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Estimation of Parameters of Burr Distribution under SSALT

M. O. Mohamed[1,*] *and Esraa Mohamed*[2]

[1]Mathematics Department, Zagazig University, Faculty of Science, Egypt
[2]Math Statistical Deparment, Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt

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Abstract: Reliability of a system under an accelerated life of step-stress with a progressive first failure sample of Burr distribution is investigated. The Burr distribution has an importance in applied engineering. Under normal circumstances, it is difficult to measure the lifetimes of the product to study the validity function. Failure rarely occurs, so we resort to the process of accelerating failure using accelerated life tests. Great results were scheduled in paper obtained using point-estimation methods such as the maximum likelihood estimator function (MLEs). Also, the mean square error (MSEs) is calculated from the estimated values. Confidence intervals were obtained with different confidence levels to study different cases of confidence intervals. The present paper asserts that the greater the size of the sample, the better and more specific the results are to the real values.

Keywords: Burr type III distribution, SSPA life tests, Progressive first failure; Censoring schemes.

1 Introduction

The step-stress model is produced when there is a combination of reliability testing and over-stress testing. The progressive stress test aims to measure the life of the product and then perform the acceleration to reach the over stress at incremental levels to find channel failure modes. The testing advantage of many supplier-buyer relationships is how easy it is to expand the contractual specifications required to perform a progressive stress test.

When adjusting the amount according to sources of pressure (for example temperature, voltage and vibration) it will increase. Therefore, the effect of each stress source on different failure modes is accelerated at different rates, and it is important to determine a good proportional increase for each pressure source.

In models of statistical reliability literature, when the experiment tests control higher stress levels, they can make accelerating life in two ways: accelerated life test (ALT) and partially accelerated life test (PALT) studies that will be used for components or units in the life tests.

Study surviving data with reliability theory which called a lifetime of data. CDF and PDF of Burr type III distribution are denoted by $F(x)$ and $f(x)$ respectively when;

$$F(x) = (1+x^{-c})^{-k}, \tag{1}$$

$$f(x) = ckx^{-(c+1)}(1+x^{-c})^{-(k+1)}, \ \ x>0, c>0, k>0. \tag{2}$$

(see ref. [1] for more details).

For one of the most important family of distribution in statistics is the Burr family distribution which has important role in the field of life testing as it is fitted to business failure data. It also has twelve types of cumulative distribution functions that provide many curves of density shapes. Several standard theoretical distributions are special cases or limiting cases of Burr family of distributions, including the Weibull, exponential logistic, generalized logistic Gompertz, normal, extreme value, and uniform distributions. For the present study, the simple closed form of these distributions has been applied to simulation studies. Some forms of the Burr distributions are related by a simple transformation: for example, the Burr type III distribution can be obtained from Burr type II distribution by replacing X with $\ln(X)$. Similarly, if we replace X with $\left(\frac{1}{X}\right)$ in Burr type III distribution, we can derive Burr type XII distribution. The Burr type I family is known as the uniform distribution. Some types of the Burr distributions have a variety of

* Corresponding author e-mail: mo1157.mm@gmail.com

shapes for density. For example, III and XII have the simple function so these two distributions are the function which best matches statistical modeling. The Burr type III distribution is more fixable than Burr type XII distribution, see [1].

Let us present studies that examine methods of obtaining the estimators of parameters for Burr-III distribution and factors of the acceleration life with SSALT with censoring data. For more studying on models of reliability function, which is the stress-strength model, we can study article [2]. The period of time in which a product has been successfully operating over a shorter period of time can be estimated using accelerated life tests and censoring data since it is used when all values of measurement are unknown. In addition, the survival times are not always observed. Some authors have addressed this point, such as Nelson who presented statistical models and methods for analyzing accelerated life-test data from step-stress tests. These models are constant stress, the gradual stress and the stress step. For example, the continuous stress was represented by [3–5]. In [6] the optimal plans for the accelerated life-tests for continuous pressure of the Lindley distribution were obtained. [7] studied estimation in the life-step-accelerated stress test of a precipitated distribution with a progressive control of the first failure. [8] estimated the parameters of the Weibull distribution under progressive stress acceleration as well as generalized force parameters of Weibull under progressive stress acceleration. [9, 10]. Authors, like [11] and [12], obtained optimal design of simple step-stress accelerated life tests for one-shot devices under exponential distribution.

Let's have independent groups $(R_1,R_2,\cdots,R_n)$, each group has h items put in a life test. For the first group, the first observed failure is selected randomly and eliminated from the test. The same action is done for the second group. Also, the group of second failure is observed and removed randomly from the test as soon as the second failure occurred $Y_{2;m,n,h}$, and finally when the m^{th} failure is $Y_{m;m,n,h}$ observed. The remaining groups R_m when $(m \le n)$ are removed from the test. Thus, $Y^R_{1;m,n,h} < \cdots < Y^R_{m;m,n,h}$ is called progressively first-failure censored order statistics with the progressive censored scheme; $R = (R_1,R_2,\cdots,R_n)$, and $n = m + \sum_{i=1}^{m} R_i$. The failure times of the $n\times h$ items were drawn from a continuous sample with cumulative function $F(y)$ and density function $f(y)$ in Equation (1) and Equation (2), respectively.

Representation the joint probability density function for $Y^R_{1;m,n,h}, Y^R_{2;m,n,h},\cdots,Y^R_{m;m,n,h}$ by

$$f_{1,2,\cdots,n}(Y^R_{1;m,n,h},Y^R_{2;m,n,h},\cdots,Y^R_{m;m,n,h}) = A(n,m-1)km \prod_{i=1}^{m} f(y^R_{1;m,n,h})[F(y^R_{1;m,n,h})]^{k(Ri+1)-1}, \quad 0<x_1,\cdots<\infty, \tag{3}$$

When

$$A(n,m-1) = n(n-R_1-1)\cdots(n-R_1-R_2-\cdots -R_{m-1}-(m-1)). \tag{4}$$

(see ref. [13] for more details).

2 Assumptions and Notations

For model of step-stress accelerated partially life (SSPALT), we use the steps as follows:

1. If we had n groups that had independent and identical data, as well as h element that had been removed from the life test at the survive time of each unit that has Burr III (c,k) distribution.
2. Stop for the test at the m^{th} the failure, where m is prenoted $(m \le n)$.
3. First, each $n\times h$ unit was operated without any changes in normal condition. If it did not fail or it was taken out of the test by breakdown time, we would have to put it under accelerated conditions.
4. The failure i^{th} and groups of surviving R_i, $i = 1,2,...,m-1$ were random number. We randomly selected groups of failure $Y^R_{1;m,n,h}$ and removed from the test. Finally, at failure m^{th} the remaining surviving groups were removed from the test and then the test was stopped.
5. Let the number of failures at normal condition is n_1 and the number of failures after time τ within stress condition be n_2. Then the censoring data based on progressive first-failure $Y^R_{1;m,n,h} < \cdots < Y^R_{m,nm,n,h}$ when $n = m + \sum_{i=1}^{m} R_i$.
6. The resulted model of random variable (i.e tampered random variable (TRV) model) indicated the tampered random variable model at the time of survival of a unit under (SSPALT) can be expressed as: $y = \begin{cases} T & \text{if } T\le\tau \\ \tau+\frac{T-\tau}{\beta} & \text{if } T>\tau \end{cases}$, where the time of life under normal condition is T, the time of stress is τ and β is the acceleration factor $(\beta > 1)$ (see ref. [14]).
7. $f(y)$ of y based on non-normal conditions is given by;

$$f(y)\begin{cases} 0, & y<0, \\ f_1(y) = cky^{-c-1}(1+y^{-c})^{-k-1}, & 0<y<\tau, \\ f_2(y) = \beta ck(\beta(y-\tau)+\tau)^{-c-1} & \\ [1+(\beta(y-\tau)+\tau)^{-c}]^{-k-1}, & \tau<y<\infty. \end{cases}$$

3 Estimation Techniques

For estimation of parameters of Burr III distribution, we followed the point and interval estimations with data based on first failure censoring data with SSPALT.

3.1 Estimation by Point

Put $y_i = Y^R_{1;m,n,h}$ the observed values of the lifetime y obtained from a progressive first-failure censoring scheme under SSPALT with censored scheme $R = (R_1, R_2, ..., R_n)$, then the maximum likelihood function of observations is:

$$L(x,k) = Ah^m \prod_{i=1}^{n_1} f_1(y_i)[1-F_1(y_i)]^{h(R_i+1)-1} \prod_{i=n_1+1}^{m} f_2(y_i)[1-F_2(y_i)]^{h(R_i+1)-1}. \quad (5)$$

By taking log function for likelihood function, it may have the form:

$$\log L(c,k) = \log A + m\log h + m\log c + m\log k + (m-n) \log\beta - (c+1)\sum_{i=1}^{n_1}\log y_i - (k+1) \sum_{i=1}^{n_1}\log(1+y_i^{-c}) + \sum_{i=1}^{n_1}[h(R_i+1)-1] \log[1-(1+y_i^{-c})^{-k} - (c+1) \sum_{i=n_1+1}^{m} \log[\beta(y_i-\tau)+\tau] - (k-1) \sum_{i=n_1+1}^{m} \log(1+[\beta(y_i-\tau)+\tau]^{-c} + \sum_{i=n_1+1}^{m}[h(R_i+1)-1]\log(1-1 + [\beta(y_i-\tau)+\tau]^{-c})^{-k}. \quad (6)$$

We can write the first derivatives w. r. t. c and k, as follows:

$$\frac{\partial \log(c,k)}{\partial c} = \frac{m}{c} - \sum_{n=1}^{n_1}\log y_i - (k+1)\sum_{n=1}^{n_1} y_i^{-c}\ln y_i + \sum_{n=1}^{n_1}[h(R_i+1)-1]\frac{k(1+y_i^{-c})^{-k-1}y_i^{-c}\ln y_i}{1-(1+y_i^{-c})^{-k}}$$

$$- \sum_{i=n_1+1}^{m}\log[\beta(y_i-\tau)+\tau] - (k+1)\sum_{i=n_1+1}^{m}\frac{[\beta(y_i-\tau)+\tau]^{-c}\ln[\beta(y_i-\tau)\tau]}{1+[\beta(y_i-\tau)+\tau]^{-c}} - \sum_{i=n_1+1}^{m} k[h(R_i+1)-1] \frac{(1+[\beta(y_i-\tau)+\tau]^{-c})^{-k-1}[\beta(y_i-\tau)+\tau]^{-c}\ln[\beta(y_i-\tau)+\tau]}{1-(1+[\beta(y_i-\tau)+\tau]^{-c})^{-k}}. \quad (7)$$

$$\frac{\partial \log(c,k)}{\partial k} = \frac{m}{k} - \sum_{i=1}^{n_1}\log(1+y_i^{-c})\sum_{i=1}^{n_1}[h(R_i+1)-1] \frac{(1+y_i^{-c})^{-k}\ln(1+y_i^{-c})}{1-(1+y_i^{-c})^{-k}} \sum_{i=n_1+1}^{m}\log(1+[\beta(y_i-\tau)+\tau]^{-c}) - \sum_{i=n_1+1}^{m}[h(R_i+1)-1] \frac{1+[\beta(y_i-\tau+\tau]^{-c})^{-k}\ln(1+[\beta(y_i-\tau)+\tau]^{-c})}{1-(1+[\beta(y_i-\tau)+\tau]^{-c})^{-k}}.$$

Put $\frac{\partial \log(c,k)}{\partial c} = 0$ and $\frac{\partial \log(c,k)}{\partial k} = 0$.
We turn to solve the nonlinear system equations for the unknown c,k numerically to find the estimated values of $(c\text{and}k)$ as it is very difficult be solved algebraically.

3.2 Interval Estimation

According to bootstrap method, estimate the interval of the unknown parameters $\theta = (c,k,\beta)$ the asymptotic $(1-\alpha)100\%$ confidence interval for θ by using method takes the form;

$$\left(\hat{c} - Z_{\frac{\alpha}{2}}\sqrt{\text{MSE}_c},\ \hat{c} + Z_{\frac{\alpha}{2}}\sqrt{\text{MSE}_c}\right), \left(\hat{k} - Z_{\frac{\alpha}{2}}\sqrt{\text{MSE}_k},\ \hat{k} + Z_{\frac{\alpha}{2}}\sqrt{\text{MSE}_k}\right), \left(\hat{\beta} - Z_{\frac{\alpha}{2}}\sqrt{\text{MSE}_\beta},\ \hat{\beta} + Z_{\frac{\alpha}{2}}\sqrt{\text{MSE}_\beta}\right). \quad (8)$$

$\widehat{\text{MSE}} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_1 - \hat{\theta})$ the bootstrap estimates for MSE.

4 Monte-Carlo Simulation

A numerical study is performed using MATHCAD2001 Package to evaluate the behavior of the point and interval estimators of the parameters for two different sampling schemes (scheme I, scheme II). The simulations are based on $N = 1000$ replications and initial values $\tau = 0.000005$, $c = 0.9$, $k = 0.9$ and $\beta = 1.1$. The simulation study is performed using Mathcad program, as follows:

1. Generate 1000 uniform $(0,1)$ random sample from Burr III under (SSALT) with progressive first failure samples according to Equation (1).
2. Obtain the MLEs and bootstrap Bayes confidence intervals using Equation (5).
3. Repeat the previous steps 1000 times.
4. Obtain the MLEs and MSEs for the distribution parameters in the two cases (scheme I and scheme II) as well as the confidence interval lengths for the parameters.

Tables 1 and 2 show the results of scheme I and Tables 3 and 4 for scheme II.

Table 1: Point Estimators and MSE for Scheme I.

h	n	m	$\hat{c}$	Mse of $\hat{c}$	$\hat{k}$	Mse of $\hat{k}$	$\hat{\beta}$	Mse of $\hat{\beta}$
1	50	20	1.550	0.451	1.736	0.700	4.938×10^{-4}	1.200
1	50	30	1.144	0.103	1.860	0.922	7.856×10^{-7}	1.210
1	50	40	0.918	0.012	1.424	0.279	6.531×10^{-7}	1.210
1	60	20	1.634	0.584	1.209	0.097	2.140×10^{-3}	1.205
1	70	30	1.695	0.655	1.677	0.607	1.999×10^{-3}	1.206
1	70	40	0.832	0.040	1.625	0.525	1.373×10^{-5}	1.210
1	80	30	1.554	0.446	0.396	0.254	1.939×10^{-4}	1.210

Table 2: The Lenfgth of 95% CI for Scheme I.

h	n	m	Confidence Interval of $\hat{c}$	Confidence Interval of $\hat{k}$	Confidence Interval of $\hat{\beta}$
1	50	20	3.324	2.793	4.310
1	50	30	0.917	3.798	4.312
1	50	40	0.426	2.084	4.312
1	60	20	2.968	1.220	4.303
1	70	30	3.259	3.042	4.305
1	70	40	1.186	2.808	4.312
1	80	30	2.547	1.975	4.311

Table 3: Point Estimators and MSE for Scheme II.

h	n	m	$\hat{c}$	Mse of $\hat{c}$	$\hat{k}$	Mse of $\hat{k}$	$\hat{\beta}$	Mse of $\hat{\beta}$
2	60	30	1.170	0.128	1.822	0.850	3.540×10^{-6}	1.210
2	60	40	0.764	0.022	1.803	0.819	1.380×10^{-6}	1.210
2	70	10	1.672	0.017	1.773	0.764	3.500×10^{-2}	1.135
2	70	20	1.219	0.155	1.871	0.943	2.520×10^{-3}	1.104
2	80	10	1.819	0.052	1.650	0.570	7.000×10^{-2}	1.061
2	80	30	1.317	0.224	1.522	0.676	1.080×10^{-3}	1.108
2	80	50	1.028	0.030	0.263	0.137	1.250×10^{-6}	1.210

Table 4: The Lenfgth of 95% CI for Scheme II.

h	n	m	Confidence Interval of $\hat{c}$	Confidence Interval of $\hat{k}$	Confidence Interval of $\hat{\beta}$
2	60	30	1.433	3.6140	4.312
2	60	40	1.519	3.6108	4.312
2	70	10	1.359	3.3570	4.215
2	70	20	1.240	3.2050	4.203
2	80	10	1.609	2.9690	4.035
2	80	30	1.066	3.1870	4.011
2	80	50	0.980	1.2650	4.012

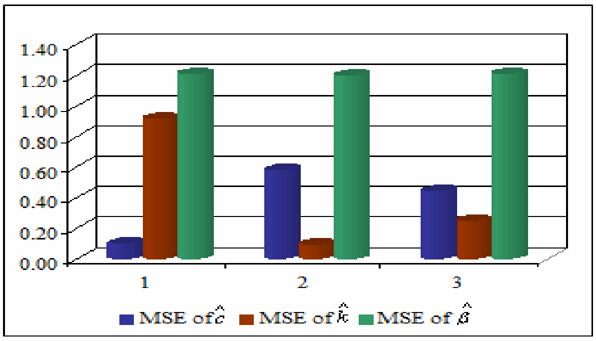

Fig. 1: Presented of MSEs Estimated Values $\hat{c}$, $\hat{k}$, $\hat{\beta}$ with value $h = 2$.

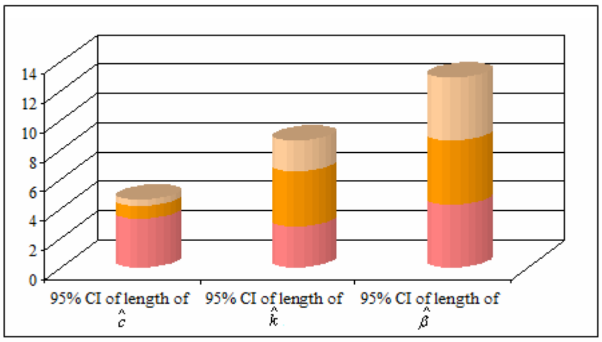

Fig. 2: Presented of MSEs Estimated Values $\hat{c}$, $\hat{k}$, $\hat{\beta}$ with value $h = 1$.

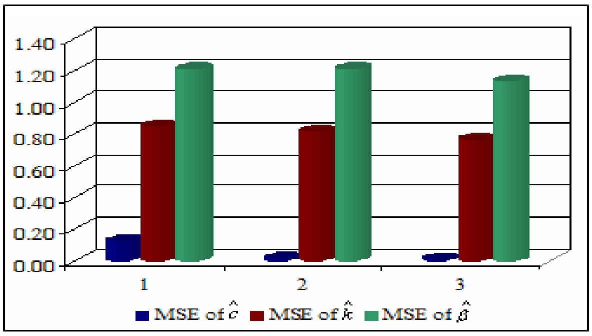

Fig. 3: Confidence Length for Estimated Values $\hat{c}$, $\hat{k}$, $\hat{\beta}$ with value $h = 1$.

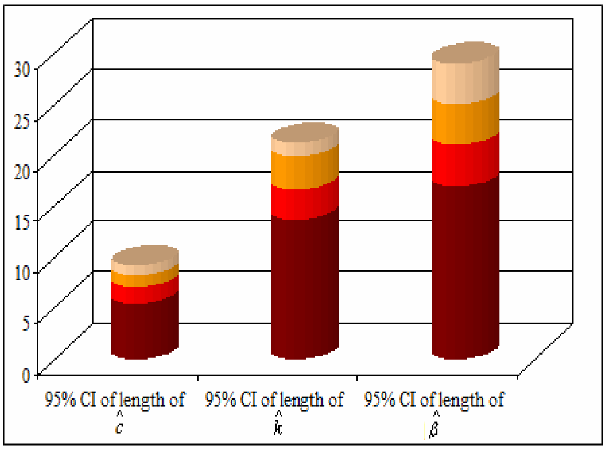

Fig. 4: Confidence Lengths for Estimated Values $\hat{c}$, $\hat{k}$, $\hat{\beta}$ with value $h = 2$.

The notation used throughout the paper is stated below:

$F(\cdot)$	CDF	T	The lifetime of the unit under normal condition
$f(\cdot)$	PDF	τ	The lifetime of the unit
x	Continuous Random variable	β	The Acceleration function
y	Random variable after acceleration	TRV	Tampered random variable
c, k	Scale Parameters	$L(\cdot,\cdot)$	Likelihood function for estimation
$Y^R_{i:m,n,h}$	Failures numbers	MSE	Mean square error
$R_1, R_2, \cdots, R_n$	Random number of the surviving groups	$Z_{\frac{\alpha}{2}}$	Z is the standard normal distribution α is the confidence interval
SSPALT	Step-stress partially accelerated life test	ALT	Accelerated life tests

5 Conclusion

In this paper, the (SSALT) with the progressive first failure sample was considered for Burr type III distribution with parametersc,k and accelerated factor β. We obtained the maximum likelihood method to estimate the parameters of our distribution. The values of MLE's were unclear in explicit form so we used the numerical study to get it. The performance of the estimators has been studied in the term of MSE by the simulation study based on different values of sample size. We studied two schemes (I and II) which were different values (I and II). The results demonstrated that the MSEs decreased by increasing the sample size. Also, the confidence interval lengths decreased by increasing the sample size. MSEs of $\hat{\beta}$ are less than both MSEs of $\hat{c}$ and of $\hat{k}$. MSEs of $\hat{\beta}$ are less than both $\hat{c}$ and $\hat{k}$. For increasing size of data, there is some little decreasing in the Mse's and the length of estimated confidence interval decreases when n increases.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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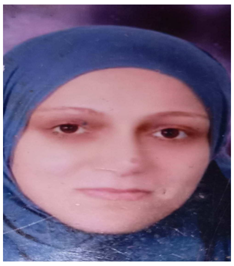

Marw Othman Mohamed specialized in the field of reliability theory, specifically in the field of studying the stress-strength model, and I do simulation with more than one model, such as Monte Carlo simulation method, as well as statistical inference based on a point and confidence, Study ways to get new distribution.

Esraa Mohamed in my Msc field, I measuring the Reliability function through a model of step-stress which is a special model from reliability, I applied simulation methods, and get the estimated values in more than one statistical inference method