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Estimation in K-Stage Step-Stress Partially Accelerated Life Tests for Generalized Pareto Distribution with Progressive Type-I Censoring

Rashad M. EL-Sagheer[1,*] *and Moaiad A. A. Khder*[2]

[1] Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt
[2] College of Arts and Science, Applied Science University (ASU), East Al Ekir, Manama, Kingdom of Bahrain

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Abstract: This article discusses k-stage step-stress partially accelerated life test under Type-I progressive interval censoring with equal inspection intervals of length. The maximum likelihood and parametric bootstrap methods are used to obtain the estimators of the model parameters. Approximate confidence intervals for the unknown parameters are constructed based on the asymptotic variances when the lifetime of a testing unit are assumed to be generalized Pareto distribution. The methods for obtaining the optimal are investigated using the variance-optimality and determinant-optimality criteria. Monte Carlo simulation studies are addressed to illustrate the proposed criteria.

Keywords: Generalized Pareto distribution, K-stage step-stress partially accelerated life test, Parametric bootstrap methods; Variance-optimality, Determinant-optimality.
Mathematics Subject Classification: 62No5; 62F10..

List of symbols:

n	Total number of units placed on test
n_i	No. of failed units at stress x_i , $i = 1,2,...,k$
m_i	No. of non-removed surviving units at the beginning of the ith stage
k	Total number of stages
x_0	The stress at use-condition
x_i	Stress level, $i = 1,2,...,k$, $x_1 < x_2 < \cdots < x_k$
R_i	No. of units withdrawn at each stress level, $i = 1,2,...,k$
τ	Time of stress change at stress x_i, $i = 1,2,...,k-1$
$\underline{\delta} \equiv \delta_i$	Acceleration factors ($\delta_i > 1,\ i = 1,2,...,k$)
π_i	Proportion of withdrawn at the ith stage
τ_D^*	Optimal τ according to the determinant-(D) optimality criterion
τ_V^*	Optimal τ according to the variance-(V) optimality criterion
cdf	Cumulative distribution function
pdf	Probability density function

1 Introduction

Because of continual improvement in manufacturing design, one often deals with high quality units that are highly reliable with a substantially long life span. It is more difficult to obtain information about the lifetime of units with high reliability at the time of testing under normal conditions. This makes the lifetime testing under normal conditions very costly and take a long time. Thus, accelerated life test (ALT) and partially accelerated life test (PALT) are the most common approaches that are used to obtain failures in a short period of time (test units are run at higher-than-usual stress conditions in order to obtain failures quickly). Units are tested at high stress levels to induce early failures and then the failure information is related to that at an operational stress level through a given stress-dependent model. When such model is unknown, the accelerated life test cannot be conducted and the PALT becomes suitable instead. In ALTs, the units are run only at accelerated conditions (stress). However, in PALTs, they are run at both use and accelerated conditions. Thus, the PALT combines both

* Corresponding author e-mail: Rashadmath@azhar.edu.eg, Rashadmath27@gmail.com

ordinary and accelerated life tests. The objective of PALTs is to collect more failure data in a limited time without using high stresses to all test units. As indicated by Nelson [1], the stress loading in an PALT can be applied various ways. They include constant-stress, step-stress, and progressive-stress.

In constant-stress, PALTs run each item at either use condition or accelerated condition only; that is, each unit is run at a constant-stress level until the test is terminated. However, in the step-stress PALTs, a test unit is subject to successively higher levels of stress. A test unit starts at a specified low stress for a specified length of time. If it does not fail, stress on it is raised and held a specified time. Thus, stress increases step by step until the test unit fails. Generally, all test units go through the same specified pattern of stress levels and test times. The simplest step-stress PALT uses only two stress levels and we call it simple step-stress PALT. The statistical inferences in this step-stress PALT have been investigated by several authors, such as Tang et al. [2], Xiong [3], Gouno et al. [4], Abdel-Hamid and Al-Hussaini [5], Ismail and Aly [6] and recently pushkarna and Saran [7] , EL-Sagheer and Hasaballah [8], EL-sagheer and Mansour [9], EL-sagheer et al. [10], as well as EL-Sagheer and Ahsanullah [11].

These experiments aim to collect more failure data in a limited time without using a high stress to all test units. As Bhattacharyya and Soejoeti [8] indicated, step-stress PALTs are practical for many problems of life testing where the test process requires a long time if the test is simply carried out under the use condition. In practice, step-stress PALTs are easier to implement and have many advantages, including time saving, economical and adaptable. To save more time and cost, ALTs or PALTs are used under censored sampling. The most common censoring schemes are type I censoring and type II censoring. They do not allow units to be removed from the test at any point other than the final termination point. However, this allowance may be needed when a compromise between reduced time of experiment and the observation of some extreme lifetimes is sought. These reasons lead us into the area of progressive censoring, which permits an efficient exploitation of the available resources by continual removal of a prespecified number of unfailed test units at the end of testing time at each stage.

The present paper aims to combine PALTs with progressive censoring and then to concentrate on the optimal choice of change points of the stress levels. Also, it explores the choice of length of the inspection interval based on results of samples from Weibull distribution. We investigate the selection of according to two competing criteria of optimality: variance (Var) optimality and determinant (D) optimality.

The layout of the paper is, as follows: Section 2 presents the details of the proposed model. Maximum likelihood estimators (MLEs) of the model parameters and the associated Fisher information matrix are derived in Section 3. In Section 4, we introduce two parametric bootstrap procedures to construct the confidence intervals for the unknown parameters. The problem of choosing the optimal length of the inspection interval will also be addressed using the variance-optimality and D-optimality criteria in Section 5. Some simulation results are presented in Section 6. Section 7 is devoted to the discussion and concluding remarks.

2 Model Description and Assumptions

Let us consider the following k-stage step-stress PALT with type-I progressive interval censoring. Suppose that n identical and independent units are simultaneously placed on a life test at stress setting x_1, and run until time τ, at which point the number of failed units n_1 is counted and R_1 surviving units are arbitrarily withdrawn from the test; starting from time τ, the test is continued on $n-n_1-R_1$ units until time 2τ and the stress changes to a higher level of stress x_2 $(x_1 < x_2)$, at which point the number of failures n_2 is counted and R_2 units are withdrawn from the test and so on. At time $k\tau$, the number of failed units n_k is counted and the surviving $R_k = n - \sum_{i=1}^{k} n_i - \sum_{j=1}^{k-1} R_i$ units are withdrawn, thereby terminating the test. Figure 1 depicts this scheme. Our objective here is to choose the optimal length of τ according to a certain optimality criterion.

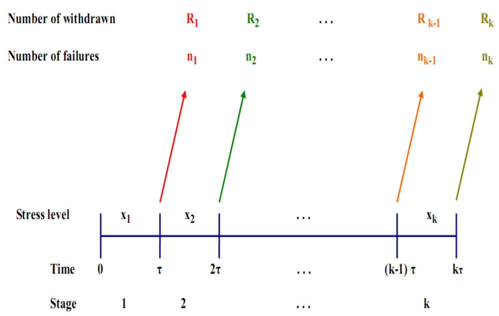

Fig. 1: k-stage step-stress PALT under type-I progressive interval censoring.

A random variable Y is said to have generalized Pareto distribution (GPD) if its probability density function is given by

$$f_{(\xi,\mu,\sigma)}(y) = \frac{1}{\sigma}\left(1+\xi\frac{y-\mu}{\sigma}\right)^{-(1/\gamma+1)}, \quad (1)$$

where $\mu, \xi \in \mathbb{R}$ and $\sigma \in (0, +\infty)$. For convenience, we reparametrized this distribution by defining $\xi/\sigma \equiv \lambda, 1/\xi \equiv \alpha$ and $\mu \equiv 0$. Therefore, probability density function (pdf) is given by

$$f(y) = \alpha\lambda(1+\lambda y)^{-(\alpha+1)};\ y>0,\ \alpha,\lambda>0. \quad (2)$$

The cumulative distribution function (cdf) is defined by

$$F(x) = 1-(1+\lambda y)^{-\alpha},\ y>0,\ \alpha,\lambda>0. \quad (3)$$

The corresponding reliability and failure rate functions of this distribution at mission time t are given, respectively by

$$S(t) = (1+\lambda t)^{-\alpha}, \qquad t>0, \quad (4)$$

and

$$h(t) = \alpha\lambda(1+\lambda t)^{-1}, \qquad t>0. \quad (5)$$

Here, α and λ are the shape and scale parameters, respectively. It is also well known that this distribution has decreasing failure rate property. This distribution is also known as Pareto distribution of the second type or Lomax distribution. Now, let us consider the k-stage step-stress PALT under type-I progressive interval censoring with equal inspection intervals of length τ. Therefore, under the assumptions of the cumulative exposure model, the cumulative distribution function is given by

$$F(y) = \begin{cases} 1-(1+\lambda y)^{-\alpha} \text{if } 0<y\le\tau, \\ 1-(1+\lambda[\tau+\delta_1(y-\tau)])^{-\alpha} \text{ if } \tau<y\le 2\tau, \\ \cdot \\ \cdot \\ \cdot \\ 1-\left(1+\lambda\left[(k-1)\tau+\delta_{(k-1)}(y-(k-1)\tau)\right]\right)^{-\alpha} \\ \quad \text{if } (k-1)\tau<y\le\infty. \end{cases} \quad (6)$$

Hence, the pdf a test unit is given by

$$f(y) = \begin{cases} \alpha\lambda(1+\lambda y)^{-(\alpha+1)} \quad \text{if } 0<y\le\tau, \\ \alpha\lambda^2\delta_1(1+\lambda[\tau+\delta_1(y-\tau)])^{-(\alpha+1)} \\ \quad \text{if } \tau<y\le 2\tau, \\ \cdot \\ \cdot \\ \cdot \\ \alpha\lambda^2\delta_{(k-1)}(1+\lambda\left[(k-1)\tau+\delta_{(k-1)}(y-(k-1)\tau)\right])^{-(\alpha+1)} \\ \text{if } (k-1)\tau<y\le\infty. \end{cases} \quad (7)$$

3 Maximum Likelihood Estimation

The maximum likelihood is one of the most important and widely used methods in statistics. The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. Furthermore, maximum likelihood estimators are versatile and applied to most models and different types of data. In addition, they provide efficient methods for quantifying uncertainty through confidence bounds. Because these estimators do not always exist in closed form, numerical techniques are used to compute them. In this Section, the point and interval estimations of the model parameters and acceleration factor are introduced using the maximum likelihood method based on progressively type-I interval censored data. Also, Fisher information matrix of the model parameters and acceleration factor are presented.

Let $n_1, n_2, \ldots, n_k$ be a progressively type-I interval-censored sample with censoring scheme $R=(R_1,R_2,\ldots,R_k)$ from a k-stage step-stress PALT. That is, the number of failed units n_i are observed while testing in the interval $((i-1)\tau, i\tau]$ at stress x_i, $i=1,2,\ldots,k$. Hence, we have the fact that $n_i|n_{i-1},\ldots,n_1 \sim \text{binomial}(m_i, F_i(\tau))$, where $m_i = n-\sum_{j=1}^{i-1} n_i - \sum_{j=1}^{i-1} R_i$ and

$$F_i(\tau) = \frac{F(i\tau)-F((i-1)\tau)}{1-F((i-1)\tau)}. \quad (8)$$

The likelihood function without normalized constant is then given by

$$L = \prod_{i=1}^{k}[F(i\tau)-F((i-1)\tau)]^{n_i}[1-F(i\tau)]^{R_i}. \quad (9)$$

The natural logarithm of the likelihood function $\ell = \ln L$ can be written as

$$\ell = \sum_{i=1}^{k}\left\{n_i \ln\left[\phi_{i-1}^{-\alpha}-\phi_i^{-\alpha}\right]-(m_i-n_i)\ln\phi_i^{-\alpha}\right\}, \quad (10)$$

where

$$\begin{aligned} \phi_{i-1} &= 1+\lambda\left[(i-1)\tau+\delta_{i-1}(y-(i-1)\tau)\right] \\ \phi_i &= 1+\lambda\left[i\tau+\delta_i(y-i\tau)\right]. \end{aligned} \quad (11)$$

Calculating the first partial derivatives of Equation (10) with respect to α, λ, and δ_i and equating each to zero, we get the likelihood equations as

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=1}^{k}\left\{\frac{\alpha n_i}{\phi_{i-1}^{-\alpha}-\phi_i^{-\alpha}}\left(\phi_i^{-(\alpha+1)}-\phi_{i-1}^{-(\alpha+1)}\right)+\alpha(m_i-n_i)\phi_i^{-1}\right\} = 0, \quad (12)$$

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{k}\left\{\frac{\alpha n_i}{\phi_{i-1}^{-\alpha}-\phi_i^{-\alpha}}\left(\phi_i^{-(\alpha+1)}\eta_i-\phi_{i-1}^{-(\alpha+1)}\eta_{i-1}\right)+\alpha(m_i-n_i)\phi_i^{-1}\eta_i\right\} = 0, \quad (13)$$

where

$$\eta_{i-1} = \left[(i-1)\tau+\delta_{i-1}(y-(i-1)\tau)\right], \quad \eta_i = \left[i\tau+\delta_i(y-i\tau)\right], \quad (14)$$

and

$$\frac{\partial \ell}{\partial \delta_i} = \sum_{i=1}^{k}\left\{ \frac{\alpha\lambda n_i}{\phi_{i-1}^{-\alpha}-\phi_i^{-\alpha}}\left[(y-i\tau)\,\phi_i^{-(\alpha+1)}-(y-(i-1)\,\tau)\,\phi_{i-1}^{-(\alpha+1)}\right] + \alpha\lambda\,(m_i-n_i)\,(y-i\tau)\,\phi_i^{-1}\right\}=0.$$

Now, we have a system of three non-linear likelihood equations (12), (13) and (15) in three unknowns α, λ and δ_i. It cannot be solved analytically. The Newton-Raphson iteration method is used to obtain the estimates. The algorithm is described, as follows:

(1):Use the method of moments or any other methods to estimate the parameters α, λ and δ_i, $i=1,2,...,k$.as starting point of iteration, denote the estimates as $(\alpha_0,\lambda_0,\delta_0)$ and set $l=0$.
(2):Calculate $\left(\frac{\partial \ell}{\partial \alpha},\frac{\partial \ell}{\partial \lambda},\frac{\partial \ell}{\partial \delta_i}\right)_{(\alpha_l,\gamma_l,\delta_l)}$ and the observed Fisher Information matrix $I^{-1}(\alpha,\lambda,\delta_i)$, given in the next paragraph.
(3):Update $(\alpha,\lambda,\delta_i)$ as

$$(\alpha_{l+1},\lambda_{l+1},\delta_{l+1})=(\alpha_l,\lambda_l,\delta_l)+\left(\frac{\partial \ell}{\partial \alpha},\frac{\partial \ell}{\partial \lambda},\frac{\partial \ell}{\partial \delta_i}\right)_{(\alpha_l,\gamma_l,\delta_l)} \times I^{-1}(\alpha,\lambda,\delta_i),\ i=1,2,...,k. \tag{15}$$

(4):Set $l=l+1$ and then go back to Step 1.
(5):Continue the iterative steps until $|(\alpha_{l+1},\lambda_{l+1},\delta_{l+1})-(\alpha_l,\lambda_l,\delta_l)|$ is smaller than a threshold value. The final estimates of $(\alpha,\lambda,\delta_i)$ are the MLE of the parameters, denoted as $(\hat{\alpha},\hat{\lambda},\hat{\delta}_i)$.

As indicated by Vander and Meeker [9], the most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the MLEs. The asymptotic variances and covariances of the MLEs, $\hat{\alpha},\hat{\lambda}$ and $\hat{\delta}_i$ are given by the entries of the inverse of the Fisher information matrix $I_{ij}=E\left[-\partial^2\ell/\partial\zeta_i\partial\zeta_j\right]$ where $i,j=1,2,3$ and $\Phi=(\zeta_1,\zeta_2,\zeta_3)=(\alpha,\lambda,\delta_i)$. Unfortunately, the exact closed forms for the above expectations are difficult to obtain. Therefore, the observed Fisher information matrix $\hat{I}_{ij}=E\left[-\partial^2\ell/\partial\zeta\partial\zeta_j\right]_{\Phi=\hat{\Phi}}$, which is obtained by dropping the expectation operator E, will be used to construct confidence intervals for the parameters, see Cohen [10]. The observed Fisher information matrix has second partial derivatives of log-likelihood function as the entries, which can be easily obtained. Hence, the observed information matrix is given by

$$\hat{I}(\alpha,\lambda,\delta_i)=\begin{pmatrix} -\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\lambda} & -\frac{\partial^2\ell}{\partial\alpha\partial\delta_i} \\ -\frac{\partial^2\ell}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell}{\partial\lambda^2} & -\frac{\partial^2\ell}{\partial\lambda\partial\delta_i} \\ -\frac{\partial^2\ell}{\partial\delta_i\partial\alpha} & -\frac{\partial^2\ell}{\partial\delta_i\partial\lambda} & -\frac{\partial^2\ell}{\partial\delta_i^2} \end{pmatrix}, \quad i=1,2,...,k,. \tag{16}$$

Therefore, the asymptotic variance-covariance matrix for the MLEs is obtained by inverting the observed information matrix $\hat{I}(\alpha,\lambda,\delta_i)$. Or equivalent

$$\hat{I}^{-1}(\alpha,\lambda,\delta_i)=\begin{pmatrix} \widehat{var(\alpha)} & cov(\alpha,\lambda) & cov(\alpha,\delta_i) \\ cov(\lambda,\alpha) & \widehat{var(\lambda)} & cov(\lambda,\delta_i) \\ cov(\delta_i,\alpha) & cov(\delta_i,\lambda) & \widehat{var(\delta_i)} \end{pmatrix}_{\downarrow(\hat{\alpha},\hat{\lambda},\hat{\delta}_i)}, \quad i=1,2,...,k,. \tag{17}$$

It is well known that under some regularity conditions, see Lawless [11], $(\hat{\alpha},\hat{\lambda},\hat{\delta}_i)$ is approximately distributed as multivariate normal with mean $(\alpha,\lambda,\delta_i)$ and covariance matrix $I^{-1}(\alpha,\lambda,\delta_i)$. Thus, the $(1-\gamma)100\%$ approximate confidence intervals (ACIs) for α , λ and δ_i can be given by

$$\left(\hat{\alpha}\pm Z_{\gamma/2}\sqrt{\widehat{var(\alpha)}}\right),\ \left(\hat{\lambda}\pm Z_{\gamma/2}\sqrt{\widehat{var(\lambda)}}\right), \left(\hat{\delta}_i\pm Z_{\gamma/2}\sqrt{\widehat{var(\delta_i)}}\right),\ i=1,2,...,k, \tag{18}$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$. The problem with applying normal approximation of the MLE is that when the sample size is small, the normal approximation may be poor. However, a different transformation of the MLE can be used to correct the inadequate performance of the normal approximation. Meeker and Escobar [12] suggested the use of the normal approximation for the log-transformed MLE. Thus, A two-sided $(1-\gamma)100\%$ normal approximation CIs for $\Omega=\alpha$, λ or δ_i are given by

$$\left(\hat{\Omega}.\exp\left\{-\frac{Z_{\frac{\gamma}{2}}\sqrt{\widehat{var(\Omega)}}}{\hat{\Omega}}\right\},\quad \hat{\Omega}.\exp\left\{\frac{Z_{\frac{\gamma}{2}}\sqrt{\widehat{var(\Omega)}}}{\hat{\Omega}}\right\}\right), \tag{19}$$

where $\hat{\Omega}=\hat{\alpha},\hat{\lambda}$ or $\hat{\delta}_i$ and $i=1,2,...,k$.

4 Bootstrap Techniques

The bootstrap is a resampling method for statistical inference. It is commonly used to estimate confidence intervals, but it can be also used to estimate bias and variance of an estimator or calibrate hypothesis tests. Also, it is evident that the confidence intervals based on the asymptotic results do not perform very well for small sample size. Hence, we propose using confidence intervals based on the parameteric bootstrap methods. We present two parametric bootstrap methods, (i) percentile bootstrap method (we call it PBM) based on the idea of Efron [13]. (ii) bootstrap-t method (we call it BTM) based on the idea of Hall [14]. For more survey of the parametric bootstrap methods, see Kreiss and Paparoditis [15]. The following steps are followed to obtain bootstrap samples for both methods:

(1):Based on the original progressively type-I interval-censored sample, $n_1, n_2, ..., n_k$ with censoring scheme $R = (R_1, R_2, ..., R_k)$ from a k-stage step-stress PALT, compute $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\delta}_i$ for $i = 1,2,...,k$.

(2):Use $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\delta}_i$ for $i = 1,2,...,k$ to generate a bootstrap sample $n_1^*, n_2^*, ..., n_k^*$ with the same values of R from generalized Pareto distribution with the distribution function is given in (6).

(3):As in Step 1 based on $n_1^*, n_2^*, ..., n_k^*$, compute the bootstrap sample estimates of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\delta}_i$ say $\hat{\alpha}^*$, $\hat{\lambda}^*$ and $\hat{\delta}_i^*$ for $i = 1,2,...,k$.

(4):Repeat the above Steps 2 and 3 N times and arrange all $\hat{\alpha}^*$, $\hat{\lambda}^*$ and $\hat{\delta}_i^*$ in ascending to obtain the bootstrap sample $(\widehat{\Psi}_l^{*[1]}, \widehat{\Psi}_l^{*[2]}, ..., \widehat{\Psi}_l^{*[N]}), l = 1,2,3$, where $\widehat{\Psi}_1^* = \hat{\alpha}^*, \widehat{\Psi}_2^* = \hat{\lambda}^*$ and $\widehat{\Psi}_3^* = \hat{\delta}_i^*$, $i = 1,2,...,k$.

4.1 Percentile bootstrap procedure

Let $\Phi(z) = P(\widehat{\Psi}_l^* \le z)$ be the cumulative distribution function of $\widehat{\Psi}_l^*$. Define $\widehat{\Psi}_{lBoot}^* = \Phi^{-1}(z)$ for given z. The approximate bootstrap $100(1-\gamma)$ confidence interval of $\widehat{\Psi}_l^*$ is given by

$$\left[\widehat{\Psi}_{lPBM}^*\left(\frac{\gamma}{2}\right), \widehat{\Psi}_{lPBM}^*\left(1-\frac{\gamma}{2}\right)\right]. \tag{20}$$

4.2 Bootstrap-t procedure

We find the order statistics $\varpi_l^{*[1]} < \varpi_l^{*[2]} < ... < \varpi_l^{*[N]}$ where

$$\varpi_l^{*[j]} = \frac{\sqrt{N}\left(\widehat{\Psi}_l^{*[j]} - \widehat{\Psi}_l\right)}{\sqrt{Var(\widehat{\Psi}_l^{*[j]})}}, \quad j = 1,2,...,N, \ l = 1,2,3, \tag{21}$$

where $\widehat{\Psi}_1 = \hat{\alpha}, \widehat{\Psi}_2 = \hat{\lambda}$, $\widehat{\Psi}_3 = \hat{\delta}_i$, $i = 1,2,...,k$ and $Var(\widehat{\Psi}_l^{*[j]})$ is obtained using the Fisher information matrix. Let $W(z) = P(\varpi_l^* < z), l = 1,2,3$ be the cumulative distribution function of ϖ_k^*. For a given z, define

$$\widehat{\Psi}_{lBTM} = \widehat{\Psi}_l + N^{-1/2}\sqrt{Var(\widehat{\Psi}_l)}W^{-1}(z). \tag{22}$$

Thus, the approximate bootstrap $100(1-\gamma)$ confidence interval of $\widehat{\Psi}_l$ is given by

$$\left[\widehat{\Psi}_{lBTM}^*\left(\frac{\gamma}{2}\right), \widehat{\Psi}_{lBTM}^*\left(1-\frac{\gamma}{2}\right)\right]. \tag{23}$$

5 Optimality Criteria

One of the aims of the present paper is to explore the choice of τ, length of the inspection interval, in k-stage step-stress PALT with type-I progressive interval censoring. In this section, we propose two selection criteria which enable one to choose the optimal value of τ.

5.1 Variance optimality

The mean lifetime is an important characteristic in reliability analysis. In a step-stress setting, the experimenter is often interested in estimating the mean life at design (use) stress with maximum precision. Let $\hat{\beta}_0$ be the MLE of mean lifetime at the design (use) stress β_0. The criterion function defined by the asymptotic variance (AVar) of the MLE of $\ln \hat{\beta}_0$ is

$$\begin{aligned}\Delta(\tau) &= AVar \ln \hat{\beta}_0 \\ &= n(1,1,x_0) I^{-1}(\alpha,\lambda,\delta)(1,1,x_0)',\end{aligned} \tag{24}$$

where x_0 is the stress at use condition. The variance optimal τ is then obtained by minimizing $\Delta(\tau)$.

5.2 Determinant optimality

The second optimal criterion is based on the determinant of the Fisher's information matrix. It has been extensively used in the context of planning life tests. If one is more interested in estimation with high precision, a more reasonable criterion should be determinant optimality, which considers the overall parameter space. It can be constructed in terms of the generalized asymptotic variance (GAV) of the MLEs of the model parameters. It is known that the GAV is proportional to reciprocal of the determinant of Fisher information matrix, so maximizing this determinant is equivalent to minimizing GAV, for more details see Bai et al. [16]. The criterion function is then defined by

$$\text{GAV}\left(\hat{\alpha},\hat{\lambda},\hat{\delta}_i\right) = \frac{1}{\left|I\left(\hat{\alpha},\hat{\lambda},\hat{\delta}_i\right)\right|}, \ i = 1,2,...,k. \tag{25}$$

Thus, the optimal length of inspection interval is chosen, so GAV is minimized. It is noted that both variance optimality and determinant optimality criteria are based on the Fisher's information matrix. These criteria have been extensively used in the design selection process for designed experiments

6 Monte Carlo Simulation Study

To investigate the optimal choice of τ, length of the inspection interval, in k-stage step-stress PALTs under type-I progressive interval censoring, simulation studies are performed using different values of sample size n equals 40, 60, 80, 100, 200, 300, 400, 500. It is assumed that the proportions removed at different stages are all equal. That is, $\pi_1 = \pi_2 = \cdots = \pi_k = \pi$. We assume that the lengths of inspection intervals are all equal for simplicity of discussion and the equi length assumption is also convenient for practitioners. Let τ_V^* and τ_D^* be optimal lengths of inspection intervals according to variance-optimality and determinant-optimality, respectively. Tables 1 and 2 present optimum τ_V^* and τ_D^* for $k = 2,3,4,5$ and π equals 0.05 and 0.10. The proposed optimality criteria can lead to better designs for conducting life tests. It provides the most efficient use of experimenter's resources. The findings are summarized, as follows:

(i)Both τ_V^* and τ_D^* decrease as k increases when π and n are fixed. That is, the larger number of stress levels, the more likely a short length of inspection interval.
(ii)The determinant-optimal length of inspection interval τ_D^* is always smaller than the variance-optimal length of inspection interval τ_V^* .
(iii)For fixed k and n, both τ_V^* and τ_D^* decrease as π increases. That is, the larger the proportion to be removed at each stage, the shorter the optimal length of the inspection interval.
(iv)Both τ_V^* and τ_D^* increase as n increases when π and k are fixed. That is, the larger number of test units n, the larger the optimal length of the inspection interval.
(v)It is shown that the second proposed criterion (determinant-optimality) can reduce the required number of failures and so reduce the total testing time without losing much precision.
(vi)When π increases, the experiment is terminated more quickly. However, it is important to note that with a much larger π, the experiments will be less informative and lead to larger standard errors in estimates.

Table 1. Optimal lengths τ_V^* and τ_D^* under k-stage step-stress PALTs and progressive type-I interval censoring with proportion of removals $\pi = 0.05$ when $\underline{\delta} = 2; 2.5; 3; 3.5$ and $\lambda = 1.5$, $\alpha = 0.2$.

	$k=2$		$k=3$		$k=4$		$k=5$	
n	τ_V^*	τ_D^*	τ_V^*	τ_D^*	τ_V^*	τ_D^*	τ_V^*	τ_D^*
40	35	32	33	29	28	25	26	21
60	47	45	44	41	42	38	39	34
80	62	58	59	54	55	50	51	47
100	76	70	68	65	62	59	55	50
200	164	149	132	127	112	108	93	82
300	235	225	166	143	123	115	109	93
400	344	324	257	236	188	164	125	111
500	437	419	360	297	243	198	165	137

Table 2. Optimal lengths τ_V^* and τ_D^* underk-stage step-stress PALTs and progressivetype-I interval censoring with proportion of removals $\pi = 0.10$ when $\underline{\delta} = 2; 2.5; 3; 3.5$ and $\lambda = 1.5$, $\alpha = 0.5$.

	$k=2$		$k=3$		$k=4$		$k=5$	
n	τ_V^*	τ_D^*	τ_V^*	τ_D^*	τ_V^*	τ_D^*	τ_V^*	τ_D^*
40	31	27	28	24	23	20	16	13
60	42	39	35	33	29	26	23	19
80	59	54	56	49	51	44	47	41
100	70	66	63	61	58	56	52	49
200	158	145	128	121	108	102	89	78
300	185	177	136	127	112	109	96	88
400	251	234	188	161	165	147	118	106
500	331	309	229	186	176	155	127	115

7 Conclusion

In this paper, we have discussed a combination of progressive censoring, step-stress PALT and interval data to develop a step-stress PALT under progressive type-I interval censoring data. The generalized Pareto lifetime distribution at each level of stress was considered. In reliability analysis of progressively interval censored life test data, for given stress levels and number of test units, determining the appropriate length of the inspection interval is an important issue for experimenters. Two optimality criteria (determinant-optimality and variance-optimality) for choosing the optimal length of the inspection interval were used for comparison purpose. In the case of certain life tests, some test units need to be removed at points other than the final termination point of the experiment and it is unpractical to screen the test units constantly. Here, the progressive interval censoring scheme permits units were removed early and inspected from time to time. Based on the fourth finding, the optimal length of the inspection interval is shorter in the case of the determinant-optimality criterion. Therefore, the determinant-optimality criterion is recommended for obtaining the optimal life test plan. These results provide valuable insight for practitioners to set up optimal life test plans under progressive type-I interval censoring.

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Competing interests

The authors declare that they have no competing interests.

References

[1] W. Nelson, *Accelerated Life Testing: Statistical Models, Data Analysis and Test Plans*. John Wiley and Sons, New York, 1990.

[2] L. C. Tang, Y. S. Sun, T. N. Goh, and H. L. Ong, Analysis of step-stress accelerated-life-test data: a new approach, *IEEE Trans. Reliability*, **45**, 69–74, 1996.

[3] C. Xiong, Inferences on a simple step-stress model with type-II censored exponential data, *IEEE Trans. Reliability*, **47**, 142–146, 1998.

[4] E. Gouno, A. Sen, and N. Balakrishnan, Optimal step-stress test under progressive type-I censoring, *IEEE Trans. Reliability*, **53**, 388–393, 2004.

[5] A. H. Abdel-Hamid and E. K. Al-Hussaini, Estimation in step-stress accelerated life tests for the exponentiated exponential distribution with type I censoring, *Comput. Statist. Data Anal.*, **53**, 1328–1338, 2009.

[6] A. A. Ismail and H. M. Aly, Optimal planning of failure-step stress partially accelerated life tests under type-II censoring, *Statist. Comput. Simul.,* **80 (12)**, 1335–1348, 2010.

[7] N. Pushkarna, J. Saran, J. Nain and Verma, K..Progressively Type-II right censored order statistics from Hjorth distribution and related inference, *Stat. Optim Inf. Comput.*, **8(2)**, 481–498, 2020.

[8] R. M. EL-Sagheer and M. H. Hasaballah, Inference of process capability index Cpy for 3-Burr-XII distribution based on progressive Type-II censoring, *Inter. J. Math. Mathal. Sci.,* 1–13, 2020.

[9] R. M. EL-Sagheer and M. M. M. Mansour, The efficacy measurement of treatment methods: an application to stress-strength model, *Appl. Math. Inf. Sci.,* **14(3)**, 487– 492, 2020.

[10] R. M. EL-Sagheer, M. A. W. Mahmoud and M. M. M. Mansour, Inferences for Weibull-Gamma distribution in presence of partially accelerated life test, *J. Stat. Appl. Prob.* , **9(1)**, 93–107, 2020.

[11] R. M. EL-Sagheer and M. Ahsanullah, Statistical inference for a step-stress partially accelerated life test model based on progressively type-II censored data from Lomax distribution, *JASS*, **21 (4)**, 307–323, 2015.

[12] G. K. Bhattacharyya, and Z. A. Soejoeti, Tampered failure rate model for step-stress accelerated life test, *Comm. Statist. Theory Methods,* **18 (5)**, 1627–1643, 1989.

[13] S. A. Vander Wiel and W. Q. Meeker, Accuracy of approx confidence bounds using censored Weibull regression data from accelerated life tests, *IEEE Trans Reliab*., **39 (3)**, 346–351, 1990.

[14] A. C. Cohen, Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples, *Technometrics*, **5**, 579–588, 1965.

[15] J. F. Lawless, *Statistical Models and Methods for Lifetime Data*, John Wiley and Sons, New York, 1982.

[16] W. Q. Meeker, and L. A. Escobar, *Statistical Methods for Reliability Data*, Wiley, New York, 1998.

[17] B. Efron, The Bootstrap and other resampling plans, In: CBMS-NSF Regional Conference Seriesin, Applied Mathematics, SIAM, Philadelphia, PA,1982.

[18] P. Hall, Theoretical comparison of Bootstrap confidence intervals, *Annal. Stat.*, **16**, 927–953, 1988.

[19] J-P Kreiss, and E. Paparoditis, Bootstrap methods for dependent data: a review, *Korean Statist. Soc.*, **40**, 357–378, 2011.

[20] D. S. Bai, J. G. Kim, and Y. R. Chun, Design of failure censored accelerated life test sampling plans for lognormal and Weibull distributions, *Engineering Optimality*, **21**, 197–212, 1993.

Rashad M. EL-Sagheer is a lecturer of Mathematical Statistics at Mathematics Department Faculty of Science AL-Azhar University Cairo Egypt. Received his B.A. degree in Mathematics from Faculty of Science AL-Azhar University Cairo Egypt in 2004. He received Master's and PhD degrees in Mathematical Statistics from Faculty of Science Sohag University Egypt in 2011 and 2014 respectively. His areas of research where he has several publications in the international journals and conferences include: Statistical inference, Theory of estimation, Bayesian inference, Order statistics, Records, Theory of reliability, censored data, Life testing and distribution theory, Statistical computing and reliability analysis. He published and Co-authored more than 47 papers in reputed international journals. He supervised for M. Sc. and PhD students.

Moaiad Ahmad Khder (Senior Member, IEEE) received the Ph.D. degree from the Faculty of Information Science and Technology, The National University of Malaysia, in 2015. He is currently an Assistant Professor with the Computer Science Department, Applied Science University, Bahrain. He has been working on the area of mobile environment, mobile database, data science and cloud computing