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Discrete Extended Erlang-Truncated Exponential Distribution and its Applications

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Abstract: A new discrete distribution with three parameters called the Discrete Extended Erlang-Truncated Exponential distribution (DEETE) is proposed by using the general approach of discretizing a continuous distribution while retaining its survival function. Some statistical properties of the DEETE, such as the quantile function, moments, moment generating function, Renyi entropy and order statistics, are studied. The maximum likelihood method is utilized to estimate the parameters of the model. Some special cases of DEETE distribution are derived. The proposed distribution is applied to two real-life count data sets and compared with some related discrete distributions. Moreover, DEETE is used to fit some data sets of COVID-19 deaths. The applications suggest that DEETE performs better than the related discrete distributions and better fits the COVID-19 deaths cases.

Keywords: Extended Erlang-truncated exponential distribution, Survival function, Maximum Likelihood Estimation, Quantile functions, Renyi entropy, Order statistics.

1 Introduction

Many researchers in different fields occasionally encounter situations where continuous random variables may not ever be measured on a continuous scale but may often be taken as discrete random variables. In life testing experiments, it is sometimes impossible or inconvenient to measure the life length of a device on a continuous scale. For example, in the case of an on/off-switching device, the switch's lifetime is a discrete random variable. In the military field, the weapons like guns, the most important is the number of times it fires until failure rather than the weapon's life. In many practical situations, reliability data are measured based on the number of runs, cycles, or shocks the device sustains before it fails. For example, In survival analysis, we may record the number of days of survival for lung cancer patients since therapy, or the times from remission to relapse are usually expressed as a number of days. In this context, the Geometric and Negative Binomial distributions are known as discrete alternatives for the Exponential and Gamma distributions, respectively. It is well known that these discrete distributions have monotonic hazard rate functions, and thus they are unsuitable for some situations. On the other hand, counted data models such as Poisson and Geometric can only cater to positive integers and zero values. More attention has been drawn to derive a discrete analogues (Discretization) of continuous distribution. There are several methods used to construct a discrete distribution from continuous ones. The first one is to study a characteristic property of a continuous distribution and build a similar property in discrete time. The second is to consider discrete lifetime as the integer part of continuous lifetime, see [1,2]. Roy [3] explored a relationships between different reliability measures and acquired a unique determination of a bivariate Geometric distribution based on a bivariate extension of a univariate characterizing proporty. He pointed out that the univariate Geometric distribution can be expressed as a discrete concentration of a corresponding Exponential distribution using a survival function. Many reliability measures and statistical properties of the distribution will stay unchanged, If discretization of a continuous life distribution can retain the same functional form of the survival function. Therefore, Roy [4] studied the discrete concentration concept and considered it as a simple approach that can generate a discrete life distribution model based on a

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continuous model, and used this idea to introduced the discrete Normal distribution. Following the same approach for discretization of continuous probability distributions, Krishna and Pundir [5] introduced the discrete Burr and Pareto distributions. Chakraborty and Chakravarty [6] derived a two-parameter discrete Gamma distribution corresponding to the continuous two parameters Gamma distribution. Nekoukhou et al. [7] introduced a discrete generalized Exponential distribution of the second type. Nekoukhou and Bidram [8] discretized the Exponentiated discrete Weibull distribution. The discrete Logistic distribution proposed by Chakraborty and Charavarty [9]. Hussain et al. [10] discussed a two-parameter discrete Lindley distribution. Alamatsaz et al. [11] introduced the discrete generalized Rayleigh distribution. Khongthip et al. [12] studied the discretization of weighted Exponential distribution and its applications. Jayakumar and Babu [13] introduced a discrete version of the additive Weibull Geometric distribution. Very recently El-Alosey [14] introduced the discrete form of the Erlang-truncated Exponential distribution (ETE) distribution. The ETE distribution proposed by El-Alosey [15] as an extension of the standard one parameter Exponential distribution. Many studies have been conducted on ETE. Mohsin [16] derived the recurrence equations for single and product moments for ETE distribution. Nasiru et al. [17] studied the generalized Erlang-truncated Exponential distribution which called the Kumaraswamy Erlang-truncated exponential distribution. Nasiru et al. [18] developed the Poisson exponentiated Erlang truncated exponential distribution. Okorie et al. [19] proposed the Marshall-Olkin generalized Erlang-truncated exponential distribution. Okorie et al. [20] introduced the transmuted Erlang-truncated exponential (TETE) distribution. Okorie et al. [21] developed the Extended Erlang-truncated exponential (EETE) distribution, studied its properties and applied the distribution to rainfall data, while Singh et al. [22] explored explicit expressions as well as some recurrence relations for single and product moments of kth lower record values from EETE distribution. Jimoh et al. [23] proposed the Gamma Log-logistic Erlang Truncated Exponential distribution. Elbatal and Aldukeel [24] examined the McDonald ETE with three shape parameters. Sayyed et al. [25] studied the effect of inspection error on cumulative sum (CUSUM) control charts for controlling the parameters of a random variable under ETE, and also they derived an expression for the parameter of the CUSUM chart. Shrahili et al. [26] studied properties and applications of beta Erlang-truncated exponential distribution. In this paper, we introduce a new discrete version of extended Erlang-truncated Exponential distribution called discrete extended Erlang-truncated exponential distribution (DEETE) using the general approach of discretizing a continuous distribution. We investigate some statistical properties of DEETE and estimate its parameters using maximum likelihood method. Moreover, we apply the proposed distribution to two real count data sets as well as the DEETE distribution is used to model some data sets regarding the deaths cases of COVID-19 collected from Gulf Cooperation Council countries including Qatar, UAE and Bahrain. The remainder of this paper is structured as follows: The new proposed distribution is presented in section 2, some statistical properties such as quantile function, moment generating function, Entropy and order statistics are demonstrated in section 3. Section 4 provides the Maximum likelihood method to estimate DEETE's parameters. An application of the DEETE to some real count data sets for the purpose of illustration is performed in section 5. Finally, section 6 provides some concluding remarks. The proposed distribution is motivated because it exhibits the three parameters which can be adapted to meet most kind of count data sets.

1.1 Discretizing a continuous distribution

The idea of discretization of a given continuous random variable first proposed by [3]. This concept is demonstrated as follows.
Given a continuous random variable X with survival function $S_X(x)$, a discrete random variable Y can be defined as equal to $[X]$ that is floor of X that is largest integer less or equal to X. The probability mass function (*pmf*) $P[Y=y]$ of Y is then given by

$$P[Y=y]=S_X(y)-S_X(y+1)$$

The *pmf* of the random variable Y thus defined may be viewed as a discrete concentration of the pdf of X, see [3, 4].

1.2 The Extended Erlang-Truncated Exponential (EETE) distribution

The EETE distribution introduced by [21]. A non-negative random variable X has an EETE distribution with parameters $\alpha>0, \beta>0$ and $\lambda>0$, (EETE(α,β,λ)), if its cumulative distribution function (*cdf*) and probability distribution function (*pdf*) are respectively given by:

$$F_{EETE}(x;\alpha,\ \beta,\ \lambda)=\left[1-e^{-\beta(1-e^{-\lambda})x}\right]^{\alpha},\ x>0 \quad (1)$$

$$f_{EETE}(x;\alpha,\ \beta,\ \lambda)=\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}\times\left[1-e^{-\beta(1-e^{-\lambda})x}\right]^{\alpha-1},\ x>0 \quad (2)$$

where α, λ and β are shape parameters.
The hazard rate functions of the EETE(α,β,λ) is:

$$h_{EETE}(x;\alpha,\ \beta,\ \lambda) = \frac{\alpha\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha-1}}{1-\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}} \quad x>0 \tag{3}$$

The survival of the EETE(α,β,λ) is:

$$S_{EETE}(x;\alpha,\ \beta,\ \lambda) = 1-\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha} \quad x>0 \tag{4}$$

By using a series expansion

$$(1-z)^a = \sum_{i=0}^{a}(-1)^i\binom{a}{i}z^i$$

The survival of the EETE(α,β,λ) in Eq.(4) can be written on the form:

$$S_{EETE}(x;\alpha,\ \beta,\ \lambda) = 1-\sum_{i=0}^{\infty}(-1)^i\binom{\alpha}{i}e^{-(1-e^{-\lambda})\beta ix} \tag{5}$$

Clearly, when $\alpha > 0$ the above infinite sums stop at α.

2 Discrete Extended Erlang-Truncated Exponential distribution

This Section pertains to derive the proposed distribution Discrete Extended Erlang-Truncated Exponential (DEETE) and exploring its the probability mass and cumulative distribution functions. Moreover, some special cases of the proposed DEETE distribution are derived.

2.1 Probability mass and cumulative distribution functions for DEETE distribution

Using the concept of discretization given in subsection 1.1, the DEETE can be derived by discretizing the re-parameterized version of the EETE distribution with parameters $\alpha > 0, \beta > 0$ and $\lambda > 0$ that given in subsection 1.2. First re-parameterization of the survival of the EETE in Eq.(4) by taking $p = e^{-(1-e^{-\lambda})}$, this leads to the formula of the *pmf* of DEETE distribution (DEETE(α,β,λ)), as follows:

$$f_{DEETE}(y;\alpha,\ \beta,\ p) = P[Y=y] = \begin{cases} \left[1-p^{\beta(y+1)}\right]^{\alpha} - \left(1-p^{\beta y}\right)^{\alpha} \\ \sum_{i=1}^{\infty}(-1)^{i+1}\binom{\alpha}{i}p^{\beta iy}\left(1-p^{\beta i}\right) \end{cases} \tag{6}$$
$$y=0,\ 1,\ 2,\ \ldots,\ \alpha\ >\ 0,\ \beta\ >\ 0,\ 0<p<1$$

It is observe that, when $\alpha > 0$ the infinite sums in Eq.(6) stop at α.
Fig. 1 shows *pmf*s of DEETE(α,β,λ) for different values of parameters α, β and p.

Proposition 2.1 The discrete generalized exponential distribution of the second type can be concluded from Eq.(6) by taking $\beta = 1$ as follows

$$f_{DGE2(\alpha,\ p)}(y;\alpha,\ ,\ p) = P[Y=y] = \begin{cases} \left(1-p^{y+1}\right)^{\alpha} - \left(1-p^{y)}\right)^{\alpha} \\ \sum_{i=1}^{\infty}(-1)^{i+1}\binom{\alpha}{i}p^{iy}\left(1-p^{i}\right) \end{cases}$$
$$y=0,\ 1,\ 2,\ \ldots,\ \alpha\ >\ 0,\ 0<p<1$$

See[7].

Proposition 2.2 The *pmf* of the discrete Erlang-truncated exponential distribution with parameters $\beta > 0$ and $0 < p < 1$ is explored, when $\alpha = 1$ in Eq.(6), as:

$$f_{DETE}(y;\ \beta,\ p) = p^{\beta y}\left(1-p^{\beta}\right), y=0,1,2,\ldots.$$

See[14].

Proposition 2.3 The distribution of the random variable Y in Eq.(6), takes the form of geometric distribution, when $\alpha = \beta = 1$, as follows:

$$f_Y(y;\ \ p) = p^y[1-p], y=0,1,2,\ldots$$

The *cdf* of DEETE(α,β,λ) is given as:

$$F_{DEETE}(u;\alpha,\ \beta,\ p) = P(Y\le u) = \begin{cases} \left[1-p^{\beta(u+1)}\right]^{\alpha}, & y\ge 0 \\ 0, & y<0 \end{cases} \tag{7}$$

Fig. 2 shows the *cdf*s of DEETE(α,β,λ) for some values of parameters
From Fig. 1, it is observed that as the parameter α increases, the mode of the distribution moves to the right, while as the parameter β increases, the probability of small numbers increase; this also can be observed from Fig. 2; that is, the *cdf* approaches one quickly. On the other hand, as p increases, the probability of large numbers increase; this can be seen from Fig. 2; that is, the *cdf* approaches one slowly.

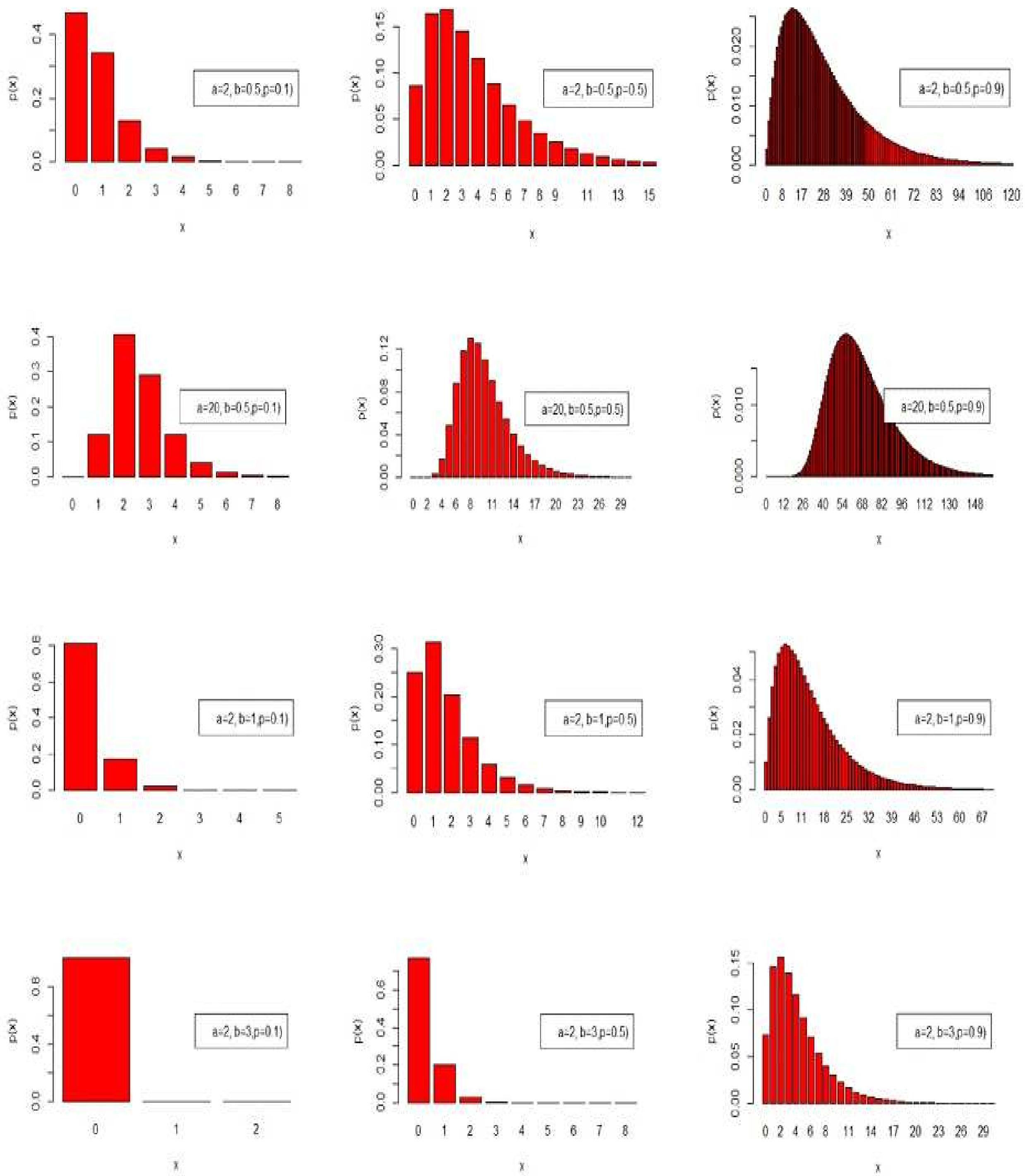

Fig. 1: The *pmf*s of DEETE(α, β, p) for possible values of α, β and p.

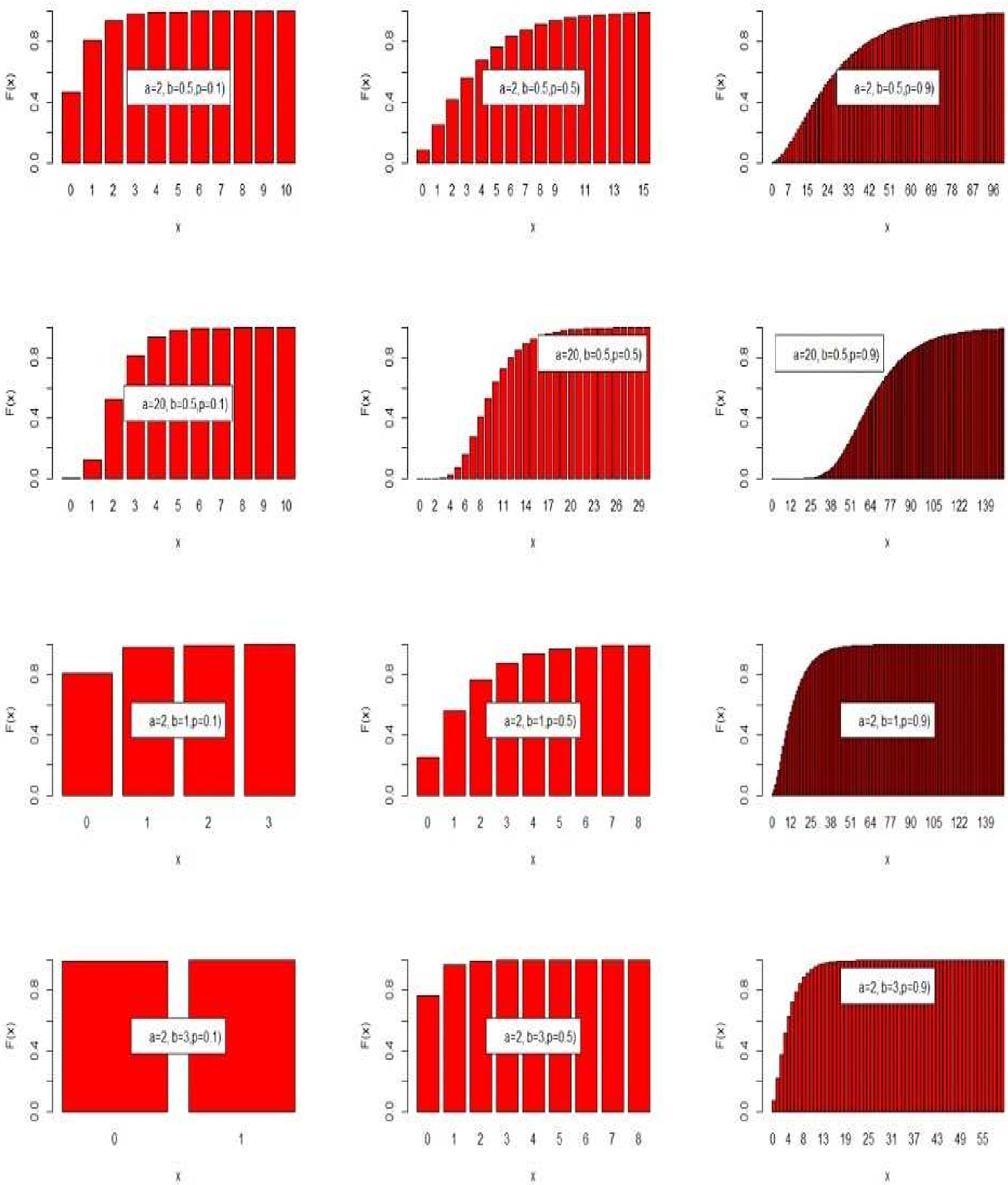

Fig. 2: The *cdf*s of DEETE(α, β, p) for possible values of α, β and p.

2.2 Survival and Hazard function

The survival function of Y is:

$$S_{DEETE}(y;\alpha,\ \beta,\ \lambda) = 1 - \left[\ 1 - p^{\beta(y+1)}\right]^{\alpha} \tag{8}$$

and the hazard function is:

$$\begin{aligned} h_{DEETE}(y;\alpha,\ \beta,\ \lambda) =& \frac{f_{DEETE}(y;\alpha,\ \beta,\ \lambda)}{S_{DEETE}(y;\alpha,\ \beta,\ \lambda)} \\ =& \frac{\left[\ 1 - p^{\beta(y+1)}\right]^{\alpha} - \left(\ 1 - p^{\beta y}\right)^{\alpha}}{1 - \left[\ 1 - p^{\beta(y+1)}\right]^{\alpha}} \end{aligned}$$

Discrete hazard rates arise in several common situations in reliability theory where clock time is not the best scale on which to describe lifetime. For example, in weapons reliability, the number of rounds fired until failure is more important than age in failure. This is the case also when a piece of equipment operates in cycles and the observation is the number of cycles successfully completed prior to failure.

3 Distributional properties

In this section, some statistical properties of the DEETE distribution were derived.

3.1 Quantile function

The quantile of order $0 < \gamma < 1$, can be obtained by inverting the *cdf* in Eq.(7) as

$$F_{DEETE}(Q;\alpha,\ \beta,\ p) = \left[\ 1 - p^{\beta(Q+1)}\right]^{\alpha}$$

then

$$\begin{aligned} F^{-1}_{DEETE}(\gamma) =& miny \in R : F_{DEETE}(y) \geq \gamma \\ =& \left[\ 1 - p^{\beta(Q+1)}\right]^{\alpha} = \gamma \end{aligned}$$

Thus, The γ^{th} quantile is

$$Q(\gamma;\alpha,\ \beta,\ p\ ) = \log_p \left[(\ 1-\gamma)^{\frac{1}{\alpha}}\right]^{\frac{1}{\beta}} - 1 \tag{9}$$

The median of DEETE can be obtained by substituting by $\gamma = \frac{1}{2}$ in Eq.(9) as follows:

$$Q_{\,0.5} = Q(\gamma;\alpha,\ \beta,\ p\ ) = \log_p \left[1 - 0.5^{\frac{1}{\alpha}}\right]^{\frac{1}{\beta}} - 1$$

3.2 The moment generating function

In this subsection, the moment generating function of a random variable Y having the DEETE distribution with parameters (α, β, p) is derived.

$$M_Y(t\ ) = E\left(e^{ty}\right) = \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \frac{1 - p^{\beta i}}{1 - p^{\beta i} e^{t}}$$

Therefore,

$$M_Y(t\ ) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \left(1 - p^{\beta i}\right) p^{\beta ik} e^{tk} \tag{10}$$

where $\sum_{k=0}^{\infty} p^{\beta ik} e^{tk} = \left(1 - p^{\beta i} e^{t}\right)^{-1}$

Thus we can calculate the first moment (the mean) of the DEETE distribution by using Eq.(10) as follows

$$E(Y) = \left.\frac{dM_Y(t\ )}{dt}\right|_{t=0} = \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \frac{p^{\beta i}}{1 - p^{\beta i}} \tag{11}$$

Also the r^{th} moment is

$$\begin{aligned} \mu_r = E(Y^r) =& \left.\frac{d^r M_Y(t\ )}{dt^r}\right|_{t=0} \\ =& \frac{d^r}{dt^r} \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \left(1 - p^{\beta i}\right) p^{\beta ik} e^{tk} \\ =& \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \left(1 - p^{\beta i}\right) k^r p^{\beta ik} \end{aligned}$$

Setting $r = 1$ in the previous equation we can get the mean as follows:

$$\begin{aligned} \mu_1 = E(Y) =& \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \left(1 - p^{\beta i}\right) \sum_{k=0}^{\infty} k p^{\beta ik} \\ =& \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \frac{p^{\beta i}}{1 - p^{\beta i}} \end{aligned}$$

where $\sum_{k=0}^{\infty} k p^{\beta jk} = \left(1 - p^{\beta j}\right)^{-2}$

The second moment, $r = 2$, about the origin is

$$\begin{aligned} \mu_2 = E\left(Y^2\right) =& \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \left(1 - p^{\beta i}\right) \sum_{k=0}^{\infty} k^2 p^{\beta ik} \\ =& \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \frac{p^{\beta i}\left(1 + p^{\beta i}\right)}{\left(1 - p^{\beta i}\right)^2} \end{aligned}$$

Thus the variance of the DEETE distribution is given by

$$\begin{aligned} V(Y) =& \mu_2 - \mu_1^2 \\ =& \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \frac{p^{\beta i}\left(1 + p^{\beta i}\right)}{\left(1 - p^{\beta i}\right)^2} \\ & - \left[\sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} \frac{p^{\beta i}}{1 - p^{\beta i}}\right]^2 \end{aligned} \tag{12}$$

It is observe that when $\alpha > 0$ (integer value), the infinite sums in Eq.(12) stop at α.
The mean and variance of DEETE distribution for different combinations of α, β, and p are computed in Table 1. It can be seen that both mean and variance increase with α and p, in contrast the mean and variance decrease as β increases. Moreover, for $p > 0.5$ the variance increases faster than that for $0 < p < 0.5$. Therefore, the three parameters of the proposed distribution can be adopted to meet most kind of count data sets.

Table 1: Mean and Variance of DEETE distribution for different values of α, β and p.

p		0.1		0.5		0.9	
α	β	Mean	Variance	Mean	Variance	Mean	Variance
2	0.5	0.81384	0.98235	3.82843	10.4853	27.9737	450.5
	1	0.21212	0.2163	1.66667	2.66667	13.7368	112.687
	3	0.002	0.002	0.26984	0.27816	4.24587	12.5908
5	0.5	1.48049	1.20089	6.08832	12.2685	42.8433	527.471
	1	0.46407	0.37014	2.79416	3.13039	21.1716	131.93
	3	0.005	0.00498	0.57393	0.44365	6.72388	14.7329
20	0.5	2.62455	1.29059	9.88088	13.3722	67.7939	575.234
	1	1.08255	0.36788	4.69044	3.40552	33.6469	143.871
	3	0.01983	0.01948	1.24489	0.4239	10.8823	16.0597

3.3 Entropy

Statistical entropy is a probabilistic measure of uncertainty or ignorance about the outcome of a random experiment and is a measure of reduction in that uncertainty. Various entropy and information indices exist, among them the Renyi entropy which has been developed and used in many disciplines and context, the Renyi entropy is defined by

$$I_R(\gamma) = \frac{1}{1-\gamma} log \left[\sum_{y=0}^{\infty} f^{\gamma}(y) \right]$$

where $\gamma > 0$ and $\gamma \neq 0$
For a random variable Y having a *pmf* of the DEETE distribution in Eq.(6), the Renyi entropy is

$$I_R(\gamma) = \frac{1}{1-\gamma} log \left[\sum_{y=0}^{\infty} f^{\gamma}(y) \right]$$
$$= \frac{1}{1-\gamma} log \left[\sum_{y=0}^{\infty} \left\{ \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} p^{\beta iy} \left(1-p^{\beta i}\right) \right\}^{\gamma} \right]$$

By using the complete multinomial expansions theorem [27], we find:

$$\left\{ \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\alpha}{i} p^{\beta iy} \left(1-p^{\beta i}\right) \right\}^{\gamma}$$
$$= \gamma! \sum_{p=1}^{M} \sum_{m_1=1}^{m-(k-1)} \sum_{m_2=m_1+1}^{m-(k-2)} \cdots \sum_{m_p=m_{p-1}+1}^{m}$$
$$\times \sum_{n_1=1}^{n-(k-1)} \sum_{n_2=1}^{n-n_1-(k-2)} \cdots$$
$$\sum_{n_{k-1}=1}^{n-n_1-n_2-\ldots-n_{k-2}-1} \sum_{n_p=n-n_1-n_2-\ldots-n_{k-1}}^{n-n_1-n_2-\ldots-n_{k-1}}$$
$$\times \prod_{i=1}^{k} \frac{\left[(-1)^{i+1} \binom{\alpha}{i} p^{\beta iy} \left(1-p^{\beta i}\right)\right]^{n_i}}{n_i!}$$

where $M = \begin{cases} n, & n < m \\ m, & n \geq m \end{cases}$
Therefore, the Renyi entropy measure can be written as

$$I_R(\gamma)$$
$$= \frac{1}{1-\gamma} \log \left\{ \sum_{y=0}^{\infty} \gamma! \sum_{p=1}^{M} \sum_{m_1=1}^{m-(k-1)} \sum_{m_2=m_1+1}^{m-(k-2)} \cdots \sum_{m_p=m_{p-1}+1}^{m} \right.$$
$$\times \sum_{n_1=1}^{n-(k-1)} \sum_{n_2=1}^{n-n_1-(k-2)} \cdots \sum_{n_{k-1}=1}^{n-n_1-n_2-\ldots-n_{k-2}-1} \sum_{n_p=n-n_1-n_2-\ldots-n_{k-1}}^{n-n_1-n_2-\ldots-n_{k-1}}$$
$$\left. \times \prod_{i=1}^{k} \frac{\left[(-1)^{i+1} \binom{\alpha}{i} p^{\beta iy} \left(1-p^{\beta i}\right)\right]^{n_i}}{n_i!} \right\}$$

When $\alpha = \beta = 1$ the same result as the geometric distribution achieved, also when $\alpha = 1$ we get the Renyi entropy of DETE, see [14].

3.4 Order Statistics

Order statistics are the most important fundamental tools in non-parametric statistics and inference. They enter the problems of estimation and hypothesis testing in a variety of ways. The aim of the present subsection is to establish some general equations regarding the DEETE distribution. More precisely, let $F_i(y;\alpha,\beta,p)$ and $f_i(y;\alpha,\beta,p)$ be the *cdf* and *pmf* of the *ith* order statistic of a random sample of size n from $DEETE(\alpha,\beta,p)$. Since,

$$F_i(y;\alpha,\beta,\ p) = \sum_{k=i}^{n} \binom{n}{k} \left[F(y;\alpha,\beta,\ p)\right]^k \times \left[1 - F(y;\alpha,\beta,\ p)\right]^{n-k} \tag{13}$$

using the binomial expansion for $[1-F(y;\beta,\ p)]^{n-k}$, we obtain the following result:

$$\begin{aligned} &F_i(y;\alpha,\beta,\ p)\\ &=\sum_{k=i}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(-1)^j\,[F(y;\alpha,\beta,\ p)]^{k+j}\\ &=\sum_{k=i}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(-1)^j\left[1-p^{\beta(y+1)}\right]^{\alpha(k+j)}\\ &=\sum_{=i}^{n}\sum_{j=0}^{n-k}\sum_{l=0}^{\alpha(k+j)}\binom{n}{k}\binom{n-k}{j}\binom{k+j}{l}(-1)^{j+l}p^{\alpha\beta l(y+1)} \end{aligned}$$

The corresponding *pmf* of the *ith* order statistic

$$f_i(y;\alpha,\beta,\ p)=F_i(y;\alpha,\beta,\ p)-F_i(y-1;\alpha,\beta,\ p)$$

for an integer value of y, then it is given by

$$\begin{aligned} &f_i(y;\alpha,\beta,\ p)\\ &=\sum_{k=i}^{n}\sum_{j=0}^{n-k}\sum_{l=0}^{k+j}\left\{\binom{n}{k}\binom{n-k}{j}\binom{k+j}{l}(-1)^{j+l+1}\right.\\ &\qquad\left.\times\,p^{\alpha\beta ly}\left(1-p^{\alpha\beta l}\right)\right\}\\ &=\sum_{k=i}^{n}\sum_{j=0}^{n-k}\sum_{l=0}^{k+j}\binom{n}{k}\binom{n-k}{j}\binom{k+j}{l}(-1)^{j+l+1}\\ &\qquad\times f_{DETE}(y;\alpha\beta l,\ p) \end{aligned}$$

where $f_{DETE}(y;\alpha\beta l,\ p)$ is the *pmf* of the DEETE distribution with parameters $\alpha\beta l$ and p. Since $f_i(y;\alpha,\beta,\ p)$ is a linear combination of a finite number of DEETE$(\alpha,\beta l,\ p)$, we may obtain some properties of order statistics from the corresponding DEETE distribution, such as moments. For example, the mean of the *ith* order statistic is given by

$$\mu_{i:n}=\sum_{k=i}^{n}\sum_{j=0}^{n-k}\sum_{l=0}^{k+j}\binom{n}{k}\binom{n-k}{j}\binom{k+j}{l}(-1)^{j+l+1}\frac{p^{\alpha\beta l}}{1-p^{\alpha\beta l}}$$

4 Maximum likelihood estimation

This section aims to derive the maximum likelihood estimation (*MLE*) for parameters of the proposed distribution DEETE.
Let $Y_1,Y_2,\dots,Y_n$ be a random sample of size n having the DEETE distribution. The log-likelihood of the DEETE distribution is

$$\ell=\sum_{j=1}^{n}log\left[(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}\right] \tag{14}$$

where $\Lambda=p^{\beta(y_j+1)}$ and $\psi=p^{\beta y_j}$
Further differentiating the log-likelihood of the DEETE distribution in Eq.(14) partially with respect to the shape parameters $\alpha,\ \beta$ and p to get the likelihood equations as

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha}&=\sum_{j=1}^{n}\left\{\frac{(1-\Lambda)^{\alpha}\log(1-\Lambda)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right.\\ &\quad\left.-\frac{(1-\psi)^{\alpha}\log[1-\psi]}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right\}\\ &=0 \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial\ell}{\partial\beta}&=\alpha\sum_{j=1}^{n}\left\{\frac{y_j(1-\psi)^{\alpha-1}\psi\log(p)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right.\\ &\quad\left.-\frac{(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda\log(p)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right\}\\ &=0 \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial\ell}{\partial p}&=\alpha\beta\sum_{j=1}^{n}\left\{\frac{y_j(1-\psi)^{\alpha-1}\psi p^{-1}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right.\\ &\quad\left.-\frac{(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda p^{-1}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right\}\\ &=0 \end{aligned} \tag{17}$$

The solutions of likelihood Eq.s(15), (16), and (17) provide the MLEs of $\alpha,\ \beta$ and p, which can be obtained by a numerical methods. Since the MLE of the vector of unknown parameters $\theta=(\alpha,\ \beta\ ,\ p)^T$ cannot be derived in closed forms, it is therefore hard to derive the exact distribution of the MLEs.
The second partial derivatives are given below

$$\begin{aligned} \frac{\partial^2\ell}{\partial\alpha^2}&=\sum_{j=1}^{n}\left\{\frac{(1-\Lambda)^{\alpha}\log[1-\Lambda]^2-(1-\psi)^{\alpha}\log[1-\psi]^2}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right.\\ &\quad\left.-\left[\frac{(1-\Lambda)^{\alpha}\ \log(1-\Lambda)-(1-\psi)^{\alpha}\log(1-\psi)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right]^2\right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2\ell}{\partial\beta^2}&=\sum_{j=1}^{n}\left\{\frac{\alpha\,(y_j+1)^2(1-\Lambda)^{\alpha-1}\Lambda\ \log(p)^2}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right.\\ &\quad-\frac{\alpha y_j^2(1-\psi)^{\alpha-1}\psi\ \log(p)^2}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\\ &\quad-\frac{\alpha\,(\alpha-1)(y_j+1)^2(1-\Lambda)^{\alpha-2}2\Lambda\ log(p)^2}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\\ &\quad-\frac{\alpha\,(\alpha-1)y_j^2(1-\psi)^{\alpha-2}\,2\psi\,log(p)^2}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\\ &\quad\left.-\left[\frac{A-B}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right]^2\right\} \end{aligned}$$

where $A = \alpha(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda\log(p)$ and $B = \alpha y_j(1-\psi)^{\alpha-1}\psi\log(p)$

$$\frac{\partial^2\ell}{\partial p^2} = \sum_{j=1}^{n}\left\{\frac{\alpha\beta y_j(\beta y_j-1)(1-\psi)^{\alpha-1}\psi\, p^{-2}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{\alpha\beta^2 y_j^2(\alpha-1)(1-\psi)^{\alpha-2}\, 2\psi\, p^{-2}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} + \frac{\alpha\beta^2(\alpha-1)(y_j+1)^2(1-\Lambda)^{\alpha-2}\, 2\Lambda p^{-2}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{\alpha\beta(y_j+1)(\beta(y_j+1)-1)(1-\Lambda)^{\alpha-1}\Lambda\, p^{-2}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \left[\frac{C-D}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right]^2\right\}$$

where $C = \alpha\beta y_j(1-\psi)^{\alpha-1}\psi\, p^{-1}$ and, $D = \alpha\beta(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda\, p^{-1}$

$$\frac{\partial^2\ell}{\partial\alpha\partial\beta} = \sum_{j=1}^{n}\left\{\frac{(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda\,\log(p)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{y_j(1-\psi)^{\alpha-1}\psi\,\log(p)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} + \frac{\alpha(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda\log(p)\log(1-\Lambda)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{\alpha y_j(1-\psi)^{\alpha-1}\,\psi\log(p)\log(1-\psi)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{G\times H}{[(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}]^2}\right\}$$

where $G = \alpha\beta y_j(1-\psi)^{\alpha-1}\psi\log(p) - \alpha\beta(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda\log(p)$ and $H = (1-\Lambda)^{\alpha}\log(1-\Lambda) - (1-\psi)^{\alpha}\log(1-\psi)$

$$\frac{\partial^2\ell}{\partial\alpha\partial p} = \sum_{j=1}^{n}\left\{\frac{\alpha\beta y_j(1-\psi)^{\alpha-1}\psi\, p^{-1}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{\alpha\beta(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda\, p^{-1}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} + \frac{\alpha\beta y_j\psi p^{-1}(1-\psi)^{\alpha-1}\log(1-\psi)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{\alpha\beta(y_j+1)\Lambda p^{-1}(1-\Lambda)^{\alpha-1}\log(\Lambda)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \left[\frac{N\times M}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right]^2\right\}$$

where $N = \alpha\beta y_j(1-\psi)^{\alpha-1}\psi p^{-1} - \alpha\beta(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda p^{-1}$ and $M = (1-\Lambda)^{\alpha}\log(1-\Lambda) - (1-\psi)^{\alpha}\log(1-\psi)$

$$\frac{\partial^2\ell}{\partial\beta\partial p} = \sum_{j=1}^{n}\left\{\frac{\alpha y_j(1-\psi)^{\alpha-1}\psi\, p^{-1}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{\alpha(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda\, p^{-1}}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} + \frac{\alpha\beta y_j^2\psi p^{-1}(1-\psi)^{\alpha-1}\log(p)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{\alpha\beta(y_j+1)^2\Lambda p^{-1}(1-\Lambda)^{\alpha-1}\log(p)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \frac{\alpha\beta(\alpha-1)y_j^2 2\psi p^{-1}(1-\psi)^{\alpha-2}\log(p)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} + \frac{\alpha\beta(\alpha-1)(y_j+1)^2 2\Lambda p^{-1}(1-\Lambda)^{\alpha-2}\log(p)}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}} - \left[\frac{Q\times U}{(1-\Lambda)^{\alpha}-(1-\psi)^{\alpha}}\right]^2\right\}$$

where $Q = \alpha\beta y_j(1-\psi)^{\alpha-1}\psi p^{-1} - \alpha\beta(y_j+1)(1-\Lambda)^{\alpha-1}\Lambda p^{-1}$ and $U = \alpha\, y_j\, \psi(1-\psi)^{\alpha}\log(p) - \alpha(y_j+1)\,\Lambda(1-\Lambda)^{\alpha-1}\log(p)$

It is known that the asymptotic distribution of the MLE $\hat{\theta}$ is [28]

$$(\hat{\theta}-\theta) \to N\left(0,\, I^{-1}(\theta)\right)$$

where $I^{-1}(\theta)$ is the inverse of the Fisher's information matrix of the unknown parameters $\theta = (\alpha,\, \beta\,,\, p)^T$ which is given as follows:

$$I_{Y(\alpha,\beta,p)}(\theta) = \begin{bmatrix} -E\left(\frac{\partial^2\ell}{\partial\alpha^2}\right) & -E\left(\frac{\partial^2\ell}{\partial\alpha\partial\beta}\right) & -E\left(\frac{\partial^2\ell}{\partial\alpha\partial p}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\alpha\partial\beta}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta^2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta\partial p}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\alpha\partial p}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta\partial p}\right) & -E\left(\frac{\partial^2\ell}{\partial p^2}\right) \end{bmatrix}$$

On the other hand, the Fisher?s information matrix can be computed by using the approximation

$$I_Y(\hat{\theta}) = \begin{bmatrix} -\left.\frac{\partial^2\ell}{\partial\alpha^2}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} & -\left.\frac{\partial^2\ell}{\partial\alpha\partial\beta}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} & -\left.\frac{\partial^2\ell}{\partial\alpha\partial p}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} \\ -\left.\frac{\partial^2\ell}{\partial\alpha\partial\beta}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} & -\left.\frac{\partial^2\ell}{\partial\beta^2}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} & -\left.\frac{\partial^2\ell}{\partial\beta\partial p}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} \\ -\left.\frac{\partial^2\ell}{\partial\alpha\partial p}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} & -\left.\frac{\partial^2\ell}{\partial\beta\partial p}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} & -\left.\frac{\partial^2\ell}{\partial p^2}\right|_{(\hat{\alpha},\hat{\beta},\hat{p})} \end{bmatrix}$$

where $\hat{\alpha}, \hat{\beta}$ and $\hat{p}$ are the MLEs of α, β and p respectively.

5 Application

In this section the proposed distribution DEETE is fitted to two right skewed real lifetime count data sets to test its goodness of fit and to examine how well it works comparing with some related distributions. MLE is utilized to estimate the parameters of DEETE. Moreover, DEETE is used to model the number of death cases of COVID-19 in some certain successive days during a given period of time.

5.1 Application of DEETE in life time count data sets

The first data set is reported by Consul and Jain [29] explains the accidents to 647 women working on Shells for 5 weeks (AWS) this data recently used by [7,30]. The second data is adduced by Ridout & Besbeas [31] represents the number of outbreaks of strikes in UK coal mining industries in four successive week periods during 1948-59 (OSU). This data was studied by [6,11]. To compare the DEETE distribution with discrete Generalized Rayleigh distribution DGR [11], discrete Burr DBD [5], discrete generalized exponentiated distribution of a second type DGE_2 [7], and Poisson distribution. Some measures of goodness of fit including the values of the log-likelihood function (-logL) and the χ^2 (chi square) statistic are calculated for the four distributions in order to verity which distribution fits better to these two data sets. The better distribution corresponds to smaller logL and χ^2 statistic. The calculations are done using R Studio statistical package. The two couples Tables 2, 3 and Tables 4, 5 respectively demonstrate the two data sets AWS and OSU together with their summary results including expected values, maximum likelihood estimates, -logL and χ^2 statistic with its corresponding P-value. From the results in Tables 2 to 5, we observe that the values of -logL for the proposed DEETE besides DGE_2 are the most minimum among the whole distributions (the smaller the better). In contrast, while the values of χ^2 statistic and their corresponding p-values demonstrate that the proposed DEETE is the most appropriate model to fit the two data sets. Moreover, Poison distribution is not good fit at all.

Table 2: The observed and expected values of AWS dataset

Count	Obs.	DEETE	DGR	DBD	DGE_2	Poisson
0	447	446.87	448.1	447.44	446.9	406.32
1	132	133.64	122.6	142.01	133.6	189.03
2	42	44.44	53	35.97	44.43	43.97
3	21	15.03	18.15	12.65	15.02	6.82
4	3	5.17	4.41	5.77	5.16	0.79
>5	2	1.85	0.72	3.16	1.84	0.07
Total	647	647	647	647	647	647

Table 3: Parameters estimates, -Log L, χ^2 statistic and p-value for the selected distributions of the AWS dataset

Model	MLEs	-Log L	χ^2 stat	p-value
DEETE	$\hat{\alpha}$=0.898, $\hat{\beta}$=2.0823, $\hat{p}$=0.594	592.183	3.445	0.6317
DGR	$\hat{\alpha}$=0.2196, $\hat{p}$=0.8123	592.544	6.172	0.2899
DBD	$\hat{\alpha}$=1.642, $\hat{\theta}$=0.1841	597.955	8.979	0.1099
DGE_2	$\hat{\alpha}$=0.898, $\hat{p}$=0.3379	592.183	3.448	0.6312
Poisson	$\hat{\lambda}$=0.4652	617.184	106.525	0.0000

Table 4: The observed and expected values of OSU dataset

Count	Obs.	DEETE	DGR	DBD	DGE_2	Poisson
0	46	46.08	47.5	47.89	46.08	57.87
1	76	75.75	69.32	80.87	75.75	57.5
2	24	26.11	31.82	18.28	26.11	28.62
3	9	6.5	6.67	6.07	6.5	9.55
>4	1	1.56	0.69	2.89	1.56	2.46
Total	156	156	156	156	156	156

Table 5: Parameters estimates, -Log L, χ^2 statistic and p-value for the selected distributions of the OSU dataset

Model	MLEs	-Log L	χ^2 stat	p-value
DEETE	$\hat{\alpha}$ =4.7995, $\hat{\beta}$=1.3016, $\hat{p}$=0.3175	187.534	1.335	0.8554
DGR	$\hat{\alpha}$=0.9415, $\hat{p}$=0.7172	188.329	3.5623	0.4685
DBD	$\hat{\alpha}$=4.6543, $\hat{\theta}$= 0.5941	192.21	4.816	0.3067
DGE_2	$\hat{\alpha}$=4.7994, $\hat{p}$=0.2247	187.534	1.3347	0.8555
Poisson	$\hat{\lambda}$=0.4652	617.184	10.029	0.0399

5.2 Application of DEETE in COVID-19 number of death case

The proposed distribution DEETE can adequately fit the number of deaths cases of COVID-19 and their corresponding number of days they occurred in a given period. A death cases data set of coronavirus from Gulf Cooperation Council countries, including Qatar, UAE, and Bahrain, is considered during the first case of coronavirus reported in each country to Nov 30, 2020. Table 6 shows the data sets with their summary results including maximum likelihood estimates, -logL and χ^2 statistic with its corresponding P-value.

6 Conclusions

This paper introduced a discretization version of extended Erlang-truncated exponential distribution called the discrete extended Erlang-truncated exponential DEETE distribution. We studied some statistical properties of the proposed distribution and estimated the model?s parameters using the Maximum likelihood method. We also derived some special cases from the DEETE. Finally, We applied DEETE on two real-life count data sets and compared it with some related discrete distributions. Moreover, we used DEETE to fit three data sets from deaths cases of COVID-19. The applications results concluded that the proposed DEETE performs better than other related discrete distributions. We recommend the proposed distribution for modelling most of the real-life count data sets adequacy because the three parameters of

Table 6: Number of deaths cases of COVID-19 and the corresponding number of days they occurred until 30 Nov 2021 in Qatar, UAE and Bahrain

Qatar data set		UAE data set		Bahrain data set	
# deaths	# days	# deaths	# days	# deaths	# days
0	156	0	93	0	124
1	58	1	70	1	67
2	31	2	67	2	41
3	18	3	31	3	31
4	9	4	17	4	7
5	4	5	8	5	9
6	0	6	7	6	2
7	1	7	4	>7	2
8	0	8	3		
9	0	9	3		
10	0	10	0		
>11	1	>11	3		
Total	**278**		**306**		**283**
$\hat{\alpha}$=0.8659, $\hat{\beta}$=2.0393		$\hat{\alpha}$=1.3283, $\hat{\beta}$=2.5176		$\hat{\alpha}$=1.1456, $\hat{\beta}$=1.4931	
$\hat{p}$=0.7042, -logL=353.5		$\hat{p}$=0.814,−logL=557.9		$\hat{p}$=0.646,-logL=429.3	
χ^2=2.1638,p-value=0.904		χ^2=9.6952,p-value=0.558		χ^2=8.5281,p-value=0.288	

the proposed distribution can be accommodated to suit most kind of count data sets.

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Availability of data and materials

Interested readers can contact the author.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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