begin{cases}({}^cD^{\frac{9}{5}}+\frac{1}{11}{}^cD^{\frac{4}{5}})u(t) = \omega_3, \quad t\in[0,1]\\ u(0)=0,\\ \frac{1}{6}u(\frac{1}{4})+\frac{1}{3}u(\frac{1}{3})+\frac{1}{5}u(\frac{1}{5}) = \int_0^{\frac{1}{30}}\frac{(\frac{1}{30}-s)^{\frac{4}{5}}}{\Gamma(\frac{9}{5})}u(s)ds,\end{cases}$$

where $\omega_3 = f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t)), q = \frac{9}{5}, k = \frac{1}{11}, a_1 = \frac{1}{6}, a_2 = \frac{1}{3}, a_3 = \frac{1}{5}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{3}, \xi_3 = \frac{1}{2}, \eta = \frac{1}{30}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{3}{4}, \beta = 1,$
which follows that

$$f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t)) = \frac{e^{-\cos^2 u(t)}}{3+\cos u(t)} + \frac{e^{-\pi t}\sin(\pi t)}{2+\cos(\pi t)}\Big(\frac{|(\phi u)(t)|}{1+|(\phi u)(t)|} + \frac{|{}^cD^{\frac{1}{2}}u(t)|}{1+|{}^cD^{\frac{1}{2}}u(t)|}\Big) + \frac{1}{2+\sin(\pi t)}\Big(\frac{|(\psi u)(t)|}{1+|(\psi u)(t)|}+\frac{|{}^cD^{\frac{3}{4}}u(t)|}{1+|{}^cD^{\frac{3}{4}}u(t)|}\Big),$$

we have that
$|f(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t))| \le \frac{9}{2}$.
Therefore, by theorem (3.5), the problem (1) has at least one solution on $[0,1]$.

5 Conclusion

We have presented some new treatments non-local multi-point boundary value problems of Caputo type sequential fractional integro-differential equations. the fixed point theory has been used to examine the existence and uniqueness results. We presented some applications to illustrate the power of the obtained results on quantum information theory is discussed. Also, to catch more interesting applications of the present research work, new examinations and applications can be investigated using master equation of more complicated quantum systems. These new discussions will be presented in a future research conducted by the author of the present paper.

Conflict of Interest

The authors declare that they have no conflict of interest.

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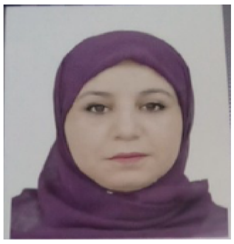

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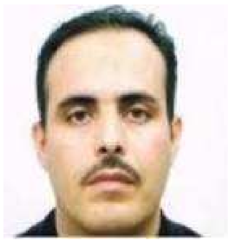

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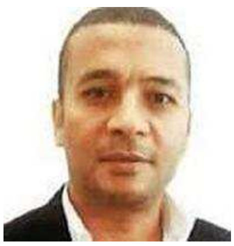

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