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Further Properties of Semi Bornological Groups

Anwar N. Imran, F. K. Alhan and J. S. Farah*

Dep. of Math., Faculty of Sciences, University Diyala, Iraq

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Abstract: In this work, we study new structure in algebraic bornological structures, its called semi bornological groups, so the problem of boundedness for this kind of groups, which cannot bornologize because the product map is not bounded was solved. Additionally, we stydy further properties of semi bornological groups. In particular, we show that every left (right) translations in semi bornological groups are bornological isomorphisms. Therefor, the semi bornological groups are homogeneous. Furthermore, to find the sufficient condition on the left bornological group or right bornological group to be semi bornological group was investigated. However, we gave the sufficient condition to make any semi bornological group to be bornological groups. Finally, in contrast with semi bornological group, we show that bornological semi group is not homogenous space.

Keywords: Bornological Set, Bornological Group, Bounded Maps, Bornological Space, Bounded set, Bornological semi group.

1 Introduction

Historically, the idea of a bounded subset of a topological vector space was introduced by von Neuman (1935), its played an important role in functional analysis that motivated the concept of more general and abstract classes of bounded sets, its called bornology. A bornological group is set with two structures group and bornology such that the product map and inverse map are bounded. Note that every group can be turned into a bornological group by providing it with the discrete bornology. However, the problem of boundedness for this kind of groups, which cannot bornologize because the inverse map is not bounded was solved by [1] in 2017, by introducing new structure bornological semi groups (BSG). This structure can be easily done by taking semi groups instead of the groups and the product map bounded. But , in this work, we deal with the problem of boundedness for this kind of groups, which cannot bornologize because the product map is not bounded by introducing new structure semi bornological group which consist of group and bornology with the condition that left (right) translations are bounded and the motivation to introduce this kind of structurezs is that left (right) translation to be a bornological isomorphism and our new structure is homogeneous. Additionally, one of the main features of homogeneous is that they behave in the same way at any point. It follows that if we know how the bornology of a semi bornological group behaves at the identity, we know this bornology wherever, these observations suggest a certain approach to bornologize a group. However, the problem to understand what is the sufficient condition on semi bornological groups to guarante that will turn semi bornological groups into bornological groups was solved (see Theorem 12) . Furthermore, that left (right) translation is bornological isomorphism just in semi bornological groups. Then, this structure is totally different from bornological semigroups (see Example 11). For more details about bornological groups we will refer to ([1], [2], [3], [4], [5], [6], [7]).

2 Semi Bornological Group

In this part we show that there is a very natural way to bornologize any group into semi bornological group which is the left bornological and right bornological group at the same time. Though the construction is standard, it provides us with a large supply of the semi bornological groups.

Definition 1 *A left bornological group is a set G with two structures:*

–*G is a group;*
–*G is a bornological set.*

Such that for any fixed element $g \in G$ the left translation $l_g : G \to G$ is bounded.

* Corresponding author e-mail: anwarnooraldeen@yahoo.com

Definition 2 *A right bornological group is a set G with two structures:*

- *G is a group;*
- *G is a bornological set.*

Such that for any fixed element $g \in G$ the right translation $r_g : G \to G$ is bounded.

The semi bornological group is a left and right bornological group at the same time. In other words, a triple $(G, *, \beta)$ is called semi bornological group (SBG) if: $(G, *)$ is group and (G, β) is a bornological set. The group operation $* : G * G \to G$ that maps (g_1, g_2) to $g_1 * g_2$ is bounded in each variable. That mean the map $* : G * G \to G$ is bounded in variable g_1 when the map $r_g : G \to G$ defined by $g_1 \to g_1 * g$ is bounded for all g in G. Similarly,$*$ is bounded in fixed point g_2 when the map $l_g : G \to G$ defined by
$g_2 \to g * g_2$ is bounded for all g in G.

Remark 3 *For a triple $(G, *, \beta)$ or simple G to be a bornological group we require G to satisfy all of the conditions for a semi bornological group. Also, ather two requirements. The product map and the inverse map given by $x \to x^{-1}$ needs to be bounded.*

Example 4 *Let G be an arbitrary infinite group and let β contain of all subsets of G having infinite complements. Then G is bornological group.*

However, G is a semi bornological group with bounded inverse. Also, it is evident that every bornological group is semi bornological group.(by Remark 3).
However, there are some interesting natural examples that are not bornological group.

Example 5 *Assume that $G = \mathbb{Z}$ has two structures. One $G = (\mathbb{Z}, +)$ is a group. And, there is a bornology β on $\mathbb{Z}$ generated by the half lines $\{n \in \mathbb{Z} : n \le a\} for a \in \mathbb{Z}$. It is clear that*

$$\beta = \{n \in \mathbb{Z} : B \subset (-\infty, a]\}$$

is a bornology on $\mathbb{Z}$. However, first we have to fixed these two structures a group with bornology which should be compatible. Know, to prove that β is a semi bornological group. We should prove that the left translation

$$l_g : \mathbb{Z} \to \mathbb{Z}$$

is bounded. Thus, for any bounded subset $B \in \beta$ and any element $n \in \mathbb{Z}$ we have that $B \subset (-\infty, b]$. Then $n * B \subset (-\infty, n + b] \subset (-\infty, c]$ for $c \in \mathbb{Z}$, hence $n * B \in \beta$. Then $(\mathbb{Z}, +, \beta)$ is a left bornological group. Since $(\mathbb{Z}, +)$ is commutative group, then the right translation is bounded. Thus $(\mathbb{Z}, +, \beta)$ is a right bornological group and it is (SBG). However, $(\mathbb{Z}, \beta)$ is not bornological group since the image of $B = (-\infty, b]$ under inverse map is $-B = (+\infty, -b] \notin \beta$, then the inverse operation $x \to x^{-1}$ is not bounded.

The group with discrete bornology is both a bornological group and semi bornological group.

Example 6 *$(S_3, \circ, \beta)$ is both SBG and BG.*

An equivalent statement in terms of bounded set will now be presented.

Definition 7 *The map $f : G * G \longrightarrow G$ that maps (g_1, g_2) to $g_1 * g_2$ is bounded in g_1 when, for each B_1 bounded set contain g_1 there exists B bounded set contain $g_1 g$ and $g \in G$ such that $B_1 g \in B$.*

An equivalent statement in terms of bounded sets will now be proved with respect to bornological group.

Proposition 8 *Let the map $f : G * G \longrightarrow G$ that maps (g_1, g_2) to $g_1 * g_2$ bounded in both variables when for all B_1 contain g_1 and B_2 contain g_2 there exists B contain $g_1 \cdot g_2$ such that $B_1 B_2 \subset B$.*

Proof. It is clear from the (Definition 7) for every bounded set B_1 contain g_1 there is bounded set B contain $g_1 \cdot g_2$ with $f(B_1) \subset B$. From this we get that $B_1 g \subset B$. Thus it is clear that $B_1 B_2 \subset B$.

The reader with some knowledge of bornological groups and bornological semigroups should notice the similarity with respect to the fundamental constructions. Obviously, every subgroup of semi bornological group, endowed with the subspace bornology, is again a semi bornological group. Moreover, if (G_i, β_i) is a semi bornological groups, $i \in I$. Then the bornological product is semi bornological group. The next result is a starting point for a rich and profound theory of semi bornological group.

Proposition 9 *In every left bornological group, every left translation l_g is bornological isomorphism.*

Proof. It is clear that in a left bornological group, every left translation l_g is a bounded and bijection. Since

$$(l_g \circ l_g^{-1})(x) = l_g(l_g^{-1}(x)) = l_g(g^{-1}(x)) = gg^{-1}(x) = l_e(x) = x = Id_G.$$

Its the identity map, so for any $g \in G, g^{-1}$ is invertible with inverse map l_g^{-1} follows that the inverse of l_g is also bounded map, that l_g is bornological isomorphism of G onto itself.

A similar argument applied when G is a right bornological group. That mean, the translation of G has a bornological isomorphism in each case. A useful result we get from this proposition is as follow.

Corollary 1 *Let G be a semi bornological group and $g_1, g_2 \in G$ there exists a bornological isomorphism f such that $f(g_1) = g_2$.*

Proof. From (Proposition 9) we know that l_g is a bornological isomorphism for all $g \in G$ by letting

$$f = r_g^{-1} g_2$$

We get

$$f(g_1) = g_2$$

as required. By (Proposition 9) we find that in every semi bornological group G all right and left translations are bornological isomorphisms of G onto itself. However, a bornological set X is said to be homogeneous if, for each $x, y \in X$, there exists a bornological isomorphism f from the space X onto itself such that $f(x) = y$. From (Proposition 9), then our new structure is homogeneous space.

Theorem 10 *Every semi bornological group is a homogenous space.*

Proof. Take any element a and b in G, and put $c = ba^{-1}$. Then $\lambda_c(a) = ca = ba^{-1}a = b$. Since, by (Proposition 9), λ_c is a bornological isomorphism which implies that the space G is homogeneous. Given a group G, it follows from (Theorem 10) that to make G into a semi bornological group, we can only use homogeneous.Therefore, one of the main for homogeneous space is that they behave same way at any point. It follows that if we know how the bornology of semi bornological groups behaves at the identity element, we know this bornology everywhere. This observation suggests a certain approach to bornologize a group G, so if bornological set is homogeneous, then it can bornologize every group to bornological group.

On the contrary, a bornological semi group, even if it has identity, then not nessasry to be homogeneous, the example below shows that (Theorem 10) cannot be extended to bornological semigroup.

Example 11 *Take the bounded unit interval $J = [0,1]$, and put $xy = \max\{x,y\}$ for all $x, y \in J$. Clearly, J with the canonical bornology, this product map is a bornological semigroup (with 0 in the role of identity). However, J is not a homogeneous space since no bornological isomorphism of J takes 0 to $1/2$ or any element to 0.*

By means of the previous example, a clear difference was given between the two structures bornological semigroups and semibornological groups.
By the next theorem, we gave sufficient condition for left or right bornological group to be a semi bornological group and it is one of the main important results in this section.

Theorem 12 *Every left bornological group G with bounded inverse is a semi bornological group.*

Proof. We can reach from x to x_a in three steps pass from x to x^{-1} by bounded inverse map, then from x^{-1} to $a^{-1}x^{-1}$, and, finally, from $a^{-1}x^{-1}$ to x_a. This mean

$$r_a = l_n \circ l_a^{-1} \circ l_n.$$

Since all the mapping on the side right of the above equality are bounded and the composition of bounded maps are bounded, it follows that the mapping r_a is also bounded.

However, by the following practical fact we gave the sufficient condition for made any semi bornological group became bornological group and it is one of the important results in this section.

Theorem 13 *Let G be a semi bornological group such that for every bounded set B contains the identity element e, there exists a bounded set A contains e satisfying the condition $A^{-1} \subset B$. Then the inverse map in G is bounded. Therefor, G is a bornological group.*

Proof. Take an element $a \in G$ and any bounded set C contains $a^{-1.}$ Since G is a left bornological group, there exists a bounded set B contains e such that $a^{-1}B \subset C$. By the assumption, there exists a bounded set A contains e such that $A^{-1} \subset B$. Then Aa is a bounded set contains a since G is a right bornological group. Now we have

$$(Aa)^{-1} = a^{-1}A^{-1} \subset a^{-1}B \subset C.$$

It follows that the inverse operation on G is bounded. Therefor the group G is a bornological group.

We describe some simple facts on families of bounded sets in semi bornological groups. We start with the proposition which shows, in left bornological groups there are an intimate relationship between the set of the form HB, where B is bounded set and H subset.

Proposition 14 *If G is a left bornological group then, for any bounded subset B of G and any subset H of G, the sets BH and HB are bounded.*

Every left translation of G is bornological isomorphism, by (Proposition 9) since $HB = \bigcup_h \in Hl_h(B)$, the conclusion follows. A similar argument in the case when G is a right bornological group. So, if G is a semi bornological group then, for any bounded subset B of G and any subset H of G, the sets BH and HB are bounded. We obtain the following corollary.

Corollary 2 *Suppose that a subgroup H of a right (or left) bornological group G contained in a non-empty bounded subset of G, then H is bounded in G.*

Proof. Let B be a bounded subset of G with $H \subset B$. A left bornological contains of group G and a bornology β on G such that β is stable under hereditary. Then, the set $H \subset B$ is bounded in G.

3 Conclusion

In this work, we constacted new structure, its called semi bornological groups to solve the proplem of boundedness for kind of groups which cannot be bornological groups becase the product map is not bounded and this new structure is homogenous and its totaly difrent from bornological semigroups. The main important result that, in every semi bornological groups all left (right) translations are bornological isomorphisms and our new structure semi bornological group is homogeneous. Furthermore, research on the problem of which sufficient condition on left or right bornological group implied that its semi bornological group are solved. Also, the sufficient condition made any semi bornological group be bornological group are gave. Furthermore, in contrast with semi bornological group, its shown that bornological semi group is not homogeneous space. Finally, alot of futher properties of semi bornological groups are given and proven .

Competing interests

The authors declare that they have no competing interests.

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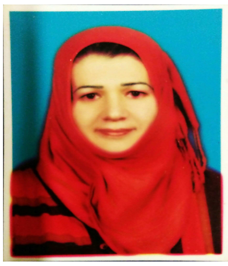

Anwar Nooruldeen She got her master in science of mathematics. Afterward she served as an assistant lecture in the faculty of engineering university of Douala. She finshed PhD in UPM Malaysia in field of pure mathematics.

Alhan Faisel, She got her Bachelor of Science in mathematics. Afterward, she served as a lecturer in Technical Institute / Baquba, Central Technical University. She study Master Degree in (Bornological Space) at University of Diyala.

Farah Jaafar Sadiq She is master student from Tikrit university, The college of Computer Science and Mathematics