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# On a Third-Order $P$-Laplacian Boundary Value Problem at Resonance on the Half-Line

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**Abstract:** In this paper, a one-dimensional third-order p-Laplacian boundary value problem at resonance on the half-line is studied. We apply the extension of Mawhin's coincidence degree theory due to Ge and Ren to obtain the existence of solutions. The results do not only generalize but also improve some known results on third-order p-Laplacian boundary value problems at resonance.

**Keywords:** Half-line, integral boundary condition, $p$-Laplacian, Resonance, Third order boundary value problem.

## 1 Introduction

In this paper, we study the third-order nonlinear boundary value problem with a $p$-Laplacian of the form:

$$\big(d(t)\varphi_p(u''(t))\big)' = h\big(t,u(t),u'(t),u''(t)\big) \text{ a.e } t \in (0,\infty), \tag{1}$$

satisfying

$$u'(0) = \sum_{i=1}^{n} \beta_i \int_0^{\eta_i} u(t)dt, u(0) = 0, \lim_{t\to\infty}(d(t)\varphi_p(u''(t))) = 0. \tag{2}$$

Where the right hand side of (1) satisfies the Carathéodory condition with respect to
$L^1[0,\infty), 0 \le \beta_i < \infty, \beta_i \in \Re, i = 1,2...,n, \sum_{i=1}^{n} \beta_i\eta_i^2 = 2.$
$d \in [C[0,\infty) \cap C^2(0,\infty)], d(t) > 0 \forall t \ge 0.$
$\varphi_p(s) = \mid s \mid^{p-2} s, p > 1$ and $0 \le \eta_i < \infty, i = 1,2...n$
Boundary value problems on the half-line have various applications in plasma physics and the theory of drain flows. Integral boundary conditions, on the other hand exist in applications such as, population dynamics, blood flow models, heat conduction, underground water flow, etc. The boundary value problem

$$(q(t)u''(t))' = g(t,u(t),u'(t),u''(t)), t \in (0,\infty)$$

$$u'(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt, u(0) = 0, \lim_{t\to\infty} q(t)u''(t) = 0$$

was studied by Iyase [6] when p =2 using the Mawhin's coincidence degree arguments. However, when $p \neq 2$ , $\varphi_p(u)$ is no longer linear with respect to $u$. In this case, Mawhin's continuation theorem cannot be applied directly as the case in [6]. From the existing results in the literature for the resonance cases, some results on second order boundary value problems with a $p$-Laplacian have been established. On the other hand, third-order boundary value problems with a $p$-Laplacian satisfying integral boundary conditions on the half-line have not received much attention. However, there have been some studies on higher order boundary value problems with a $p$-laplacian, on bounded domains. For some results on boundary value problems with a $p$-Laplacian, see, [2,4,8,10,12,13,14,15] and the references therein.

The boundary value problem (1)-(2) is called a problem at resonance if
$Tu = (d(t)\varphi_p(u''(t)))' = 0$ has nontrivial solutions under the boundary conditions (2), i.e when $\dim\ker T \geqslant 1$. When $\ker T = 0$, the differential operator is invertible. In this case, the problem is said to be at non-resonance. The rest of this paper is organized as follows: In Section 2, we recall some background definitions and technical results. Section 3 is devoted to proving the main existence results. In Section 4 an example is presented to illustrate our result.

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## 2 Some definitions and Technical results

In this section, we introduce some definitions and lemmas that will be used in the subsequent sections which include Ge-Ren's continuation theorem.

**Lemma 2.1** *[5] Let $\varphi_p(s) = \mid s \mid^{p-2} s$. Then $\varphi_p$ has the following properties*

*(i)$\varphi_p$ is continuous, monotonically increasing and invertible with $\varphi_p^{-1} = \varphi_q, q > 1$ a real constant such that $\frac{1}{p} + \frac{1}{q} = 1$.*
*(ii)$\mid \varphi_p(u) \mid = \varphi_p(\mid u \mid), u\varphi_p(u) \geq 0$, for $u \in \Re$.*
*(iii)$\varphi_p(u+v) \leq (\varphi_p(u) + \varphi_p(v))$, $1 \leq p < 2$*
*(iv)$\varphi_p(u+v) \leq 2^{p-2}(\varphi_p(u) + \varphi_p(v)), p \geq 2$.*

***Definition 2.2*** *The map $h : [0,\infty) \times \Re^n \to \Re$ is $L^1$-Carathéodory if the following conditions hold*

*(i)for each $u \in \Re^n$, the mapping $t \to f(t,u)$ is Lebesgue measurable.*
*(ii)for a.e $t \in [0,\infty)$, the mapping $u \to f(t,u)$ is continuous on $R^n$.*
*(iii)for each $r > 0$, there exists an $\alpha_r \in L^1[0,\infty)$ such that for a.e $t \in [0,\infty)$ and every $u$ such that $\mid u \mid \leq r$, we have $\mid f(t,u) \mid \leq \alpha_r(t)$.*

***Definition 2.3*** *Let $X$ and $Z$ be Banach Spaces. A continuous operator $T : X \cap \ dom\, T \to Z$ is called quasi-linear if and only if $Im\, T$ is a closed subset of $Z$ and $\ker T$ is linearly homeomorphic to $\Re^n$.*

***Definition 2.4*** *Let $X$ be a Banach space with $X_1 \subset X$ a subspace. A mapping $Q : X \to X_1$ is called a semi-projector if $Q$ satisfies*

*(i)$Q^2 u = Qu, u \in X$.*
*(ii)$Q(\lambda u) = \lambda Qu, u \in X, \lambda \in R$.*

***Definition 2.5*** *$N_\lambda : \overline{\Omega} \to Z, \lambda \in [0,1]$ is said to be $T$-compact in $\overline{\Omega}$ if there exists a subspace $Z_1 \subset Z$ with $dim Z_1 = dim \ker T$ and an operator $S : \overline{\Omega} \times [0,1] \to X$ continuous and compact such that for $\lambda \in [0,1]$*

$$(I-Q)N_\lambda(\overline{\Omega}) \subset ImT \subset (I-Q)Z \tag{3}$$

$$QN_\lambda u = 0, \lambda \in (0,1) \text{ iff } QNu = 0, u \in \Omega \tag{4}$$

$$S(.,0) \text{ is the zero operator} \tag{5}$$

$$S(.,\lambda) \mid_{A_\lambda} = (I-P) \mid_{A_\lambda} \text{ where} A_\lambda = \{u \in \Omega : Tu = N_\lambda u\} \tag{6}$$

$$T[P + S(.,\lambda)] = (I-Q)N_\lambda \tag{7}$$

*Where $P : X \to X$ is a projector and $Q$ is a Semi-projector such that $Im\, P = \ker T$ and $Im\, Q = Z_1$*

***Theorem 1.*** *[4] Let $X$ and $Z$ be two Banach spaces with norms $\| \,.\, \|_X$ and $\| \,.\, \|_Z$ respectively and $\Omega \subset X$ be an open and bounded set. Suppose $T : X \cap \ dom\, T \to Z$ is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Z, \lambda \in [0,1]$ is T-Compact. In addition if*

*(1)$Tu \neq N_\lambda u$, for $\lambda \in (0,1), u \in \ dom\, T \cap \partial\Omega$.*
*(2)deg $\{JQN, \Omega \cap \ker T, 0\} \neq 0$.*

*Where $J : Im\, Q \to \ker T$ is a homeomorphism with $J(\theta) = \theta$, where $\theta$ is the origin and $N_1 = N$. Then the abstract equation $Tu = Nu$ has at least one solution in $\overline{\Omega}$.*

*Let $AC[0,\infty)$ be the space of absolutely continuous functions on $[0,\infty)$. We shall use the following spaces*

$$X = \{u : [0,\infty) \to \Re \mid u, d\varphi_p(u'') \in AC[0,\infty), \lim_{t\to\infty} e^{-t} \mid u^{(i)}(t) \mid$$

$$\text{exists}, 0 \leq i \leq 2, (d\varphi_p(u''))' \in L^1[0,\infty) \text{ and } \varphi_p\left(\frac{1}{d}\right) \in L^1[0,\infty)\} \tag{8}$$

*With the norm*

$$\| u \| = \max\left[\sup_{t\in[0,\infty)} e^{-t} \mid u(t) \mid, \sup_{t\in[0,\infty)} e^{-t} \mid u'(t) \mid, \sup_{t\in[0,\infty)} e^{-t} \mid u''(t) \mid\right] \tag{9}$$

*Then $X$is a Banach Space. We let $Z = L^1[0,\infty)$ endowed with the norm*

$$\| y \|_1 = \int_0^\infty \mid y(t) \mid dt, y \in Z.$$

*To prove the compactness of the operator $T$ we use the following compactness criterion.*

***Theorem 2.*** *[1] Let $X$ be the space of all bounded continuous vector valued functions on $[0,\infty)$ and $V \subset X$. Then $V$ is relatively compact in $X$ if the following conditions hold:*

*(i)$V$ is bounded in $X$;*
*(ii)the functions from $V$ are equicontinuous on any compact interval of $[0,\infty)$;*
*(iii)the functions from $V$ are equiconvergent at infinity.*

*We introduce the mapping $T : domT \subset X \to Z$ defined by*

$$Tu = (d(t)\varphi_p(u''(t)))', t \in [0,\infty) \tag{10}$$

*where*

$$dom\, T = \left\{u \in X : u'(0) = \sum_{i=1}^n \beta_i \int_0^{\eta_i} u(t)dt,\ u(0) = 0, \lim_{t\to\infty} d(t)\varphi_p(u''(t)) = 0\right\}$$

*We define the operator $N_\lambda : X \to Z$ by $N_\lambda u(t) = \lambda h(t,u(t),u'(t),u''(t))$.*
*Then (1.1) - ( 1.2) takes the form $Tu = N_\lambda u$ when $\lambda = 1$*

***Lemma 2.6*** *If $\sum_{i=1}^n \beta_i \eta_i^2 = 2$ then*

*(i)*$\ker T = \{u \in domT : u(t) = ct, c \in R, t \in [0,\infty)\}$.
*(ii)$Im\, T = \{y \in Z : \sum_{i=1}^n \beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q \left(\frac{1}{d(r)}\right) \varphi_q\left(\int_r^\infty y(\tau)d\tau\right) drdsdt = 0\}$.*
*(iii)$T : domT \to Z$ is a quasi-linear operator.*

***proof:*** *It is easily verified that (i) holds.*
*To prove (ii), Let $y \in Z$ and consider the equation*

$$(d(t)\varphi_p(u''(t)))' = y(t) \tag{11}$$

*Then using (1.2) we obtain*

$$d(t)\varphi_p(u''(t)) = -\int_t^\infty y(\tau)d\tau.$$

*Thus*

$$u''(t) = -\varphi_q\left(\frac{1}{d(t)}\right)\varphi_q\left(\int_t^\infty y(\tau)d\tau\right)$$

*or*

$$u(t) = -\int_0^t\int_0^s \varphi_q\left(\frac{1}{d(r)}\right)\varphi_q\left(\int_r^\infty y(\tau)\tau\right) drds + tu'(0) \tag{12}$$

*In view of (2) and $\sum_{i=1}^n \beta_i\eta_i^2 = 2$ we obtain*

$$\sum_{i=1}^n \beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{d(r)}\right)\varphi_q\left(\int_r^\infty y(\tau)d\tau\right) drdsdt = 0 \tag{13}$$

*if (11) holds, then*
*$u(t) = ct - \int_0^t\int_0^s \varphi_q\left(\frac{1}{d(r)}\right)\varphi_q(\int_r^\infty y(\tau)d\tau)drds$ is a solution of (11), where $c \in \Re$.*
*Thus*

$$Im\, T = \left\{ y \in Z : \sum_{i=1}^n \beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{d(r)}\right)\varphi_q\left(\int_r^\infty y(\tau)d\tau\right)drdsdt. = 0 \right\}$$

*Hence, we have* $\dim\ker T = 1 < \infty$, *$Im\, T \subset Z$ is closed. Therefore, $T$ is a quasi-linear operator.*

***Lemma 2.7*** *If $h$ is a $L^1$-Carathéodory function then $N_\lambda : \overline{V} \to Z$ is T-compact in $\overline{V}$ for $V \subset X$ an open and bounded subset with the origin $\theta \in V$*

***Proof*** *Define the continuous operator $Q : Z \to Z$ by*

$$Qy(t) = \rho(t)\sum_{i=1}^n \beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{d(r)}\right) \varphi_q\left(\int_r^\infty y(\tau)d\tau\right) drdsdt \tag{14}$$

*where*

$$\rho(t) = \frac{e^{-t}}{\sum_{i=1}^n \beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q\left(\frac{e^{-r}}{d(r)}\right) drdsdt} \tag{15}$$

*It is easily deduced that $Q^2y = Qy$ and $Q(\lambda y) = \lambda Qy$ for $y \in Z, \lambda \in \Re$. Thus, $Q$ is a semi-projector with* $\dim\ker T = \dim ImQ = 1$.
*From the definition of $Q$ we can derive (3), (4) and (5). To establish conditions (6) and (7), define*

$$S(u,\lambda)(t) = -\int_0^t\int_0^s [\varphi_q(\frac{1}{d(r)})\varphi_q(\int_r^\infty \lambda(h(\tau,u(\tau),u'(\tau),u''(\tau) - (Qh)(\tau))d\tau)]drds. \tag{16}$$

*Let $P : X \to \ker T$ be defined by*

$$Pu(t) = u'(0)\ t, t \in [0,\infty). \tag{17}$$

*For any $u \in A_\lambda = \{u \in \overline{V} : Tu = N_\lambda u\}$*

$$\lambda h(t,u(t),u'(t),u''(t)) = (d(t)\varphi_p(u''(t)))' \in Im\, T \subset \ker Q.$$

*Thus,*

$$\begin{aligned} S(u,\lambda)(t) &= -\int_0^t\int_0^s \left[\varphi_q\left(\frac{1}{d(r)}\right)\varphi_q\left(\int_r^\infty \lambda(h(\tau,u(\tau),u'(\tau),u''(\tau)\right) - (Qh)(\tau))d\tau\right]drds \\ &= \int_0^t\int_0^s \left[\varphi_q\left(\frac{d(r)}{d(r)}\right)u''(r)drds\right] \\ &= \int_0^t\int_0^s u''(\tau)d\tau ds = u(t) - tu'(0) = (I-P)u(t). \end{aligned} \tag{18}$$

*Also,*

$$T\left[Pu+S(u,\lambda)\right](t)$$
$$=\left\{d(t)\varphi_p[u'(0)t-\int_0^t\int_0^r\varphi_q\left(\frac{1}{d(s)}\right)\varphi_q\right.$$
$$\left.\left(\int_s^\infty\left(\lambda h(\tau,u(\tau),u'(\tau),u''(\tau))-\lambda(Qh)(\tau)\right)d\tau\right)dsdr]''\right\}'$$
$$=\left[-d(t)\varphi_p\varphi_q\left(\frac{1}{d(t)}\right)\varphi_q\left(\int_t^\infty(\lambda h(\tau,u(\tau),u'(\tau),u''(\tau))\right.\right.$$
$$\left.\left.-\lambda(Qh)(\tau))d\tau\right)\right]'$$
$$=\lambda h(t,u(t),u'(t),u''(t))-\lambda Qh(t,u(t),u'(t),u''(t))$$
$$=[(I-Q)N_\lambda(u)](t). \tag{19}$$

*This verifies (6) and (7). Next, we show that $S$ is relatively compact for any $\lambda\in[0,1]$. Let $V\subset X$ be bounded, that is there exists an $r>0$ such that $r=\sup\{\|u\|: u\in V\}$. Since $h:[0,\infty)\times\Re^3\to\Re$ is $L^1$ Carathèodory, there exits $\alpha_r\in L^1[0,\infty)$ such that for all $u\in V$ and a.e $t\in[0,\infty)$.*

$$|\, h(t,u(t),u'(t),u''(t))\,|\leq\alpha_r(t) \tag{20}$$

*For $u\in V$*

$$e^{-t}\,|\,S(u,\lambda)\,|\leq\sup_{t\in[0,\infty)}e^{-t}t\,\|\,\varphi_q\left(\frac{1}{d}\right)\|_1\,\varphi_q[\|\,\alpha_r\,\|_1 + \|\,Qh\,\|_1] \tag{21}$$

$$e^{-t}\,|\,S'(u,\lambda)\,|\leq\sup_{t\in[0,\infty)}e^{-t}\,\|\,\varphi_q\left(\frac{1}{d}\right)\|_1\,\varphi_q[\|\,\alpha_r\,\|_1 + \|\,Qh\,\|_1]$$

$$=\|\,\varphi_q\left(\frac{1}{d}\right)\|_1\,\varphi_q[\|\,\alpha_r\,\|_1+\|\,Qh\,\|_1] \tag{22}$$

$$e^{-t}\,|\,S''(u,\lambda)\,|\leq\sup_{t\in[0,\infty)}e^{-t}\,\|\,\varphi_q\left(\frac{1}{d}\right)\|_\infty\,\varphi_q[\|\,\alpha_r\,\|_1 + \|\,Qh\,\|_1]$$
$$=\|\,\varphi_q\left(\frac{1}{d}\right)\|_\infty\,\varphi_q[\|\,\alpha_r\,\|_1+\|\,Qh\,\|_1\,.] \tag{23}$$

*Therefore, from (2.19), (2.20) and (2.21) we obtain*

$$\|\,S(u,\lambda)\,\|<\max\left\{\sup_{t\in[0,\infty)}e^{-t}t\,\|\,\varphi_q\left(\frac{1}{d}\right)\|_1,\right.$$
$$\left.\|\,\varphi_q\left(\frac{1}{d}\right)\|_\infty\right\}\varphi_q[\|\,\alpha_r\,\|_1+\|\,Qh\,\|_1]$$
$$=\max\left\{\sup_{t\in[0,\infty)}e^{-t}t,1,\frac{\|\,\varphi_q\left(\frac{1}{d}\right)\|_\infty}{\|\,\varphi_q\left(\frac{1}{d}\right)\|_1}\right\}$$
$$\|\,\varphi_q\left(\frac{1}{d}\right)\|_1\,[\varphi_q\,\|\,\alpha_r\,\|_1+\|\,Qh\,\|_1]$$
$$=A_1\,\|\,\varphi_q\left(\frac{1}{d}\right)\|_1\,[\varphi_q\,\|\,\alpha_r\,\|_1+\|\,Qh\,\|_1]$$
$$=L_1 \tag{24}$$

*$S(.,\lambda)$ is therefore uniformly bounded in $X$.*
*Now for any $t_1,t_2\in[0,B].B\in(0,\infty)$ with $t_1<t_2$, $u\in V$, we have*

$$|e^{-t_2}S(u,\lambda)(t_2)-e^{-t_1}S(u,\lambda)(t_1)|$$
$$=\left|\int_{t_1}^{t_2}\left[e^{-\tau}S(u,\lambda)(\tau)\right]'d\tau\right|$$
$$\leq 2(t_2-t_1)\,\|\,S(u,\lambda)\,\|$$
$$\leq 2(t_2-t_1)L_1\to 0\ as\ t_1\to t_2,$$

$$|e^{-t_2}S'(u,\lambda)(t_2)-e^{-t_1}S'(u,\lambda)(t_1)|$$
$$=\left|\int_{t_1}^{t_2}\left[e^{-\tau}S(u,\lambda)(\tau)\right]'d\tau\right|$$
$$=\left|\int_{t_1}^{t_2}\left[-e^{-\tau}S'(u,\lambda)(\tau)+e^{-\tau}S''(u,\lambda)(\tau)\right]d\tau\right|$$
$$\leq 2(t_2-t_1)\,\|\,S(u,\lambda)\,\|$$
$$\leq 2(t_2-t_1)L_1\to 0\ as\ t_1\to t_2,$$

$$\begin{aligned}
&|e^{-t_2}\varphi_p(S''(u,\lambda)(t_2)-e^{-t_1}\varphi_p(S''(u,\lambda))(t_1)| \\
&=\left|\frac{-e^{-t_2}}{d(t_2)}\int_{t_2}^{\infty}\lambda[h(\tau,u(\tau),u'(\tau),u''(\tau)) \right.\\
&\quad -(Qh)(\tau)]d\tau \\
&\quad +\frac{e^{-t_1}}{d(t_1)}\int_{t_1}^{\infty}\lambda\Big[h(\tau,u(\tau),u'(\tau),u''(\tau)) \\
&\quad \left. -(Qh)(\tau)\Big]d\tau\right| \\
&\le\left|\frac{e^{-t_2}}{d(t_2)}-\frac{e^{-t_1}}{d(t_1)}\right|\int_{t_2}^{\infty}|\lambda[h(\tau,u(\tau),u'(\tau),u''(\tau)) \\
&\quad -(Qh)(\tau)]|d\tau \\
&\quad +\frac{e^{-t_1}}{d(t_1)}\int_{t_1}^{t_2}|\lambda[h(\tau,u(\tau),u'(\tau),u''(\tau) \\
&\quad -(Qh)(\tau)]\mid d\tau \\
&\le\|\frac{1}{d}\|_{\infty}^{2}|d(t_1)e^{-t_2}-d(t_2)e^{-t_1}|\int_{t_2}^{\infty}|\Big[\alpha_r(s) \\
&+|Qh|(s)\Big]|ds+\|\frac{1}{d}\|_{\infty}\int_{t_1}^{t_2}|\Big[\alpha_r(s)+Qh(s)\Big]|ds \\
&\le\|\frac{1}{d}\|_{\infty}^{2}|d(t_1)e^{-t_2}-d(t_2)e^{-t_1}|[\|\alpha_r\|_1+\|Qh\|_1] \\
&+\|\frac{1}{d}\|_{\infty}\int_{t_1}^{t_2}[\alpha_r(s)+|Qh|(s)]ds\to 0 \text{ as } t_1\to t_2.
\end{aligned}$$

*This implies that*

$$|e^{-t_2}S''(u,\lambda)(t_2)-e^{-t_1}S''(u,\lambda)(t_1)|\to 0 \text{ as } t_1\to t_2$$

*Therefore, $S(u,\lambda)(V)$ is equicontinuous on every compact subset of $[0,\infty)$.*
*Next, we establish that $S(.,\lambda)(V)$ is equiconvergent at infinity.*
*For $u\in V$, we have*

$$\begin{aligned}
&e^{-t}|S(u,\lambda)(t)| \\
&\quad =e^{-t}|\int_0^t\int_0^s\Big[\varphi_q\left(\frac{1}{d(r)}\right)\varphi_q\Big(\int_r^{\infty}\lambda[h(\tau,u(\tau),u',u''(\tau)) \\
&\qquad -(Qh)(\tau)]d\tau\Big)\Big]drds| \\
&\quad \le e^{-t}t\|\varphi_q\left(\frac{1}{d}\right)\|_1\varphi_q[\alpha_r\|_1+\|Qh\|_1]\to 0 \text{ as } t\to\infty,
\end{aligned}$$

$$\begin{aligned}
&e^{-t}|S'(u,\lambda)(t)| \\
&\quad =e^{-t}|\int_0^t\varphi_q\left(\frac{1}{d(s)}\right)\varphi_q[\int_s^{\infty}\lambda(h(\tau,u(\tau),u'(\tau),u''(\tau)) \\
&\qquad -(Qh)(\tau))d\tau]ds| \\
&\quad \le e^{-t}\|\varphi_q\left(\frac{1}{d}\right)\|_{\infty}\varphi_q[\|\alpha_r\|_1+\|Qh\|_1] \\
&\qquad \to 0 \text{ as } t\to\infty,
\end{aligned}$$

$$\begin{aligned}
&e^{-t}|S''(u,\lambda)(t)| \\
&\quad =e^{-t}|\varphi_q\left(\frac{1}{d(t)}\right)\varphi_q[\int_t^{\infty}\lambda(h(\tau,u(\tau),u'(\tau),u''(\tau)) \\
&\qquad -(Qh)(\tau))d\tau]| \\
&\quad \le e^{-t}\|\varphi_q\left(\frac{1}{d}\right)\|_{\infty}\varphi_q[\|\alpha_r\|_1+\|Qh\|_1] \\
&\qquad \to 0 \text{ as } t\to\infty.
\end{aligned}$$

*This shows that $S(u,\lambda)(V)$ is equiconvergent at infinity. Since all the conditions of theorem 2.2 are satisfied, the set $S(u,\lambda)(V)$ is relatively compact. The continuity of the mapping $S(u,\lambda)$ follows from the Lebesgue dominated convergence theorem.*

## 3 Main Result

***Theorem 3.*** *Let $h$ be a $L^1$ - Carathéodory function. Assume that the following conditions hold.*

*(A0)* $\sum_{i=1}^{n}\beta_i\eta_i^2=2,\ \sum_{i=1}^{n}\beta_i\int_0^{\eta_i}\int_0^t\int_0^s\varphi_q\left(\frac{e^{-r}}{d(r)}\right)drdsdt\neq 0.$

*(A1)There exists $M_1>0$ such that for $u\in domT/KerT$ satisfying $|u'(t)|>M_1$ for $t\in[0,\infty)$ we have $QN_\lambda u\neq 0$.*

*(A2)There exist positive functions $a_1,a_2,a_3,r\in L^1[0,\infty)$such that.*

$$|h(t,u_1,u_2,u_3)|\le e^{-t(p-1)}\Big[a_1(t)|u_1|^{p-1}+a_2(t)|u_2|^{p-1}+a_3(t)|u_3|^{p-1}\Big]+r(t) \tag{25}$$

*(A3)There exists $M_2>0$ such that for every $c\in R$ with $|c|>M_2$ we have either*

$$c\sum_{i=1}^{n}\beta_i\int_0^{\eta_i}\int_0^t\int_0^s\varphi_q\left(\frac{1}{d(r)}\right)\varphi_q\Big(\int_r^{\infty}\lambda h(\tau,c\tau,c,0)d\tau\Big)drdsdt>0 \tag{26}$$

*or*

$$c\sum_{i=1}^{n}\beta_i\int_0^{\eta_i}\int_0^t\int_0^s\varphi_q\left(\frac{1}{d(r)}\right)\varphi_q\Big(\int_r^{\infty}\lambda h(\tau,c\tau,c,0)d\tau\Big)drdsdt<0. \tag{27}$$

*Then the BVP (1)- (2) has at least one solution provided*

$$2^{2(q-2)}\|\varphi_q\left(\frac{1}{d}\right)\|_1 A_1\sum_{i=1}^{3}\|a_i\|_1^{q-1}<1 \text{ for } 1<p<2 \tag{28}$$

*or*

$$\| \varphi_q\left(\frac{1}{d}\right) \|_1 A_1 \sum_{1=i}^{3} \| a_i \|_1^{q-1} < 1 \; for \; p \geq 2. \tag{29}$$

*To prove theorem 3.1, we first derive some Lemmas.*

***Lemma 3.1****Let*
$W_1 = \{u \in domT/KerT : Tu = N_\lambda u \; for \; some \; \lambda \in (0,1].\}$

*Then $W_1$ is a bounded set.*
***proof:****Let $u \in W_1$. Assume that $Tu = N_\lambda u$. Then $QN_\lambda u = 0$. Therefore, from (A1) there exists $t_0 \in [0,\infty)$ such that*

$$| u'(t_0) | \leq M_1. \tag{30}$$

*Then*

$$| u'(0) | = | u'(t_0) - \int_0^{t_0} u''(s)ds | \leq M_1 + \| u'' \|_1 \tag{31}$$

*For $u \in W_1$, $(I-P)u \in domT \bigcap KerP$. Thus, from (18) and (24)*
*we have,*

$$\| (I-P)u \| = \| S(u,\lambda) \| < L_1$$

*where $L_1$ is defined in (24).*
*From the definition of $P$ we have*

$$Pu(t) = u'(0)t, (Pu)'(t) = u'(0), t \in [0,\infty)$$

*Hence, from (31) we obtain*

$$\begin{aligned} \| Pu \| &= \max\left\{ \sup_{t\in[0,\infty)} e^{-t}t \,| u'(0) |, | u'(0) | \right\} \\ &= \max\left\{ \sup_{t\in[0,\infty)} e^{-t}t, 1 \right\} | u'(0) | < A_1 | u'(0) | \\ &< A_1 \left[M_1 + \| u'' \|_1\right] = A_1 \| u'' \|_1 + A_1 M_1 \end{aligned} \tag{32}$$

$$\begin{aligned} \| u \| &= \| Pu + (I-P)u \| \\ &\leq \| Pu \| + \| (I-P)u \| \\ &\leq \| u'' \|_1 A_1 + L_1 + M_1 A_1 \\ &= \| u'' \| A_1 + L_2. \end{aligned} \tag{33}$$

*where $L_2 = L_1 + M_1 A_1$.*
*If $p < 2$ then from (12), (25) and Lemma 2.1 we get*

$$\begin{aligned} \| u'' \|_1 &= \int_0^\infty \left| \varphi_q\left(\frac{1}{d(t)}\right) \varphi_q\Big( \int_t^\infty \lambda h(\tau, u(\tau), u'(\tau), u''(\tau)) d\tau \Big) \right| dt \\ &\leq \left\| \varphi_q\left(\frac{1}{d}\right) \right\|_1 \varphi_q[\| a_1 \|_1 \| u \|^{p-1} \\ &+ \| a_2 \|_1 \| u \|^{p-1} + \| a_3 \|_1 \| u \|^{p-1} + \| r \|_1] \\ &\leq \left\| \varphi_q(\frac{1}{d}) \right\|_1 2^{q-2} \varphi_q[\| a_1 \|_1 \| u \|^{p-1} \\ &+ \| a_2 \|_1 \| u \|^{p-1}] + \varphi_q\left[\| a_3 \|_1 \| u \|^{p-1} + \| r \|_1\right] \\ &\leq \left\| \varphi_q\left(\frac{1}{d}\right) \right\|_1 2^{2(q-2)} \Big[ \sum_{i=1}^{3} \| a_i \|_1^{q-1} \| u \| \\ &+ \| r \|_1^{q-1} \Big]. \end{aligned}$$

*Using (28) and (33) we derive*

$$\left[1 - 2^{2(q-2)} \| \varphi_q(\frac{1}{d}) \|_1 \Sigma_{i=1}^3 \| a_i \|_1^{q-1} A_1 \right] \| u'' \|_1$$

$$\leq \| \varphi_q(\frac{1}{d}) \|_1 2^{2(q-2)} \left[ \sum_{i=1}^{3} \| a_i \|_1^{q-1} L_2 + \| r \|_1^{q-2} \right]$$

*From (28) we conclude that there exists $L_3 > 0$ such that*

$$\| u'' \|_1 < L_3 \tag{34}$$

*Therefore, from (33) we obtain*

$$\| u \| < L_4, L_4 > 0. \tag{35}$$

*Similarly, if $p \geq 2$*

$$\begin{aligned} \| u'' \|_1 &\leq \| \varphi_q\left(\frac{1}{d}\right) \|_1 \left[ \sum_{i=1}^{3} \| a_i \|^{q-1} \| u \| + \| r \|_1^{q-1} \right] \\ &\leq \left\| \varphi_q\left(\frac{1}{d}\right) \right\|_1 \Big\{ \sum_{i=1}^{3} \| a_i \|_1^{q-1} \left[A_1 \| u'' \|_1 + L_2\right] \\ &+ \| r \|_1^{q-1}] \Big\} \end{aligned}$$

*or*

$$\left(1 - \| \varphi_q\left(\frac{1}{d}\right) \|_1 \sum_{i=1}^{3} \| a_1 \|_i^{q-1} A_1 \right) \| u'' \|_1$$

$$\leq \| \varphi_q\left(\frac{1}{d}\right) \|_1 \left[ \sum_{i=1}^{3} \| a_i \|^{q-1} L_2 + \| r \|_1^{q-1}. \right]$$

*From (29) we conclude that there exists $L_5 > 0$ such that*

$$\| u'' \|_1 < L_5. \tag{36}$$

*Using (33) we again obtain $L_6 > 0$ such that $\| u \| < L_6$*
*Therefore, $W_1$ is bounded.*

***Lemma 3.2*** *Let* $W_2 = \{u \in \ker T : N_\lambda u \in Im\ T\}$. *Then* $W_2$ *is bounded.*
***Proof:*** *We have for* $c \in R$ *and* $t \in [0,\infty), u(t) = ct$ *and* $N_\lambda u \in\ Im\ T$ *implies* $N_\lambda u \in \ker Q$.
*Hence,*

$$\sum_{i=1}^{n}\beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{d(r)}\right)\varphi_q bigg( \int_r^\infty h(\tau, c\tau, c, 0)d\tau\bigg) drdsdt = 0.$$

*By (A3) we obtain*

$$|\, c \,| < M_2$$

*Therefore, for* $u \in W_2$

$$\|\, u \,\| = \max\left\{\sup_{t\in[0\infty)} e^{-t}t, 1\right\} |\, c \,| < A_1 M_2$$

*We therefore conclude that* $W_2$ *is bounded.*

*Define* $J : ImQ \to \ker T$ *by*

$$J(c\rho(t)) = ct \ or\ J^{-1}(ct) = c\rho(t)$$

*If (26) holds, let*

$$W_3 = \{u \in KerT : \lambda u + (1-\lambda)JQN_\lambda u = 0, \lambda \in [0,1]\}$$

*Then*

$$-\lambda J^{-1}u = (1-\lambda)QN_\lambda u$$

*or*

$$-\lambda c\rho(t) = (1-\lambda)\rho(t)\sum_{i=1}^{n}\beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{d(r)}\right) \times \varphi_q\left(\int_r^\infty h(\tau, c\tau, c, 0)d\tau\right) drdsdt$$

*if* $\lambda = 1$ *then* $c = 0$ *and if* $|\, c \,| > M_2$ *then from (26) we have*

$$0 > -\lambda c^2 = (1-\lambda)c\sum_{0}^{n}\beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{d(r)}\right) \times \varphi_q\left(\int_r^\infty h(\tau, c\tau, c, 0)d\tau\right) drdsdt > 0$$

*which is a contradiction. Therefore,* $W_3$ *is bounded. If (27) holds we set*

$$W_3 = \{u \in \ker T : -\lambda u + (1-\lambda)JQN_\lambda u = 0, \lambda \in [0,1]\}$$

*Using the same arguments as above we obtain that* $W_3$ *is bounded.*
*Let* $W$ *be open and bounded such that* $W_1 \cup W_2 \cup W_3 \subset W$.
*Then from the above Lemmas, we can deduce that*

$$Tu \neq N_\lambda u, (u,\lambda) \in [domT \cap \partial W] \times (0,1)$$
$$Let\ H(u,\lambda) = \lambda u + (1-\lambda)JQN_\lambda u.$$

*It is easily checked that* $H(u,\lambda) \neq 0$ *for* $U \in \partial W \cap \ker T$.
*Hence,*

$$\begin{aligned} deg(JQN\,|_{KerT}, W\cap KerT, 0) &= deg(H(.,0), W\cap KerT, 0)\\ &= deg(H(.,1), W\cap KerT, 0)\\ &= deg(\pm I, W\cap KerT, 0) \neq 0. \end{aligned}$$

*From theorem 2.1 we can conclude that* $Tu = Nu$ *has a solution in*
$domT \cap W$. *Therefore, (1)- (2) has at least one solution.*

## *4 Example*

*Consider the boundary value problem*

$$\left[d(t)\varphi_p(u''(t))\right]' = e^{-3t}\left[1 + \frac{|\,u(t)\,|^3}{4(1+t)^2} + \frac{|\,u'(t)\,|^3}{8(1+t)^3} + cos^2 t\frac{|\,u''(t)\,|^3}{16(1+t)^4}\right] \quad (37)$$

$$u'(0) = \sum_{1}^{2}\beta_i \int_0^{\eta_i} u(t)dt, u(0) = 0, \lim_{t\to\infty}(d(t)\varphi_p(u''(t)) = 0 \quad (38)$$

*Here,*
$d(t) = e^{3t}, p = 4, q = \frac{4}{3}, \beta_1 = 4, \beta_2 = 9, \eta_1 = \frac{1}{2}, \eta_2 = \frac{1}{3}$

$$h(t,u,u',u'') = e^{-3t}\left[1 + \frac{|\,u(t)\,|^3}{4(1+t)^2} + \frac{|\,u'(t)\,|^3}{8(1+t)^3} + cos^2 t\frac{|\,u''\,|^3}{16(1+4)^4}\right]$$

$\sum_{i=1}^{2}\beta_i\eta_i^2 = 2$. *It is easily checked that* $h : [0,\infty) \times \Re^3 \to \Re$ *is an* $L^1$*- Carathéodory function*

$$\sum_{i=1}^{2}\beta_i \int_0^{\eta_i}\int_0^t\int_0^s \varphi_q\left(\frac{e^{-r}}{e^{3r}}\right) drdsdt$$

$$= \sum_{i=1}^{2}\beta_i \int_0^{\eta_i}\int_0^t\int_0^s e^{-4r} drdsdt \neq 0$$

*Assumption(A0) is satisfied.*
*Clearly,* $(h(t,x,y,z) > 0$ *for all* $(t,x,y,z) \in [0,\infty) \times \Re^3$.
*Thus,* $QN_\lambda u \neq 0$ *on* $[0,\infty)$ *for all* $u \in domT/\ker T$.
*Assumption (A1) is verified .*

$$|\, h(t,u,u',u''\,| \leq e^{-3t}\left(1 + \frac{|\,u\,|^3}{4(1+t)^3} + \frac{|\,u'\,|^2}{8(1+t)^3} + \frac{|\,u''\,|^3}{16(1+t)^4}\right)$$

*Here*
$a_1(t) = \frac{1}{4(1+t)^2}, a_2(t) = \frac{1}{8(1+t)^3}, a_3(t) = \frac{1}{16(1+t)^4}$

*This verifies assumption (A2). To verify (A3) we have*

$$4c\int_0^{1/2}\int_0^t\int_0^s \varphi_q\left(\frac{1}{e^{3r}}\right)$$
$$\times\varphi_q\left(\int_r^\infty e^{-3\tau}\left[1+\frac{|c\tau|^3}{4(1+\tau)^2}+\frac{|c|^3}{8(1+\tau)^3}\right]d\tau\right)drdsdt$$
$$+9c\int_0^{1/3}\int_0^t\int_0^s \varphi_q\left(\frac{1}{e^{3r}}\right)$$
$$\times\varphi_q\left(\int_r^\infty e^{-3\tau}\left[1+\frac{|c\tau|^3}{4(1+\tau)^2}+\frac{|c|^3}{8(1+\tau)^3}\right]d\tau\right)drdsdt$$
$$\leq 4c\int_0^{1/2}\int_0^t\int_0^s \varphi_q\left(\frac{\frac{1}{3}e^{-3r}}{e^{3r}}\right)drdsdt$$
$$+9c\int_0^{1/3}\int_0^t\int_0^s \varphi_q\left(\frac{1/3e^{-3r}}{e^{3r}}drdsdt\right)=.$$
$$c\left[\frac{4}{3^{1/3}}\int_0^{1/2}\int_0^t\int_0^s e^{-2r}drdsdt+\frac{9}{3^{1/3}}\int_0^{1/3}\int_0^t\int_0^s e^{-2r}drdsdt\right].$$

*Assumption (26) or (27) are satisfied respectively if $c>1$ or $c<-1$, i.e. if $|c|>1$.*

*Finally, we have* $\sum_{i=1}^3 \| a_i \|_1=\frac{1}{4}+\frac{1}{16}+\frac{1}{48}=\frac{1}{3}$, $\left\|\varphi_q\left(\frac{1}{d}\right)\right\|_1=1, \left\|\varphi_q\left(\frac{1}{d}\right)\right\|_\infty=1$

*Therefore, for $p\geq 2$, we have from (29)*

$\| \varphi_q\left(\frac{1}{d}\right) \|_1 \sum_{i=1}^3 \| a_i \|_1 A_1=\frac{1}{3}<1$ *where* $A_1=1$

*Thus from theorem 3.1 we conclude that the BVP (37)-(38) has at least one solution.*

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